Chapter 4

Abelian varieties

4.1 Abelian varieties

An abelian variety is a higher dimensional version of an elliptic curve. Here is the definition we use: a (complex) abelian variety is a complex torus, i.e. a quotient $X = \mathbb{C}^g/L$ where L is a lattice, that embeds into some complex projective space as a complex manifold. The significance of the last condition stems from Chow's theorem.

Theorem 4.1.1 (Chow)**.** *A complex submanifold of $\mathbb{P}^n_{\mathbb{C}}$ is a nonsingular projective algebraic variety i.e. it is the zero set of a collection of homogeneous polynomials.*

Corollary 4.1.2. *An abelian variety X is a projective variety. Furthermore the group operations $+ : X \times X \to X$ and $- : X \to X$ are morphisms of algebraic varieties*

For the last statement, we apply Chow's theorem to the graphs of $+$ and $-$. Abelian varieties can be defined over arbitrary fields. In this case, the statement in the corollary is taken as the definition.

Although we saw that every elliptic curve is projective, it is not true for arbitrary tori. To formulate sufficient conditions, we modify what we did before, but now we replace the element τ in the upper half, with matrix $\Omega \in \mathbb{H}_g$. Given such a matrix, we can define the lattice

$$L_\Omega = \mathbb{Z}^g + \Omega\mathbb{Z}^g$$

The following is basically due to Riemann and Lefschetz

Theorem 4.1.3 (Riemann-Lefschetz)**.** *If $L \subseteq L_\Omega$ is a sublattice, then $A = \mathbb{C}^g/L$ is an abelian variety.*

Corollary 4.1.4. *A Jacobian is an abelian variety.*

A sketch of a proof will occupy the rest of this section. For simplicity, we treat the case where $L = L_\Omega$

As a first step, we define the Riemann theta function on $\mathbb{C}^g$ by

$$\theta(z) = \sum_{n\in\mathbb{Z}^g} \exp(\pi i n^t \Omega n + 2\pi i n^t z)$$

This is a generalization of the Jacobi function. Since $Im\,\Omega$ has positive eigenvalues, $\|\exp(\pi i n^t \Omega n)\| \to 0$ rapidly as $\|n\| \to \infty$. So the series converges to a holomorphic function. It is quasi-periodic in the sense that

$$\theta(z+n) = \theta(z)$$

$$\theta(z+\Omega n) = \exp(-\pi i n^t \Omega n - 2\pi i z^t n)\theta(z)$$

for $n \in \mathbb{Z}^g$. See [Mumford, Lectures on theta I] for details.

For each natural number ℓ, define V_ℓ to be the space of holomorphic functions satisfying

$$f(z+n) = f(z)$$

$$f(z+\Omega n) = \exp(-\pi i \ell n^t \Omega n - 2\pi i \ell z^t n) f(z)$$

Of course V_1 contains θ. Given $a, b \in \mathbb{C}^g$, let

$$\theta_{ab}(z) = \theta(z+a+b)\theta(z-a)\theta(z-b)$$

One sees that $\theta_{ab}(z) \in V_3$.

Lemma 4.1.5. $\dim V_\ell = \ell^g$

Sketch. When $\ell = 1$, $f(z) \in V_1$ satisfies $f(z+n) = f(z)$. Therefore it can be expanded in Fourier series

$$f_n(z) = \sum a_n \exp(2\pi i n^t z)$$

The second functional equation

$$f(z+\Omega n) = \exp(-\pi i n^t \Omega n - 2\pi i z^t n) f(z)$$

implies $f(z) = a_0\theta(z)$. For general ℓ, the analogue of the last equation yields a recurrence condition on the Fourier coefficients with ℓ^g free choices. □

Lemma 4.1.6. *Given $z_0 \in \mathbb{C}^g$, there exists $f(z) \in V_3$ such that $f(z_0) \neq 0$.*

Proof. Since $\theta \neq 0$, we can find a, b such that $\theta_{a,b}(z_0) \neq 0$. □

Choose a basis $f_1, \ldots, f_{3^g} \in V_3$, then the previous lemma shows that the map $\iota : A \to \mathbb{P}^{3^g-1}$ given by

$$\iota(z) = [f_1(z), \ldots, f_{3^g}(z)]$$

defined everywhere. To finish the proof of theorem 4.1.3, we will show that this is an embedding.

