Complex Algebraic Geometry, Abelian Varieties, and Modular varieties

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Chapter 1

Riemann surfaces

1.1 Hyperelliptic integrals and curves

As all of us learn in calculus, that integrals involving square roots of quadratic polynomials can be evaluated by elementary methods. For higher degree polynomials, this is no longer true, and this was a subject of intense study in the 18th and 19th centuries by Euler, Legendre, Abel,.... An integral of the form

$$\int \frac{p(x)}{\sqrt{f(x)}}dx \tag{1.1}$$

is called elliptic if $p(x)$ is polynomial and $f(x)$ is a polynomial of degree 3 or 4, and hyperelliptic if f has higher degree.

A big advance in the above study involved switching from real to complex analysis. But the really big step was due to Riemann, who introduced the geometric point of view in the mid 19th century. He suggested that we should really be looking at the curve X^o defined by

$$y^2 = f(x)$$

in $\mathbb{C}^2$. When $f(x) = \prod(x - a_i)$ has distinct roots (which we assume from now on), X^o is a nonsingular affine algebraic curve. A bit more precisely, let

$$F(x, y) = y^2 - f(x)$$

Nonsingularity means that the gradient $\nabla f = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y})$ does not vanish on the zero set

$$X^o = V(F) := \{(x, y) \mid F(x, y) = 0\}$$

Since we are working over $\mathbb{C}$, we can also regard it as a Riemann surface. We will give the precise definition shortly, but intuitively it is is something which locally looks like $\mathbb{C}$. To see that it is the case for X^o, we need to invoke the implicit function theorem as explained later.

It is convenient to add points at infinity to make it a compact Riemann surface X called a (hyper)elliptic curve. To make this rigorous, we first take the projective closure of X^o

$$\overline{X} = \{[x, y, z] \in \mathbb{P}^2_{\mathbb{C}} \mid F^h(x, y, z) = 0\}$$

where $\mathbb{P}^2_{\mathbb{C}} = (\mathbb{C}^3 - \{0\})/\mathbb{C}^*$ is the complex projective plane, and F^h is the homogenization of F (the smallest degree homogeneous polynomial such that $F^h(x, y, 1) = F(x, y)$). $\overline{X}$ is a projective algebraic curve. In general, $\overline{X}$ could be singular, but let us ignore this for now, and suppose that $X = \overline{X}$ is nonsingular.[1] $\mathbb{C}^3 - \{0\}$ will inherit a Hausdorff topology from $\mathbb{C}^3$. Taking the quotient topology makes $\mathbb{P}^2$ into a compact Hausdorff space (which is very different from the Zariski topology). We give X the induced topology, then it is compact and Hausdorff.

1.2 Riemann surfaces

It is time to give a rigorous definition.[2] A *Riemann surface* or a *one dimensional complex manifold* or a *nonsingular complex curve* (these terms are interchangeable) consists of the following data:

1. A metrizable topological space X.
2. An open cover $\{U_i\}$ of X.
3. A collection of homeomorphisms $\phi_i : U_i \to \Delta$ to a disk, such that $\phi_i \circ \phi_j^{-1}$ are holomorphic.

The sets U_i are called coordinate disks or charts, and the composition $z \circ \phi_i$ (which is usually just written as z_i or just z_i) is called a local coordinate. We call $x_i = Re\, z_i, y_i = Im\, z_i$ the real coordinates. We can define higher dimensional complex manifolds and C^∞ manifolds in the same way, with Δ replaced by a ball in $\mathbb{R}^n$ or a product of disks in $\mathbb{C}^n$. Condition 3 is that $\phi_i \circ \phi_j^{-1}$ is either C^∞ or holomorphic. A function of several variables is holomorphic if it is continuous and holomorphic in each variable, when the other variables are fixed.

Let us consider some examples. Obviously:

Example 1.2.1. *Any open subset of $\mathbb{C}$ gives a Riemann surface.*

The first nontrivial example that we learn in basic complex analysis is

Example 1.2.2. *The Riemann sphere S^2 consists of the sphere with $U_0 = S^2 -$ (south pole) and $U_\infty = S^2 -$ (north pole). The function ϕ_0 is given by stereographic projection. If $z = z_0$ is the coordinate on U_0, the coordinate z_∞ on U_∞ satisfies $z_\infty = z^{-1}$, when it make sense. Note that algebraic geometers prefer to think of this as $\mathbb{P}^1_{\mathbb{C}}$.*

[1] For people who know what this means, in general one can always blow up $\overline{X}$ obtain a nonsingular curve X.

[2] As far as I can tell, this goes back to Weyl. His 1913 book on Riemann surfaces gave the first completely rigourous treatment of this topic.

