Chapter 6

Moduli space of abelian varieties

6.1 The action of the symplectic group

We want to generalize the constructions given earlier for elliptic curves to higher dimensions. The first step is to replace the usual upper half plane with the Siegel upper half plane. Recall that this

$$\mathbb{H}_g = \{\Omega \in Mat_{g\times g}(\mathbb{C}) \mid \Omega = \Omega^T, \operatorname{Im}(\Omega) > 0\}$$

This is an open subset of the space of symmetric matrices. So its dimension is $g(g+1)/2$. Given $\Omega \in H_g$ we can construct a lattice $L_\Omega = \mathbb{Z}^g + \Omega\mathbb{Z}^g$ and an abelian variety $A_\Omega = \mathbb{C}^g/L_\Omega$. This carries a principal polarization E_Ω given by the matrix

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{6.1}$$

with respect to the basis of L_Ω obtained by combining the standard basis of $\mathbb{Z}^g$ with the columns of Ω. This was explained before in detail.

For now the key point, is that $\Omega \in \mathbb{H}_g$ gives a pair (A_Ω, H_Ω) consisting of an abelian variety with a principal polarization. This is a key difference from the one dimensional theory, and that is that in order to get a good moduli theory, we need to keep track of a polarization. Given two such pairs $(V_i/L_i, E_i)$, by an isomorphism we mean a linear isomorphism $f : V_1 \xrightarrow{\sim} V_2$ such that $\phi(L_1) = L_2$ and $E_1(v,w) = E_2(\phi(v), \phi(w))$.

Lemma 6.1.1. *Given any g dimensional principally polarized abelian variety (A, E), there exists $\Omega \in H_g$ and an isomorphism $(A,E) \cong (A_\Omega, E_\Omega)$.*

Sketch. We just outline the argument, because it is essentially the same as what we did to obtain the normalized period matrix for a Riemann surface. Write

$A = V/L$. One of the conditions for E to be principal polarization guarantee we can choose a basis $\alpha_1, \dots \beta_g$ for L satisfying

$$E(\alpha_i, \alpha_j) = E(\beta_i, \beta_j) = 0, E(\alpha_i, \beta_j) = \delta_{ij}$$

Then observe that the remaining conditions for a polarization guarantee that we can choose a basis $\omega_1, \dots, \omega_g$ for L such that the change of basis matrix from the second to the first basis is (I, Ω) with $\Omega \in \mathbb{H}_g$.

□

Given (A, E) as above, there are many Ω's which satisfy $(A, E) \cong (A_\Omega, E_\Omega)$. The next problem is to deal with the nonuniqueness. A point of $\mathbb{H}_g$ gives rise to a polarized abelian variety with a preferred basis for the lattice, given the columns of (Ω, I). We need to mod out the choice of basis. It is important to restrict to change of bases which are compatible with the polarization. For any commutative ring R (e.g. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$) we define the symplectic group

$$Sp_{2g}(R) = \left\{ M \in GL_{2g}(R) \mid M^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \right\}$$

In other words, this is the group of matrices with preserves the symplectic form E.

Lemma 6.1.2. *Given* $\Omega \in \mathbb{H}_g$ *and* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{R})$

$$M \cdot \Omega := (A\Omega + B)(C\Omega + D)^{-1} \in \mathbb{H}_g$$

This defines an action of $Sp_{2g}(\mathbb{R})$ *on* $\mathbb{H}_g$*.*

Proof. For M as above, one checks the following identities: A^TC and B^TD are symmetric, and $A^TD - C^TB = I$. Let $M(\Omega) = (A\Omega + B)(C\Omega + D)^{-1}$. After expanding, using the above identities, and canceling, we obtain

$$(C\Omega + D)^T (M(\Omega) - M(\Omega)^T)(C\Omega + D) = \Omega - \Omega^T = 0$$

Therefore $M(\Omega)$ is symmetric. Similarly

$$(C\Omega + D)^T \operatorname{Im} M(\Omega)(C\Omega + D) = \operatorname{Im} \Omega > 0$$

which implies that $M(\Omega)$ is positive definite. □

One can put the Siegel space in the more general framework of symmetric spaces using the following:

Lemma 6.1.3. *The action of* $Sp_{2g}(\mathbb{R})$ *on* $\mathbb{H}_g$ *is transitive and the stabilizer of* iI *is*

$$\left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid AB^T = BA^T, AA^T + BB^T = I \right\} \cong U_g(\mathbb{R})$$

where the isomorphism is given by sending

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + iB$$

Proof. Let $\Omega = X + iY \in H_g$. Since Y is symmetric and positive definite, we can find an $A \in GL_g(\mathbb{R})$ so that $Y = AA^T$. Then $M = \begin{pmatrix} A & X(A^T)^{-1} \\ 0 & (A^T)^{-1} \end{pmatrix}$ sends iI to Ω. The formula for the stabilizer can be checked by calculation. $\square$

Corollary 6.1.4. *Thus* $\mathbb{H}_g \cong Sp_{2g}(\mathbb{R})/U_g(\mathbb{R})$.

6.2 The moduli space of abelian varieties

We define

$$\mathcal{A}_g = Sp_{2g}(\mathbb{Z})\backslash\mathbb{H}_g = Sp_{2g}(\mathbb{Z})\backslash Sp_{2g}(\mathbb{R})/U_g(\mathbb{R})$$

Theorem 6.2.1. *Given* $\Omega, \Omega' \in \mathbb{H}_g$*, the pairs* $(A_\Omega, E_\Omega), (A_{\Omega'}, E_{\Omega'})$ *are isomorphic if and only if there exists* $M \in Sp_{2g}(\mathbb{Z})$ *with* $\Omega' = M \cdot \Omega$*. In particular,* $\mathcal{A}_g$ *can be identified with the set of isomorphism classes of* g*-dimensional principally polarized abelian varieties.*

Proof. An isomorphism $(A_\Omega, E_\Omega) \cong (A_{\Omega'}, E_{\Omega'})$ is induced by an invertible $g \times g$ complex matrix Φ such that (a) $\Phi L_\Omega = L_{\Omega'}$ and (b) Φ takes E_Ω to $E_{\Omega'}$. Condition (a) implies that there exists integer $g \times g$ matrices A, B, C, D such that

$$\begin{aligned} \Omega' &= \Phi A \Omega + \Phi B \\ I &= \Phi C \Omega + \Phi D \end{aligned} \tag{6.2}$$

Condition (b) implies that $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z})$. We can solve (6.2) to obtain $\Omega' = M \cdot \Omega$. Conversely, if $\Omega' = M \cdot \Omega$, with M as above, then (6.2) holds with $\Phi = (C\Omega + D)^{-1}$. Therefore Φ gives an isomorphism $(A_\Omega, E_\Omega) \cong (A_{\Omega'}, E_{\Omega'})$.

$\square$

We will give a more conceptual proof later. Our next goal is to understand the geometry of $\mathcal{A}_g$ better.

Lemma 6.2.2. *The action of* $Sp_{2g}(\mathbb{Z})$ *on* $\mathbb{H}_g$ *is properly discontinuous. Therefore the quotient is a Hausdorff space.*

Proof. Given compact sets $K_1, K_2 \subset H_g$, we have to show that $S = \{M \in Sp_{2g}(\mathbb{Z}) \mid M(K_1) \cap K_2 \neq \emptyset\}$ is finite. Let us identify $H_g = Sp_{2g}(\mathbb{R})/U_g(\mathbb{R})$ as above. Note that the group $U_g(\mathbb{R})$ is compact, so that the projection $p : Sp_{2g}(\mathbb{R}) \to H_g$ is proper. $M \in Sp_{2g}(\mathbb{Z})$ lies in S if and only if $Mp^{-1}K_1 \cap p^{-1}K_2 \neq \emptyset$ if and only if $M \in T = (p^{-1}(K_1))^{-1} \cap p^{-1}(K_2)$. Now T is compact because it is the image of $K_1 \times K_2$ under $(M_1, M_2) \mapsto M_1^{-1}M_2$. Therefore S is the intersection of a compact set with a discrete set, so it's finite. $\square$

The action of $Sp_{2g}(\mathbb{Z})$ on $\mathbb{H}_g$ is not free, so $\mathcal{A}_g$ will have singularities. The solution as before is to pass to a suitable subgroup. Given an integer $n > 0$, define the principal congruence group by

$$\Gamma(n) = \ker[Sp_{2g}(\mathbb{Z}) \to Sp_{2g}(\mathbb{Z}/n\mathbb{Z})]$$

Proposition 6.2.3. *Let $n \geq 3$. Suppose that γ is an automorphism of a principally polarized abelian variety (A, E) which acts trivially on the lattice mod n. Then $\gamma = 1$. The action for $\Gamma(n)$ on $\mathbb{H}_g$ is free.*