It is convenient to switch to more geometric language. A *divisor* on A is a formal integer linear combination of hypersurfaces in it. Given a nonzero theta function f, which is irreducible in the sense that it can't be factored, its zero set $D = Z(f)$ will be invariant under L, so it defines a divisor in A. In particular, $\Theta = Z(\theta) \subset A$ is called the theta divisor. (We omit the proof of irreducibility, but it's true.)

Given $a \in A$, let $t_a : A \to A$ denote translation by a. Let t_a^*D denote the preimage of D under this operation. A little thought, shows that $t_a^*D = D - a$.

Lemma 4.1.7. *$b \in t_a^*D$ if and only $a \in t_b^*D$.*

Proof. Both sides are equivalent to $a + b \in D$. □

Lemma 4.1.8. *If $a \neq 0$, then $t_a^*\Theta \neq \Theta$.*

Then the divisor of $\theta_{a,b}$ is

$$\Theta_{a,b} = t_{a+b}^*\Theta + t_{-a}^*\Theta + t_{-b}^*\Theta$$

Given an irreducible divisor $D \subset A$ defined by f, let $D_{smooth} \subset D$ denote the set of smooth points (the set where ∇f is nonzero). To each $x \in D_{smooth}$, define the *Gauss map* by $G(x) = [\nabla f(x)] \in \mathbb{P}^{g-1}$. In other words, $G(x)$ is the tangent space to D at x viewed as a subspace of $\mathbb{C}^g$.

Lemma 4.1.9. *The Gauss map of Θ is nonconstant.*

Proof. Intuitively, this comes to the fact that the Gauss map of a hypersurface is nonconstant unless it's "linear", but Θ clearly isn't. We refer to pp 81-82 [Birkenhake-Lange, Complex Abelian Varietes] for a detailed proof. □

Corollary 4.1.10. *Given a nonzero vector $v \in \mathbb{C}^g$, there exists a point $x \in \Theta_{smooth}$ such that v is not tangent to Θ at x.*

Proof of theorem 4.1.3. We need to show that ι is injective. It suffices to show that given points $x \neq y$, we can find $\Theta_{a,b}$ containing x but not y. Since $x - y \neq 0$, $t_{x-y}^*\Theta \neq \Theta$ by lemma 4.1.8. Therefore $t_x^*\Theta \neq t_y^*\Theta$. Consequently, there exists $a \in A$ with $-a \in t_x^*\Theta$ and $-a \notin t_y^*\Theta$. This implies that $x \in t_{-a}^*\Theta$ and $y \notin t_{-a}^*\Theta$ by lemma 4.1.7. A similar argument shows that there exists b such that $y \in t_{-b}^*\Theta \cup t_{a+b}^*\Theta$. Therefore $\Theta_{a,b}$ contains x but not y.

To finish the proof, we need to show that the derivative of ι is injective at all points. In the language of divisors, it is enough to prove that given $x \in A$ and a tangent vector v to A at x, there exists a, b such that $x \in \Theta_{a,b}$ is a nonsingular point and v is not tangent to $\Theta_{a,b}$ at x. This follows from corollary 4.1.10. □

It is worth noting that in the language of algebraic geometry, the above proof shows that 3Θ is a very ample divisor, and therefore that Θ is ample.

Corollary 4.1.11. *A Jacobian is an abelian variety.*

4.2 Riemann forms

We want to characterize lattices of the form $L \subseteq L_\Omega = \mathbb{Z}^g + \Omega\mathbb{Z}^g$, with $\Omega \in \mathbb{H}_g$, in coordinate free language.

Definition 4.2.1. *A* polarization *or* Riemann form *(the terms are interchangeable) on a lattice $L \subset V$, in a finite dimensional complex vector space, is a nondegenerate real skew-symmetric bilinear form $E : V \times V \to \mathbb{R}$ such that*

(a) $E(u, v) \in \mathbb{Z}$, when $u, v \in L$.

(b) There exist a positive definite hermitian form H on $\mathbb{C}^g$, such that $E = Im\, H$.

This is called a principal polarization if in addition $\det E = 1$.