Lemma 1.2.3. *A nonsingular affine algebraic curve has the structure of a Riemann surface in a natural way.*

Proof. Let $f(x,y)$ be a polynomial such that ∇f does not vanish on $X = V(f)$. Let $(x_i, y_i) \in X$ be a point such that $\frac{\partial f}{\partial x}(x_i, y_i) \neq 0$ (resp. $\frac{\partial f}{\partial y}(x_i, y_i) \neq 0$) , then the holomorphic implicit function [Griffiths-Harris, p 19] says that there exits open sets $U_i = \{|x - x_i| < \epsilon_i\}$, $V = \{|y - y_i| < \delta_i\}$ and a holomorphic function $g : U_i \to V_i$ such that $X \cap U_i \times V_i = \{(x, y) \mid y = g(x)\}$ (resp. with roles of x and y reversed). Then the collection $\{X \cap U_i \times V_i\}$ gives an open cover with ϕ_i given by projection to U_i. □

Lemma 1.2.4. *A nonsingular algebraic curve in the projective plane has the structure of a Riemann surface in a natural way.*

Proof. Let $X \subset \mathbb{P}^2$ be a nonsingular curve. Let $U_i = \{[x_0, x_1, x_2] \mid x_i \neq 0\}$, where x_i are homogeneous coordinates. Then $U_i \cong \mathbb{C}^2$ where for example when $i = 0$, the bijection is given by $[x_0, x_1, x_2] \mapsto (x_1/x_0, x_2/x_0)$. Under this bijection, X maps to a nonsingular affine curve. We can now apply the previous lemma. □

Example 1.2.5. *Let $L \subset \mathbb{C}$ be a lattice, which means $L = \mathbb{Z}\alpha + \mathbb{Z}\beta$, where α and β are $\mathbb{R}$-linearly independent, e.g. $\alpha = 1, \beta = i$. Consider $X = \mathbb{C}/L$. Topologically, this is a torus. Choose a disk Δ centered at 0 and contained in the parallelogram with corners $\pm\alpha/2, \pm\beta/2$. For any $p \in X$, lift it to $\tilde{p} \in \mathbb{C}$, and let U be the image of $\Delta + \tilde{p}$. This gives a coordinate disk,*

We will see other examples that later. Given a Riemann surface X and an open set $U \subseteq X$. A function $f : U \to \mathbb{C}, \mathbb{R}$ is holomorphic (resp. C^∞) if its restriction to any coordinate disk is given by a holomorphic (resp. C^∞) function of the local coordinate z (coordinates x, y).

Theorem 1.2.6. *If X is a compact connected Riemann surface, then a holomorphic function on it is constant.*

Proof. Let $f : X \to \mathbb{C}$ be holomorphic. Since X is compact, $|f|$ must attain a maximum somewhere, say $p \in X$. Let $c = f(p)$ and $Z = \{q \in X \mid f(q) = c\}$. Then Z is closed. Choose $q \in Z$, and choose a coordinate disk $\Delta \subset X$ containing q. Since $|f|$ has a maximal value at an interior point of Δ, the maximum principle from complex analysis tells us that $f|_\Delta$ must be constant. Therefore $Z \supset \Delta$, which implies that it is open. Since Z is open and closed, and X is connected, we must have $Z = X$. □

A continuous map $f : X \to Y$ between Riemann surfaces is called *holomorphic* if it can be expressed as a holomorphic function of local coordinates. More precisely, for any $p \in X$, choose coordinates $\phi_p : U \xrightarrow{\sim} V \subset \mathbb{C}$ and $\psi_q : U' \xrightarrow{\sim} V' \subset \mathbb{C}$ at p and $q = f(p)$, then $\psi_q \circ f \circ \phi_p^{-1}$ should be holomorphic. A map f is an isomorphism if it is bijective and both f and f^{-1} are holomorphic. Clearly $f : X \to \mathbb{C}$ is holomorphic in the current sense if it $f \in \mathcal{O}(X)$. A holomorphic function $f : X \to \mathbb{P}^1$ is called a meromorphic function on X. The restriction of a meromorphic function to a coordinate disk is a meromorphic in the usual sense.

Theorem 1.2.7. *A nonconstant holomorphic between compact connected Riemann surfaces is surjective.*

Proof. Given $f : X \to Y$ as above, $f(X)$ is closed since X is compact. On the other hand, $f(X)$ is also open by basic complex analysis. So $f(X) = Y$. □

1.3 A little sheaf theory

To completely check that something is a manifold or Riemann surface using the previous standard definition can get a little tedious. This means that details tend to get omitted in practice. We give an alternative definition based on sheaf theory which is sometimes easier to use and pretty natural for an algebraic geometer. But to avoid spending a lot of time on foundations, we will just talk about sheaves of functions.[3] Given a topological space X and a set S, a *presheaf* of S-valued functions on X, is a collection $\mathcal{F}(U)$ of functions $U \to S$, for open sets U, such that $f|_V \in \mathcal{F}(V)$ whenever $f \in \mathcal{F}(U)$ and $V \subset U$. We say that $\mathcal{F}$ is a *sheaf* if $f : U \to S$ lies in $\mathcal{F}(U)$ if $f|_{U_i} \in \mathcal{F}(U_i)$ for any open cover of U. If S has an abelian group or ring, we say that $\mathcal{F}$ is sheaf of abelian groups or rings if $\mathcal{F}(U)$ is an abelian group or ring under pointwise operations. If $\mathcal{F}$ is a sheaf on X, then for any open $U \subset X$, the restriction $\mathcal{F}|_U(V) = \mathcal{F}(V)$, for $V \subseteq U$, gives a sheaf on U.