Proof. We assume that $\gamma \neq 1$. Then it has finite order, which we can assume is a prime p, by replacing γ a power. Then by assumption, $1 - \gamma = n\phi$ where $\phi \in End(A)$. Let ζ be a nontrivial eigenvalue of γ, and let η be the corresponding eigenvalue of ϕ. ζ is a primitive pth root of unity and η is an algebraic integer in the cyclotomic field $\mathbb{Q}(\zeta)$. We have a relation $n\eta = 1 - \zeta$. Taking the norm with respect to $\mathbb{Q}(\zeta)/\mathbb{Q}$ yields an equality of integers

$$n^{p-1}N(\eta) = (1-\zeta)(1-\zeta^2)\ldots(1-\zeta^{p-1}) = p$$

But this impossible because p is prime and $n \geq 3$. This proves the first part.

Suppose that $\gamma \in \Gamma(n)$ fixes a point of $\Omega \in \mathbb{H}_g$. Then γ will be an automorphism of (A_Ω, E_Ω) which is trivial mod n. Therevfore $\gamma = 1$. $\square$

A level n-structure on an abelian variety $A = \mathbb{C}^g/L$ is a choice of symplectic basis of

$$H^1(A, \mathbb{Z}/n\mathbb{Z}) \cong Hom(L, \mathbb{Z}/n\mathbb{Z})$$

Let

$$\mathcal{A}_{g,n} = \Gamma(n)\backslash\mathbb{H}_g$$

Theorem 6.2.4. *Suppose $n \geq 3$. Then $\mathcal{A}_{g,n}$ is a complex manifold. The semidirect product $\Gamma(n) \ltimes \mathbb{Z}^{2g}$ acts naturally on $\mathbb{H}_g \times \mathbb{C}^g$ and the quotient can be equipped with a family of sections and polarizartions so that it becomes the universal family of principally polarized g-dimensional abelian varieties with level n-structure. In particular $\mathcal{A}_{g,n}$ is a fine moduli space.*

Outline. Since $\Gamma(n)$ acts freely on $\mathbb{H}_g$, $A_{g,n}$ is a manifold. The remaining statements are similar to case of $g = 1$ discussed earlier. $\square$

The moduli space allows us to study the behaviour of a typical abelian variety. Recall that a set in a complete metric space is called meagre if it is a countable union of nowhere dense sets. Such sets should be viewed as very small, and in particular the Baire category theorem says that a meagre set has a nonempty complement.

Proposition 6.2.5. *There exists a set $U \subset \mathcal{A}_g$ which is the complement of meagre set and such that for any $(A, E) \in U$*

(a) (A, E) is not a product of polarized abelian varieties of smaller dimension

(b) $End(A) = \mathbb{Z}$.

Proof. We just explain the proof for (a). Let

$$R' = \bigcup_{g>h>0} \mathbb{H}_h \times \mathbb{H}_{g-h} \subset \mathbb{H}_g$$

and let R denote the union of all translates of R under $Sp_{2g}(\mathbb{Z})$, Then R is meagre in $\mathbb{H}_g$, and the same is true for its image $\bar{R} \subset \mathcal{A}_g$. Let U be the complement of $\bar{R}$. □

6.3 $\mathcal{A}_g$ is an algebraic variety

At the moment, we have an analytic construction of $\mathcal{A}_g$, which shows that it is almost a complex manifold; more precisely a quotient of a manifold by a finite group. In fact, it turns out to be a quasi-projective algebraic variety. The first step goes back to Satake, who constructed an explicit analytic compactification $\mathcal{A}_g^* \supset \mathcal{A}_g$ now called the Satake compactification. It is also called the Bailey-Borel compactifcation because these authors gave a more general construction a few years later. Before saying more, it is helpful to recall what happened when $g = 1$. Then we found that $\mathcal{A}_1 \cong \mathbb{C}$. This can be compactified by adding a point to get $\mathcal{A}_1^* = \mathbb{P}_1$. Before taking the quotient, we should add ∞, and all its translates to $\mathbb{H}$. This amounts to taking $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. In order to get $SL_2(\mathbb{Z})\backslash\mathbb{H}^* = \mathbb{P}^1$ as a topological space, we need to put a somewhat strange topology on $\mathbb{H}^*$. The basic neighbourhoods of ∞ are the strips $\operatorname{Im} z > c$, $c \in \mathbb{R}_+$, and for other points on the boundary we take translates of these.