Lemma 4.2.2. *If E is a polarization, then H above is uniquely determined.*

Proof. We leave it as linear algebra exercise to show that

$$H(x, y) = E(ix, y) + iE(x, y)$$

□

Lemma 4.2.3. *Suppose $L = L_\Omega = \mathbb{Z}^g + \Omega\mathbb{Z}^g$ is a lattice with $\Omega \in \mathbb{H}_g$. Let $e_1, \dots e_g$, be the standard basis of $\mathbb{Z}^g$. We can extend this to basis of L, by taking e_{g+j} to be the jth column of Ω. This can also be viewed as a* real *basis of $\mathbb{C}^g$. Let $E : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{R}$ be the real bilinear form with matrix*

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{4.1}$$

with respect to this basis. Then E is a principal polarization on L

Proof. The form E is clearly integer valued on L with $\det E = 1$. The form

$$H(u, v) = u^t (Im\Omega)^{-1} \bar{v}$$

is positive hermitian because the matrix $(Im\Omega)^{-1}$ is positive definite symmetric. We can check that $E = Im\, H$ by calculation: for $j, k \le g$,

$$Im\, H(e_j, e_k) = Im[e_j^t (Im\Omega^{-1}) e_k] = 0$$

$$Im\, H(e_j, e_{g+k}) = Im[(e_j^t (Im\,\Omega^{-1})(Re\,\Omega e_k - \sqrt{-1} Im\,\Omega e_k)] = -\delta_{jk}$$

Finally, by the spectral theorem, we can assume that $Im\,\Omega$ is the diagonal matrix $diag(\tau_1, \dots, \tau_g)$. Then if $j \ne k$

$$Im\, H(e_{g+j}, e_{g+k}) = 0$$

and

$$Im\, H(e_{g+j}, e_{g+j}) = Im[\tau_j e_j^t (\tau_j^{-1}) \bar{\tau}_j e_j] = 0$$

□

Corollary 4.2.4. *A sublattice of L_Ω carries a not necessarily principal polarization.*

We omit the proof, but the previous lemma has converse.

Lemma 4.2.5. *If L has a polarization, then after choosing bases of $\mathbb{C}^g$ and L, we have $L \subseteq L_\Omega$ for some $\Omega \in \mathbb{H}_g$. If the polarization is principal, we can choose a basis such that $L = L_\Omega$.*

Theorem 4.2.6 (Riemann-Lefschetz). *$\mathbb{C}^g/L$ is an abelian variety if and only if L possesses a polarization.*

The key fact, we need is

Theorem 4.2.7. *There is an isomorphism $H^2(\mathbb{P}^N_C, \mathbb{Z}) \cong \mathbb{Z}$. Under the natural embedding $H^2(\mathbb{P}^N_{\mathbb{C}}, \mathbb{Z}) \subset H^2_{dR}(\mathbb{P}^N_{\mathbb{C}}, \mathbb{C})$, a generator is represented by the differential form*

$$\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(|\zeta_0|^2 + \ldots + |\zeta_N|^2)$$

where $\zeta_0, \ldots$ are homogeneous coordinates.

We want to make a few comments about the formula.

1. The definition of $\partial, \bar{\partial}$ are similar to what we did what we did in dimension one. Given f, collect the terms of df involving $d\zeta_0, \ldots$ (resp. $d\bar{\zeta}_0, \ldots$) to get ∂f (resp. $\bar{\partial} f$).

2. Homogeneous coordinates are not true coordinates. To get those, we take ratios ζ_i/ζ_j on the sets $U_j = \{\zeta_j \neq 0\}$. We can rewrite

$$\omega|_{U_j} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(|\zeta_0/\zeta_j|^2 + \ldots + |\zeta_N/\zeta_j|^2)$$

   in terms of the true coordinates.

3. The significance of the normalization is that ω can be rewritten as a real differential form such that

$$\int_{\mathbb{P}^1} \omega = 1$$

   where $\mathbb{P}^1$ is the line defined by $z_2 = \ldots z_N = 0$. This is why the class of ω generates $H^2(\mathbb{P}^N, \mathbb{Z})$, and in fact it gives the preferred generator.

4. ω is a *positive* $(1,1)$ form. This means that

$$\omega = \sqrt{-1}\sum_{jk} h_{jk} d(\zeta_j/\zeta_0) \wedge \overline{d(\zeta_k/\zeta_0)}$$

   where h_{jk} is a positive definite Hermitian matrix. To see this observes that ω is invariant under the action of the unitary group $U(N+1)$, and after some calculation that at $[1, 0, \ldots, 0]$ $h_{jk} = (1/2\pi)\delta_{jk}$.

5. Finally one could ask where ω comes from. Without defining what it means, the answer is that ω is the Kähler form for the Fubini-Study metric.