The following examples are sheaves of rings of $\mathbb{C}$-valued functions:

1. The collection of all continuous functions $C(U)$ on a space X.
2. The collection of C^∞ functions $C^\infty(U)$ on a C^∞ manifold X.
3. The collection of all holomorphic functions $\mathcal{O}(U)$ on a complex manifold X.

The following is a presheaf but not a sheaf.

1. The collection of constant functions on $\mathbb{C}$ (or almost any space X).

Let k be a field. By a ringed space over k, we mean a pair $(X, \mathcal{F})$ consisting of a topological space X and a sheaf of algebras of k-valued functions on it. A morphism $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ between k-ringed spaces is a continuous map such that $g \in \mathcal{G}(U)$ implies that $f^*g := g \circ f \in \mathcal{F}(f^{-1}U)$. This is called an isomorphism if f^{-1} exists and is also a morphism.

We will be interested in the following examples:

[3] In fact, this is not a real restriction, because any sheaf is isomorphic to a sheaf of functions to an appropriate target.

1. (X, C^∞_X) is an $\mathbb{R}$-ringed space, where X is (real) C^∞ manifold. A morphism $f : (X, C^\infty_X) \to (Y, C^\infty_Y)$ is the same thing as C^∞ map of manifolds. An isomorphism is the same thing as a diffeomorphism.

2. $(X, \mathcal{O}_X)$ is a $\mathbb{C}$-ringed space, where X is a complex manifold. A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the same thing as holomorphic map of manifolds (and in particular Riemann surfaces). An isomorphism is the same thing as biholomorphism (we will just use the word isomorphism).

Finally, we have the following alternative definition of Riemann surfaces etc.

Proposition 1.3.1. *A Riemann surface is the same thing a ringed space $(X, \mathcal{O}_X)$ over $\mathbb{C}$, such that X is metrizable and such that it is locally isomorphic to $(\Delta, \mathcal{O}_\Delta)$, where $\Delta \subset \mathbb{C}$ is a the unit disk. More precisely, there exists an open cover $\{U_i\}$ such that $(U_i, \mathcal{O}_X|_{U_i}) \cong (\Delta, \mathcal{O}_\Delta)$. Similar statements hold for C^∞ and complex manifolds.*

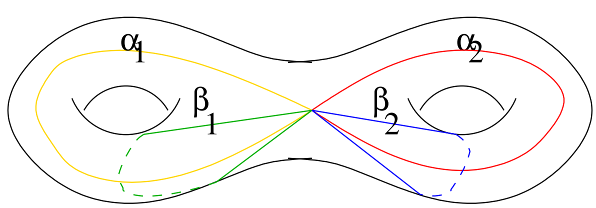
Exercise 1.3.2. *Let Γ be a subgroup of the automorphism group of Riemann surface X, and assume that the action is free and properly discontinuous (this means every point p has a closed nbhd K such that $\gamma(K) \cap K = \emptyset$ for $\gamma \neq 1$). Let $Y = X/\Gamma$ with quotient topology, let $\pi : X \to Y$ denote the projection,and let $\mathcal{O}_Y(U) = \mathcal{O}_X(\pi^{-1}U)^\Gamma$ the ring of invariant functions. With the help of the last proposition, show that $(Y, \mathcal{O}_Y)$ is a Riemann surface.*

1.4 Topological invariants

If X is a Riemann surface, then the connected components are also Riemann surfaces. So we may usually restrict our attention to the connected surfaces (and this assumption is not always stated explicitly).

Theorem 1.4.1 (Topological classification)**.** *A compact connected Riemann surface is homeomorphic to a sphere with g handles. The number $g \in \mathbb{N}$ is called the genus.*

For example,



has genus 2. This is really a theorem topology about compact oriented 2-manifolds. The proof can be found in several places, such as Seifert and Threllfal's classic "Lectures in topology", which originally written in the 1930's.

Although the genus is the key topological invariant, it is not the only one. The other invariant we want to discuss is the first Betti number. We first define from the de Rham point of view. In calculus, given C^∞ functions f, g on an open set $U \subset \mathbb{R}^2$, we define an expression

$$\omega = f(x,y)dx + g(x,y)dy$$

to be a C^∞ differential form of degree 1 or simply 1-form on U. Let $\mathcal{E}^1(U)$ denote the vector space of 1-forms on U. A basic question is when can we find a C^∞ function h such that

$$\omega = dh := \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy$$

A necessary condition is that the 2-form

$$d\omega := \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = 0$$

One says that ω is closed if the last condition holds, and exact if $\omega = df$. A closed form need not be exact as can be seen using $U = \mathbb{R}^2 - \{0\}$ and ω equals "$d\theta$" (note that θ is not a true function on U). We measure the failure by introducing the first de Rham cohomology

$$H^1_{dR}(U,\mathbb{R}) = \frac{\{\omega \in \mathcal{E}^1(U) \mid d\omega = 0\}}{\{df \mid f \in C^\infty(U)\}}$$