Now let us suppose that $g > 1$. We describe $\mathcal{A}_g^*$ as a set. As a first step, we switch to the disk model.

Lemma 6.3.1. *$\mathbb{H}_g$ is isomorphic to*

$$D_g = \{Z \in Mat_{g\times g}(\mathbb{C}) \mid Z^T = Z, I - \bar{Z}Z > 0\}$$

by sending

$$\Omega \mapsto (\Omega - \sqrt{-1}I)(\Omega + \sqrt{-1}I)^{-1}$$

Let $\overline{D}_g$ denote the closure of D_g in the space of symmetric matrices. For $r < g$, we can embed $D_r \hookrightarrow \overline{D}_g$ by identifying it with

$$\left\{\begin{pmatrix} Z & 0 \\ 0 & I \end{pmatrix} \mid Z \in D_r\right\}$$

Let D_g^* denote D_g union the of images of D_r under $Sp_{2g}(\mathbb{Z})$ for all $r < g$. Then define

$$\mathcal{A}_g^* := Sp_{2g}(\mathbb{Z})\backslash D_g^*$$

The action of $Sp_{2g}(\mathbb{Z})$ on each boundary component D_r factors through the action of $Sp_{2r}(\mathbb{Z})$. Thus

$$\mathcal{A}_g^* = \mathcal{A}_g \amalg \mathcal{A}_{g-1} \amalg \mathcal{A}_{g-2} \ldots$$

which for the moment is only a set. However, it has more structure. To formulate the result, we need to say what an analytic space is. A reduced analytic space is roughly speaking a compact manifold with singularities. The basic example is a pair $(Z, \mathcal{O}_Z)$ consisting of the zero set Z of a collection of holomorphic functions $f_1, \ldots, f_r$ on the ball $B \subset \mathbb{C}^n$, and $\mathcal{O}_Z$ the sheaf of holomorphic functions on B restricted to this. In general, a reduced analytic space is a ringed space $(X, \mathcal{O}_X)$ consisting of a metrizable space X and a sheaf of complex valued functions $\mathcal{O}_X$ which is locally isomorphic, at each point, to one of the basic examples.

Theorem 6.3.2 (Satake, 1957)**.** *The set D_g^* carries an $Sp_{2g}(\mathbb{Z})$-invariant topology, such that the quotient topology on $\mathcal{A}_g^*$ is compact Hausdorff. Furthermore, when $g > 1$ we have a sheaf of functions $\mathcal{O}_{\mathcal{A}_g^*}$, such that $f \in \mathcal{O}_{\mathcal{A}_g^*}(U)$ if and only if its pullback to the preimage of U is $\mathbb{H}_g$ is holomorphic. The pair $(\mathcal{A}_g^*, \mathcal{O}_{\mathcal{A}_g^*})$ is a reduced analytic space*

To finish the story define a *Siegel modular form* of weight k to be a holomorphic function $f : \mathbb{H}_g \to \mathbb{C}$ such that

$$\forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z}),\ f((A\Omega + B)(C\Omega + D)^{-1}) = \det(C\Omega + D)^k f(\Omega)$$

This generalizes the notion of modular form discussed earlier when $g = 1$. However, unlike that case, when $g > 1$ there are no conditions at infinity.

Theorem 6.3.3 (Bailey, 1962)**.** *For some (in fact infinitely many) k, there exists a set $f_0, \ldots f_N$ of Siegel modular forms of weight k such that there is an embedding $\mathcal{A}_g \hookrightarrow \mathbb{P}^N$ given by $\Omega \mapsto [f_0(\Omega), \ldots, f_N(\Omega)]$. Furthermore, the closure of $\mathcal{A}_g$ is isomorphic to $\mathcal{A}_g^*$.*

By applying Chow's theorem, we get

Corollary 6.3.4. *$\mathcal{A}_g^*$ is a projective variety and $\mathcal{A}_g$ is a quasi-projective variety.*

Note that $\mathcal{A}_g^*$ is very singular when $g > 1$. So it is not a nice compactification from the view point of geometry. In the 1970's, Mumford and his collaborators constructed a family of compactifications called toroidal compactifications. These are often nonsingular, but are somewhat complicated to describe since they involve both representation theory and combinatorics.