Let $A = \mathbb{C}^g/L$. Choose a basis $e_1, \ldots, e_{2g}$ for L, the dual basis e_i^* for L^*. Then using the Künneth formula, we can easily compute cohomology,

Lemma 4.2.8. *We isomorphisms*

$$H^1(A, \mathbb{Z}) \cong L^* = \bigoplus \mathbb{Z}e_i^*$$

$$H^2(A, \mathbb{Z}) \cong \wedge^2 L^* = \bigoplus \mathbb{Z}e_i^* \wedge e_j^*$$

$$\ldots$$

$$H^1_{dR}(A, \mathbb{C}) = \bigoplus \mathbb{C}e_i^*$$

$$H^2_{dR}(A, \mathbb{C}) = \bigoplus \mathbb{C}e_i^* \wedge e_j^*$$

$$\ldots$$

The last set of isomorphisms imply that any de Rham cohomology class is represented by a unique differential form with constant coefficients. If ω is closed p-form, then its cohomology class is represented by the p-form with constant coefficients given by

$$I(\omega) = \sum_{j_1 < \ldots < j_p} \left(\int_{e_{j_1} \times \ldots \times e_{j_p}} \omega \right) e_{j_1}^* \wedge \ldots \wedge e_{j_p}^*$$

where $e_{j_i} \times \ldots$ is understood as the subtorus of A given by the projection of the product of line segments $[0, e_{j_1}] \times \ldots \subset \mathbb{C}^g$.

Sketch of proof of theorem 4.2.6. The "if" direction was theorem 4.1.3 above. Let us briefly explain the converse. By definition we have an embedding $A \subset \mathbb{P}^N_{\mathbb{C}}$. The class $[\omega] \in H^2(\mathbb{P}^N, \mathbb{Z})$ restricts to a class $E \in H^2(A, \mathbb{Z})$ which can be viewed as an integer valued skew symmetric form on L using the isomorphism $H^2(A, \mathbb{Z}) \cong \wedge^2 L^*$. To verify the remaining axiom, under the embedding $H^2(A, \mathbb{Z}) \subset H^2_{dR}(A, \mathbb{C})$, E can be represented by the differential given as the restriction of ω on $\mathbb{P}^N$ above. The form $\omega|_A$ will continue to be a positive $(1,1)$ form. A bit of thought using the previous formula shows that $I(\omega|_A)$ will remain a positive $(1,1)$ form. In other words

$$I(\omega|_A) = \sqrt{-1} \sum_{jk} h_{jk} dz_j \wedge d\bar{z}_j$$

where $H = (h_{jk})$ is positive definite hermitian matrix with constant coefficients. This will satisfy $E = ImH$. □

4.3 Examples of abelian varieties and tori

4.3.1 Non abelian tori

Let $g > 1$. A lattice in $\mathbb{C}^g$ is given $2g$ $\mathbb{R}$-linearly independent vectors $v_1, \ldots v_{2g}$. Two tori are isomorphic if the underlying lattice of one of them can be taken to the other by a linear automorphism. Therefore we can assume that $(v_1, \ldots v_g)$ is the identity. Without trying to be too rigorous, this shows that a g dimensional torus depends on g free parameters $v_{g+1}, \ldots, v_{2g}$. On the other hand a g dimensional abelian variety depends on a choice of $\Omega \in \mathbb{H}_g$, which involves $g(g+1)/2$ parameters. Thus there should (and does !) exist g-dimensional tori which are not abelian varieties.

4.3.2 Dual tori

Let $L \subset V$ be a lattice in a finite dimensional complex vector space, then $A = V/L$ is a complex torus. A function $f : V \to \mathbb{C}$ is antilinear if it is additive, and $f(av) = \bar{a}f(v)$ for $a \in \mathbb{C}$, $v \in V$. Let

$$\hat{V} = \{f : V \to \mathbb{C} \mid f \text{ is antilinear } \}$$

This is naturally a complex vector space of the same dimension as V called the antilinear dual. It is easy to see that

$$\hat{L} = \{f \in \hat{V} \mid Im\, f(\hat{L}) \subseteq \mathbb{Z}\}$$

is a lattice. We define the *dual torus* by

$$\hat{A} = \hat{V}/\hat{L}$$

Suppose that E is a polarization with associated hermitian form H, then we can define an isomorphism

$$\phi_E : V \cong \hat{V}, \quad \phi_E(v) = H(v, -)$$

This satisfies $\phi(L) \subseteq \hat{L}$ with equality if E is principal. In general, we can always find an integer $N > 0$ such that $N\hat{L} \subset \phi(L)$. Restricting the polarization given by E to $\hat{L}$ under this embedding yields a polarization on $\hat{L}$. Therefore we can conclude that:

Proposition 4.3.3. *If A is an abelian variety, then so is $\hat{A}$. If A is principally polarized then $A \cong \hat{A}$.*

4.3.4 Albanese varieties

We have see that Jacobians are principally polarized abelian varieties. The construction can be generalized as follows. Let X be a smooth projective variety (over $\mathbb{C}$) of dimension n. The Hodge theorem in higher dimensions shows that

$H_1(X,\mathbb{Z})/torsion$ embeds as a lattice in $H^0(X,\Omega^1_X)^*$ by sending $\gamma \to \int_\gamma$ as before. The Albanese torus

$$Alb(X) = \frac{H^0(X,\Omega^1_X)^*}{H_1(X,\mathbb{Z})/torsion}$$

We claim that this is an abelian variety. To prove this, we need to construct a polarization. Fix an embedding $X \subset \mathbb{P}^N$. Choose $n-1$ hyperplanes $H_1,\ldots,H_{n-1}$ in general position. Bertini shows that $C = X \cap H_1 \cap \ldots H_{n-1}$ is a smooth curve. We can now obtain a (possibly nonprincipal) polarization E on $Alb(X)$ by restricting to C. At the level of differential forms

$$E(\alpha,\beta) = \int_C \alpha|_C \wedge \beta|_C$$

Therefore $Alb(X)$ is an abelian variety.

When $x_0 \in X$, one can define a holomorphic map $\alpha : X \to Alb(X)$ by

$$\alpha(x) = \int_{x_0}^{x} \mod H_1(X,\mathbb{Z})$$

exactly as in the one dimensional case. This is a really important construction in algebraic geometry, since it allows us to partially reduce questions about X to abelian varieties which are easier to understand. When X is an abelian variety, α gives an isomorphism $X \cong Alb(X)$. So in particular, every abelian variety is an Albanese.

4.3.5 Picard varieties

The construction $J^\vee(X)$ also generalizes to higher dimensions Let X be a smooth projective variety. Using the exponential sequence as before, we get an exact sequence

$$\ldots H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}^*_X) \to H^2(X,\mathbb{Z})$$

Again using the Hodge theorem for X, one can see that

$$Pic^0(X) := \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})} = \ker[H^1(X,\mathcal{O}^*_X) \to H^2(X,\mathbb{Z})]$$

is a torus call the Picard torus. The previous argument for Riemann surfaces generalizes to show that

$$Pic^0(X) = \widehat{Alb(X)}$$

Therefore this is also an abelian variety. Unlike the one dimensional case $Alb(X)$ and $Pic^0(X)$ are generally not isomorphic. If X is an abelian variety, $Pic^0(X) \cong \hat{X}$.

4.4 Further comments about Jacobians

As we have seen, given a compact Riemann surface X of genus g, we can associate an abelian variety $J(X)$ with a canonical polarization E. What is equivalent, and more geometric, is to consider the theta divisor $\Theta \subset J(X)$. Under Poincaré duality, the class $E \in H^2(J(X), \mathbb{Z}) \cong H_{2g-2}(J(X), \mathbb{Z})$ is the homology class $[\Theta]$. The divisor has a very nice geometric interpretation:

Theorem 4.4.1 (Riemann)**.** *Up to translation Θ is the image $\alpha(X^{g-1})$ (which is traditionally denoted by W_{g-1}). In particular, $[\Theta] = [W_{g-1}]$*

We won't give the proof. It can be found in Griffiths and Harris. The polarized Jacobian is a complete invariant:

Theorem 4.4.2 (Torelli)**.** *$J(X)$ with its canonical polarization determines X.*

In the simplest case, where the genus $g = 1$, $X \cong J(X)$, so the polarization is not needed. But otherwise, it is. There exists examples of nonisomorphic curves with isomorphic Jacobians (but the isomorphism won't respect polarizations). When $g = 2$, Torelli's theorem follows immediately from Riemann's theorem, because it implies $X \cong \Theta$. In general, X is also reconstructed using Θ, but the proof is more involved. Again we refer to Griffiths and Harris for the details.

Finally, we might wonder about the relationship between Jacobians and arbitrary abelian varieties. In fact, is not true is that every abelian variety is a Jacobian. But what is true is given an abelian variety A, we can always find a surjective homomorphism $J(X) \to A$ from a Jacobian. This trick was used often enough that Mumford, in the introduction to his book on abelian varieties says, "Rather stubbornly I wanted to prove that the theory of abelian varieties could be developed without the crutch of 'reduction to Jacobians' ".