The first Betti number $b_1(U)$ is

$$\dim_{\mathbb{R}} H^1_{dR}(U,\mathbb{R})$$

We extend these definitions to a Riemann surface X as follows: A C^∞ 1-form ω on X is an assignment of a 1-form in the above sense $\omega_i = f_i dx_i + g_i dy_i$, for every system of real coordinates. These are required to be compatible with coordinate changes in the sense that

$$\omega_j = f_i\left(\frac{\partial x_i}{\partial x_j}dx_j + \frac{\partial x_i}{\partial y_j}dy_j\right) + g_i\left(\frac{\partial y_i}{\partial x_j}dx_j + \frac{\partial y_i}{\partial y_j}dy_j\right)$$

(This is pretty much the classical approach. There are coordinate free approaches which, however, take more time to set up.) The other notions extend in the same way, and we can define $H^1_{dR}(X,\mathbb{R})$ and the first Betti number as above. Letting $\mathcal{E}^0(X) = C^\infty(X)$ and $\mathcal{E}^2(X)$ denote the space of 0-forms and 2-forms. We define the other de Rham cohomologies by

$$H^0_{dR}(X,\mathbb{R}) = \{f \in \mathcal{E}^0(X) \mid df = 0\}$$

$$H^2_{dR}(X,\mathbb{R}) = \{\omega \in \mathcal{E}^0(X) \mid d\omega = 0\}$$

It is easy to see that $df = 0$ implies that f is constant. Therefore

$$H^0_{dR}(X,\mathbb{R}) \cong \mathbb{R}$$

It is also true that

$$H^2_{dR}(X,\mathbb{R}) \cong \mathbb{R}$$

but this is harder to see. Thus the dimensions

$$b_0(X) = b_2(X) = 1$$

The Euler characteristic of any topological space (for which b_i is defined and $\sum b_i < \infty$) is given by

$$e(T) = \sum (-1)^i b_i(T)$$

Therefore

$$e(X) = b_0(X) - b_1(X) + b_2(X) = 2 - b_1(X)$$

One nice feature of the Euler characteristic is the following "inclusion-exclusion" property

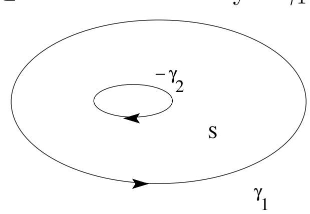
Theorem 1.4.2. *If $T = U \cup V$ is a union of open sets,*

$$e(T) = e(U) + e(V) - e(U \cap V)$$

Proof. This follows from the Mayer-Vietoris sequence and the additivity of dim for exact sequences. □

Exercise 1.4.3. *Use the above theorem to show that $e(X) = 2 - 2g$ when X is a genus g compact surface. Conclude that $b_1(X) = 2g$.*

There is a dual point of view, which is more geometric. Roughly speaking the first homology $H_1(X,\mathbb{Z})$ has generators consisting of closed oriented C^∞ paths, or loops, in X. Two loops γ_1, γ_2 define the same element of $H_1(X,\mathbb{Z})$ if there exists subsurface $S \subset X$ whose boundary is $\gamma_1 - \gamma_2$.



For a more complete treatment, see for example Hatcher's Algebraic Topology. Given a closed 1-form ω, Stokes' theorem shows that

$$\gamma \mapsto \int_\gamma \omega$$

gives a well defined element of

$$Hom(H_1(X,\mathbb{Z}),\mathbb{R})$$

Theorem 1.4.4 (de Rham)**.** *The above map gives an isomorphism*

$$H^1_{dR}(X,\mathbb{R}) \cong Hom(H_1(X,\mathbb{Z}),\mathbb{R})$$

Therefore $rank H_1(X,\mathbb{Z}) = 2g$. In fact, it is known that $H_1(X,\mathbb{Z})$ is torsion free, therefore it is isomorphic to $\mathbb{Z}^{2g}$. In the genus 2 example depicted above, the loops a_1, a_2, b_1, b_2 denote the generators.

1.5 Elliptic curves

Let $L \subset \mathbb{C}$ be a lattice, i.e. subgroup spanned by two $\mathbb{R}$-linearly independent numbers ω_i. The torus $E = \mathbb{C}/L$ is called an *elliptic curve*. We will see below that such a curve can be realized as cubic curve in the plane. Since $E \cong \mathbb{C}/\omega_1^{-1}L$, there is no loss in assuming that $\omega_1 = 1$, and that $Im(\omega_2) > 0$ (replace ω_2 by $-\omega_2$ if necessary). A translate of a parellogram having corners $0, 1, \omega_2, 1+\omega_2$ will be referred to as a fundamental parallelogram.

Now consider complex function theory on E. Any function on E can be pulled back to a function f on $\mathbb{C}$ such that

$$f(z+\lambda) = f(z), \quad \lambda \in L \tag{1.2}$$

A meromorphic function satisfying this is called a doubly periodic function or an *elliptic function* with respect to L.

Proposition 1.5.1. *Any holomorphic elliptic function is constant.*

First proof. An holomorphic elliptic function is a holomorphic function on E. Earlier we proved that holomorphic functions on compact Riemann surfaces are constant. □

Second proof. A holomorphic elliptic function is a bounded entire function. This is constant by Liouville's theorem. □

So to get interesting elliptic functions, we must have poles. For example:

Theorem 1.5.2. *The Weirerstrass $\wp$-function*

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in L - \{0\}} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right]$$

is an even elliptic function with double poles at points of L and no other singularites.