Mumford gave a completely different proof of quasi-projectivity in the mid 1960's which showed much more:

Theorem 6.3.5 (Mumford, 1965)**.** *There exists a quasiprojective scheme $\mathcal{A}_{g,\mathbb{Z}}$ over* Spec $\mathbb{Z}$*, such that for any field k,*

$$Hom_{schemes}(\operatorname{Spec} k, \mathcal{A}_{g,\mathbb{Z}}) = \textit{the set of iso. classes of } g\textit{-dim ppav over } k$$

where ppav = principally polarized abelian varieties.

This is not an easy theorem. In fact, Mumford got the Fields medal partly for this work. The fact that this is defined over $\mathbb{Z}$ is essential for many number theoretic applications. For example, Faltings used it in an essential way to prove:

Theorem 6.3.6 (Faltings, 1983). *If X is a smooth projective curve of genus at least 2 defined over a number field K, then X has only finitely many K-rational points.*

Corollary 6.3.7. *Suppose that $f(x, y, z) \in K[x, y, z]$ has degree 4 or more and that it defines a nonsingular curve in $\mathbb{P}^2_{\bar{K}}$. Then $f = 0$ has only finitely many solutions in $\mathbb{P}^2_K$.*

Finally in 1990, Faltings and Chai gave an arithmetic construction of the various compactifications discussed above. So these are also defined over $\mathbb{Z}$.

6.4 Hodge structures

Although the previous constructions were very explicit, it is easy to get lost in the matrix computations, and lose sight of what is really happening. We want to give an alternate basis free description of $\mathbb{H}_g$ and $\mathcal{A}_g$. It is helpful to backtrack to where we first encountered this, namely from the cohomology of a compact Riemann surface X of genus g. To X we associated a cohomology group

$$H^1(X, \mathbb{C}) \cong H^1(X, \mathbb{Z}) \otimes \mathbb{C}$$

with an intersection pairing

$$E : H^1(X, \mathbb{Z}) \otimes H^1(X, \mathbb{Z}) \to \mathbb{Z}$$

and a distinguished g dimensional subspace

$$H^{10} = H^0(X, \Omega^1) \subset H^1(X, \mathbb{C})$$

All of this data satisfies the Hodge decomposition

$$H^{10} \oplus \overline{H}^{10} = H^1(X, \mathbb{C})$$

and the Riemann bilinear relations. This leads to the following definitions.

Definition 6.4.1. *A Hodge structure of type $\{(1, 0), (0, 1)\}$ consists of a finitely generated free abelian group $H_{\mathbb{Z}}$, with subspace $H^{10} \subset H := H_{\mathbb{Z}} \otimes \mathbb{C}$ satisfying the Hodge decomposition as above, A (principal) polarization on this is a nondegenerate (unimodular) skew-symmetric pairing $E : H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \to \mathbb{Z}$ satisfying the 2 Riemann relations*

$$\forall v, w \in H^{10}, E(v, w) = 0 \tag{6.3}$$

$$\forall v \in H^{10} - \{0\}, -iE(v, \bar{v}) > 0 \tag{6.4}$$

To shorten things, we will usually drop the phrase "of type $\{(1, 0), (0, 1)\}$". We note that the axioms imply that there is an integer g such that $\dim H^{10} = g$ and $rank H_{\mathbb{Z}} = 2g$. Finally, observe that there is some redundancy in the above axioms. If $H^{10} \subset H$ is a g-dimensional subspace satisfying the Riemann bilinear relations, then $H^{10} \cap \overline{H}^{10} = 0$ so the Hodge decomposition follows. Hodge structures form category where a morphism is $\mathbb{Z}$-linear map of lattices $H_{\mathbb{Z}} \to G'_{\mathbb{Z}}$ which takes $H^{10} \to G^{10}$. These completely captures the linear algebra aspects of abelian varieties because:

Lemma 6.4.2. *There is an equivalence of categories between the category of polarizable Hodge structures and the category of abelian varieties.*

Proof. In one direction, given a Hodge structure $H_{\mathbb{Z}}$, $H/(H_{\mathbb{Z}}+H^{10})$ is an abelian variety. ☐

In view of the last lemma, $\mathcal{A}_g$ can be identified with the set of isomorphism classes of $2g$-dimensional principally polarizable Hodge structures. We give a more explicit description below.