First we need the following, which can be proved using elementary analysis.

Lemma 1.5.3. *If $k > 2$, the series, called an Eisenstein series,*

$$\sum_{\lambda \in L - \{0\}} \frac{1}{\lambda^k}$$

converges absolutely.

Proof of theorem. Some elementary manipulations lead to an inequality

$$\left|\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right| = \left|\frac{2\lambda z - z^2}{\lambda^2(z-\lambda)^2}\right| \le \frac{const}{|\lambda|^3}$$

when z lies in a bounded subset of $\mathbb{C} - L$. Therefore we see that the series for $\wp(z)$ converges uniformly on compact sets away from L by the previous lemma.

The series shows that $\wp(-z) = \wp(z)$, so it is even. By uniform convergence, we can differentiate term by term to get

$$\wp'(z) = -2\sum_{\lambda\in L}\frac{1}{(z-\lambda)^3}$$

This is clearly elliptic. Therefore

$$\wp(z+\lambda) = \wp(z) + c(\lambda)$$

for some $c(\lambda)$ which is independent of z. Choosing $z = -\lambda/2$, and using the evenness of $\wp(z)$ shows that $c(\lambda) = 0$. This implies that $\wp(z)$ is elliptic. Clearly it has a double pole at 0. and therefore at all points of L.

□

We will need the following below.

Lemma 1.5.4. *Let $f(z)$ be a nonzero elliptic function, and P a fundamental parallelogram such that the poles of $f(z)$ do not lie on the boundary of P. Then the sum of orders of f within P equals the sum of the orders of the poles.*

Proof. By complex analysis the difference between the above orders is

$$\frac{1}{2\pi i}\int_{\partial P}\frac{f'(z)}{f(z)}dz$$

Double periodicity of f implies that the integral along opposite sides of P cancel.

□

Exercise 1.5.5. *Use the fact that $\wp'(z)$ is an odd elliptic function to prove that $S = \{\frac{1}{2}, \frac{\omega_2}{2}, \frac{1+\omega_2}{2}\}$ are zeros of this function. Choose a fundamental parallelogram P' with 3 corners given by the above points. Let $P = P' + \epsilon(1+\omega_2)$ with $\epsilon > 0$ small, so that the points S lie in the interior of P. Use the lemma to show that the zeros of $\wp'(z)$ in P are exactly the points in S.*

The next step is to relate this to algebraic geometry by embedding E into projective space. Denote the image of 0 in $\mathbb{C}$ by 0 as well.

Theorem 1.5.6.

(a) $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ for the appropriate choice of constants g_i.

(b) The affine and projective algebraic curves defined by

$$y^2 = 4x^3 - g_2x - g_3$$

and

$$zy^2 = 4x^3 - g_2xz^2 - g_3z^3$$

are nonsingular.

(c) The map $z \mapsto (\wp(z), \wp'(z)) \in \mathbb{C}^2$, gives a well defined map of E minus 0 to $\mathbb{C}^2$. This gives an isomorphism between $E - \{0\}$ (resp. E) and the affine (projective) cubic defined in (b).

Sketch. See [Silverman, Arithmetic of elliptic curves, chap VI] for complete details. The idea for (a) is to choose the constants so that the difference $(\wp')^2 - 4\wp^3 - g_2\wp - g_3$ vanishes at 0. But then it is elliptic with no poles, so it is constant. Therefore it vanishes everywhere.

Using the previous exercise and (a), we can see that $4x^3 - g_2x - g_3$ has 3 distinct roots. It is easy to see using that the curves defined in (b) are nonsingular.

Let $\phi(z) = (\wp(z), \wp'(z))$. Clearly this factors through $\Phi : E - \{0\} \to \mathbb{C}^2$ and the image lies within the affine cubic given in (b). We will be content to prove that Φ is injective. Suppose not. Then $\phi(z_1) = \phi(z_2)$ for $z_1 - z_2 \notin L$. Let P be a fundamental parallelogram which is symmetric about 0. After translating P slightly, we can assume without loss of generality that $\pm z_1, z_2$ lie in the interior of P. The function $f(z) = \wp(z) - \wp(z_1)$ is even, so it must vanish at $\pm z_1$ and z_2. Since $f(z)$ has a double pole at 0 and not other poles in P, we can conclude by the previous lemma that $f(z)$ can have at most 2 zeros. This forces $z_2 = -z_1$. Since $\wp'(z)$ is an odd function, $\wp'(z_2) = -\wp'(z_1)$. If $\wp'(z_1) \neq 0$, then $\phi(z_1) \neq \phi(z_2)$, which is a contradiction. Therefore $\wp'(z_1) = 0$, which implies that z_1 is a double zero of f. Therefore $z_2 = z_1$. In a nbhd of 0, $\phi(z) = [z^3\wp(z), z^3\wp'(z), z^3] \in \mathbb{P}^2$. Since $\wp'(z)$ has a triple pole at 0, this shows that we have a holomorphic extension with $\phi(0) = [0, 1, 0]$. This gives an injective holomorphic map Φ from E to the projective cubic in (b). Since Φ is nonconstant, it also surjective. This almost proves that it is an isomorphism. To make sure, we need to check that the derivative everywhere nonzero. See Silverman for this. □

In summary, an elliptic curve really is an algebraic curve, which can be realized as a plane cubic. In the purely algebraic theory, which works over any field, one starts with the latter.

1.6 Jacobi's Theta function

The alternative approach to getting interesting holomorphic functions on a lattice is to relax the periodicity (1.2). This leads to the theory of theta functions. The higher dimensional analogue will play an important role later. Basically, we want holomorphic functions that satisfy

$$f(z + \lambda) = (\text{some factor})f(z)$$

which we refer to as quasi-periodicity with respect to $L = \mathbb{Z} + \mathbb{Z}\tau$ with $\tau = \omega_2$ in the upper half plane. Similar ideas are in the theory of automorphic forms. We can obtain elliptic functions by taking ratios of two such functions with the same factors. To make it more precise, we want

$$f(z + \lambda) = \phi_\lambda(z)f(z) \tag{1.3}$$

where $\phi_\lambda(z)$ is a nowhere zero entire function. To guarantee nonzero solutions, we require some compatibility conditions

$$f(z+(\lambda_1+\lambda_2)) = \phi_{\lambda_1+\lambda_2}(z)f(z)$$

$$f((z+\lambda_1)+\lambda_2) = \phi_{\lambda_2}(z+\lambda_2)\phi_{\lambda_1}(z)f(z)$$

which suggests that we should impose

$$\phi_{\lambda_1+\lambda_2}(z) = \phi_{\lambda_2}(z+\lambda_1)\phi_{\lambda_1}(z)$$

This is called the 1-cocycle identity. As it turns out, there is a cheap way to get solutions, choose a nowhere zero function $g(z)$ and let $\phi_\lambda(z) = g(z+\lambda)/g(z)$ such as cocycle is called a coboundary. From the point of view of constructing interesting solutions of (1.3), it is not very good. Any solution would be a constant multiple of $g(z)$. Taking ratios of two of these functions would result in a constant.

The problem of constructing cocycles which are not coboundaries can be solved using the machinery of group cohomology. The set of cocycles modulo coboundaries forms the cohomology group $H^1(L, \mathcal{O}(\mathbb{C})^*)$. There is a connecting homomorphism[4]

$$c_1 : H^1(L, \mathcal{O}(\mathbb{C})^*) \to H^2(L, \mathbb{Z}) \cong \wedge^2 L^*$$

to the space of alternating integer valued forms on L. Given a cocycle ϕ_λ, to show that it is not a coboundary is enough to show that the image of c_1 is nonzero. Fortunately this can be done explicitly. Since ϕ_λ is entire and nowhere 0, we can take a global logarithm $\psi_\lambda(z) = \log \phi_\lambda(z)$. Then

$$F(\lambda_1, \lambda_2) = \frac{1}{2\pi i}[\psi_{\lambda_1+\lambda_1}(z) - \psi_{\lambda_2}(z+\lambda_2) - \psi_{\lambda_1}(z)] \in \mathbb{Z}$$

gives an integer valued function such that

$$c_1(\phi_\bullet)(\lambda_1, \lambda_2) = F(\lambda_1, \lambda_2) - F(\lambda_2, \lambda_1)$$

One can check that the exponential of

$$\psi_{n\tau+m}(z) = -n^2\pi i\tau + 2\pi inz$$

gives a cocycle whose image under c_1 is nonzero. With this choice, we can find an explicit solution to (1.3). The Jacobi θ-function is given by the Fourier series

$$\theta(z) = \sum_{n\in\mathbb{Z}} \exp(\pi in^2\tau + 2\pi inz) = \sum_{n\in\mathbb{Z}} \exp(\pi in^2\tau)\exp(2\pi inz)$$

[4]As the notation suggests, it is a version of the first Chern class. People familiar with group cohomology can verify the last isomorphism for $H^2(L, \mathbb{Z})$ using the Koszul resolution, otherwise take it as a blackbox.

Writing $\tau = x + iy$, with $y > 0$, shows that on a compact subset of the z-plane the terms are bounded by $O(e^{-n^2 y})$. So uniform convergence on compact sets is guaranteed. This is clearly periodic

$$\theta(z+1) = \theta(z)$$

In addition it satifies the function equation

$$\begin{aligned}\theta(z+\tau) &= \sum \exp(\pi i n^2\tau + 2\pi i n(z+\tau)) \\ &= \sum \exp(\pi i(n+1)^2\tau - \pi i\tau + 2\pi i n z) \\ &= \exp(-\pi i\tau - 2\pi i z)\theta(z)\end{aligned}$$

and more generally

$$\theta(z + n\tau + m) = \exp(\psi_{n\tau+m}(z))\theta(z)$$

We can get a larger supply of quasiperiodic functions by translating. Given a rational number b, define

$$\theta_{0,b}(z) = \theta(z+b)$$

Then

$$\theta_{0,b}(z+1) = \theta_{0,b}(z), \quad \theta_{0,b}(z+\tau) = \exp(-\pi i\tau - 2\pi i z - 2\pi i b)\theta(z)$$

We can construct elliptic functions by taking ratios: $\theta_{0,b}(Nz)/\theta_{0,b'}(Nz)$ is a (generally nontrivial) elliptic function when $b, b' \in \frac{1}{N}\mathbb{Z}$. More generally given rational numbers $a, b \in \frac{1}{N}\mathbb{Z}$, we can form the theta functions with characteristics

$$\theta_{a,b}(z) = \exp(\pi i a^2\tau + 2\pi a(z+b))\theta(z + a\tau + b) \tag{1.4}$$

Fix $N \geq 1$, and let V_N denote the set of linear combinations of these functions.