Lemma 6.4.3. *Let $H_{\mathbb{Z}} = \mathbb{Z}^{2g}$ with the standard skew-symmetric form E_{std} given by the matrix (6.1). Let $\mathbb{H}'_g$ denote the set of Hodge structures on $H_{\mathbb{Z}}$ polarized by E_{std} (i.e. the set of subspaces $H^{10} \subset \mathbb{C}^{2g} = H_{\mathbb{Z}} \otimes \mathbb{C}$ satisfying the previous conditions). Then there is a bijection between $\mathbb{H}_g$ and $\mathbb{H}'_g$ given by $\Omega \mapsto \{(v, -\Omega v) \mid v \in \mathbb{C}^g\}$*

This is really a lemma in linear algebra. The proof is not hard, and we will skip it. Instead we will refine this result. Since $\mathbb{H}_g$ is an open subset of the space of $g \times g$ complex matrices, it has the structure of a complex manifold. This is also true for $\mathbb{H}'_g$, although in a less obvious way. The set of the g-dimensional subspaces $H^{01} \subset \mathbb{C}^{2g}$ satisfying the first Riemann relation (6.3) is parameterized by a projective algebraic variety $LGr(g)$ sometimes called the Lagrangian Grassmanian. $LGr(g)$ is nonsingular becomes the symplectic group $Sp_{2g}(\mathbb{C})$ acts on it by translating subspaces, and this action can be seen to be transitive. Now observe that $\mathbb{H}'_g$ is an open subset of $LGr(g)$, because the second Riemann relation (6.4) is an open condition. When $g = 1$, $LGr(1) = \mathbb{P}^1$, and the embedding $\mathbb{H}_1 \subset \mathbb{P}^1$ is the usual one. The subgroup $Sp_{2g}(\mathbb{R}) \subset Sp_{2g}(\mathbb{C})$ preserves $\mathbb{H}'_g$ because it preserve both Riemman relations. A sharper version of the previous lemma is that:

Lemma 6.4.4. *We have an isomorphism $\mathbb{H}_g \cong \mathbb{H}'_g$ as complex manifolds, and this is compatible with $Sp_{2g}(\mathbb{R})$-actions.*

We can regard an element of $\mathbb{H}'_g$ as a Hodge structure with a preferred symplectic isomorphism $H_{\mathbb{Z}} \cong \mathbb{Z}^{2g}$. We can view

$$\mathcal{A}_g = Sp_{2g}(\mathbb{Z}) \backslash \mathbb{H}'_g$$

and the projection $\mathbb{H}'_g \to \mathcal{A}_g$ forgets the isomorphism.

Finally, let us say a few words about the nonprincipally polarized case.

Theorem 6.4.5 (Frobenius)**.** *Given a nondegenerate skew-symmetric form $E : \mathbb{Z}^{2g} \otimes \mathbb{Z}^{2g} \to \mathbb{Z}$, we can find positive integers $d_1 | d_2 | \ldots$ and a basis such that E*

is represented by

$$E_{std}(d_1, d_2, \ldots) = \begin{pmatrix} 0 & 0 & d_1 & 0 \\ 0 & 0 & 0 & d_2 \\ & \ddots & & \ddots \\ -d_1 & 0 & 0 & 0 \\ 0 & -d_2 & 0 & 0 \\ & \ddots & & \ddots \end{pmatrix}$$

The set of integers $(d_1, \ldots)$ is called the type of the polarization. Fix the type, and modify $\mathbb{H}'_g$ to denote the set of Hodge structures on $H_{\mathbb{Z}} = \mathbb{Z}^{2g}$ polarized by $E_{std} = E_{std}(d_1, \ldots)$. Define the symplectic group

$$\Gamma = Sp(\mathbb{Z}^{2g}, E_{std}) = \{A \in GL_{2g}(\mathbb{Z}) \mid A^T E_{std} A = E_{std}\}$$

Then the moduli space of abelian varieties of type d_i is

$$\mathcal{A}_g(d_1, d_2, \ldots) = \Gamma \backslash \mathbb{H}'_g$$