Lemma 1.6.1. *Given nonzero $f \in V_N$, it has exactly N^2 zeros in the parallelogram with vertices $0, N, N\tau, N+\tau$.*

Sketch. Complex analysis tells us that the number of zeros is given by the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)dz}{f(z)}$$

over the boundary of the parallelogram. This can be evaluated to N^2 using the identities $f(z+N) = f(z)$, $f(z+N\tau) = Const.\exp(-2\pi i N z)f(z)$ following from (1.4). □

These can be used to construct a projective embedding different from the previous.

Theorem 1.6.2. *Choose an integer $N > 1$ and the collection of all θ_{a_i,b_i}, as (a_i, b_i) runs through representatives of $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$. The the map of $\mathbb{C}/L$ into $\mathbb{P}^{N^2-1}$ by $z \mapsto [\theta_{a_i,b_i}(z)]$ is an embedding.*

Sketch. Suppose that this is not an embedding. Say that $f(z_1) = f(z_1')$ for some $z_1 \neq z_1'$ in $\mathbb{C}/L$ and all $f \in V_N$. By translation by $(a\tau + b)/N$ for $a, b \in \frac{1}{N}\mathbb{Z}$, we can find another such pair z_2, z_2' with this property. Since $\dim V_N = N^2$, we can find additional points $z_2, \ldots z_{N^2-3}$, distinct in $\mathbb{C}/NL$, so that

$$f(z_1) = f(z_2) = f(z_3) = \ldots f(z_{N^2-3}) = 0$$

for some $f \in V_N - \{0\}$. Notice that we are forced to also have $f(z_1') = f(z_2') = 0$ which means that f has at least $N^2 + 1$ zeros which contradicts the lemma.

Further details can be found in [Mumford, Lectures on Theta I] □

Note that the smallest such embedding lands in $\mathbb{P}^3$, so it is different from what we obtained before.

Chapter 2

Hodge theory

2.1 Cauchy-Riemann operator

Let $U \subset \mathbb{C}$ be an open set Let $C^\infty(U)$ (resp. $C^\infty_{\mathbb{R}}(U)$) denote the space of complex (resp. real) valued functions. Similarly, we work with complex valued differential forms, where $\mathcal{E}^1(U)$ (resp. $\mathcal{E}^1_{\mathbb{R}}(U)$) denotes the space of complex (resp. real) valued 1-forms. Note that $\mathcal{E}^1(U)$ is a module over $C^\infty(U)$. If

$$z = x + iy$$

as usual, and introduce complex valued differential forms

$$dz = dx + idy,\ d\bar{z} = dx - idy$$

Therefore

$$dx = \frac{1}{2}(dz + d\bar{z})$$

$$dy = \frac{1}{2i}(dz - d\bar{z})$$

Given a C^∞ function $f : U \to \mathbb{C}$, the total differential

$$df = f_x dx + f_y dy = \frac{1}{2}(f_x - if_y)dz + \frac{1}{2}(f_x + if_y)d\bar{z}$$

This suggests that we should introduce the operators

$$\partial f = \frac{1}{2}(f_x - if_y)dz$$

$$\bar{\partial} f = \frac{1}{2}(f_x + if_y)d\bar{z}$$

so that

$$d = \partial + \bar{\partial}$$

If we set $u = Re\, f, v = Im\, f$, then

$$\bar{\partial} f = \frac{1}{2}[(u_x - v_y) + i(u_y + v_x)]d\bar{z}$$

This makes it clear that the condition $\bar{\partial} f = 0$ is equivalent to the Cauchy-Riemann equations. Therefore

Lemma 2.1.1. *$f \in C^\infty(U)$ is holomorphic if and only if $\bar{\partial} f = 0$.*

We let $\mathcal{E}^{10}(U) \subset \mathcal{E}^1(U)$ (resp. $\mathcal{E}^{01}(U) \subset \mathcal{E}^1(U)$) be the submodule spanned by dz (resp. $d\bar{z}$). We call these forms of type $(1,0)$ or $(0,1)$. We have

$$\mathcal{E}^1(U) = \mathcal{E}^{10}(U) \oplus \mathcal{E}^{01}(U)$$

and ∂ (resp. $\bar{\partial}$) is just d followed by projection to these submodules.

We now want to show that of this make sense on a Riemann surface X. Given two overlapping coordinate disks U and V with local coordinates z and ζ, we see that ζ is a holomorphic function of z and visa versa. Therefore

$$d\zeta = \partial\zeta = \frac{\partial \zeta}{\partial z} dz$$

$$dz = \partial z = \frac{\partial z}{\partial \zeta} d\zeta$$

Therefore

$$\mathcal{E}^{10}(U \cap V) = C^\infty(U \cap V)dz = C^\infty(U \cap V)d\zeta$$

We can now define $\mathcal{E}^{10}(X) \subset \mathcal{E}^1(X)$ to be the space of 1-forms whose restriction to any coordinate disk U_i lies $\mathcal{E}^{10}(U_i)$. The previous equality shows that this is well defined. We define $\mathcal{E}^{01}(X)$ to be the space of complex conjugates of $(1,0)$-forms. We can see that any form in $\mathcal{E}^1(X)$ has a unique decomposition into a sum of $(1,0)$-form and $(0,1)$-form. Therefore

$$\mathcal{E}^1(X) = \mathcal{E}^{10}(X) \oplus \mathcal{E}^{01}(X)$$

We define ∂f (resp. $\bar{\partial} f$) to be the projection of df to the first (resp. second) factor. A $(1,0)$-form is called *holomorphic* if its restriction to any coordinate disk with coordinate z is $f(z)dz$ with f holomorphic. We let $\Omega^1(X)$ denote the space of holomorphic 1-forms.

2.2 Harmonic forms

Fix a compact (connected) Riemann surface X. Let us suppose that the genus is g. As before $C^\infty(X)$ and $\mathcal{E}^p(X)$ will now denote the spaces of complex valued C^∞ functions and complex valued forms. We these conventions, we can define complex valued de Rham cohomology as before

$$H^1_{dR}(X, \mathbb{C}) = \frac{\{\alpha \in \mathcal{E}^1(X) \mid d\alpha = 0\}}{\{df \mid f \in C^\infty(X)\}}$$

This is isomorphic to $H^1_{dR}(X,\mathbb{R})\otimes\mathbb{C} = \mathbb{C}^{2g}$. Note the formula and similar ones appear more uniform, if we set

$$\mathcal{E}^0_X = \mathcal{E}^{00}_X = \mathbb{C}_X$$

We note that Riemann surfaces have a canonical orientation: if x, y are real and imaginary parts of a complex coordinate z, then $dx \wedge dy$ is positively oriented. The orientation allows us to integrate two forms on X. Given $\alpha, \beta \in \mathcal{E}^1(X)$, define

$$(\alpha,\beta) = \int_X \alpha\wedge\beta$$

Stokes' theorem and properties of the wedge product shows that this gives a well defined skew symmetric pairing

$$(\ ,\ ) : H^1_{dR}(X,\mathbb{C}) \times H^1_{dR}(X,\mathbb{C}) \to \mathbb{C}$$

For people familiar with it, this is dual to the (complexified) intersection pairing on $H_1(X,\mathbb{Z})$

An element of de Rham cohomology is really an equivalence class. *Does such a class have a distinguished representative?* The answer will turn out to be yes. To describe it, let us introduce a $C^\infty(X)$-linear operation called the Hodge star given locally by $*dx = dy$, $*dy = -dx$. This is amounts to multiplication by i in the cotangent planes, so it is globally well defined operation. We have the following basic properties

Lemma 2.2.1. *$\mathcal{E}^1(X)$ has an inner product given by*

$$\langle\alpha,\beta\rangle = (\alpha, *\bar\beta) = \int_X \alpha\wedge *\bar\beta$$

Proof. One can see that

$$(fdx + gdy)\wedge *\overline{(hdx + kdy)} = (f\bar h + g\bar k)dx\wedge dy$$

$$(fdx + gdy)\wedge *\overline{(fdx + gdy)} = (|f|^2 + |g|^2)dx\wedge dy$$

This implies the basic properties including positive definiteness. □

Corollary 2.2.2 (Poincaré duality)**.** *The bilinear form $(\ ,\ )$ is nondegenerate.*

Remark 2.2.3. *The topological form of Poincaré duality gives the stronger result that the intersection pairing on $H_1(X,\mathbb{Z})$ is unimodular. This means that H_1 has a basis, called a symplectic basis, such that the pairing is represented by*

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

We will use this later on.

Definition 2.2.4. *We define a* 1*-form* α *to be co-closed if* $d(*\alpha) = 0$. *It is harmonic if it is both closed and co-closed, i.e.* $d\alpha = d(*\alpha) = 0$. *A form is called co-exact if it equals* $*df$.

The reason for the name will be explained later on. The basic properties are given by:

Proposition 2.2.5.

(a) A harmonic 1*-form is a sum of a* $(1,0)$ *harmonic form and* $(0,1)$ *harmonic form.*

(b) A $(1,0)$*-form is holomorphic if and only it is closed if and only if is harmonic.*

(c) A $(0,1)$*-form is harmonic if and only it is antiholomorphic i.e. its complex conjugate is holomorphic.*

(d) A 1*-form is co-closed (resp. closed) forms if and only if it is orthogonal the space of exact (resp. co-exact) forms. Therefore a* 1*-form is harmonic if and only if it is orthogonal to the direct spaces of*

Proof. If α is a harmonic 1-form, then $\alpha = \alpha' + \alpha''$, where $\alpha' = \frac{1}{2}(\alpha + i * \alpha)$ is a harmonic $(1,0)$-form and $\alpha'' = \frac{1}{2}(\alpha - i * \alpha)$ is a harmonic $(0,1)$-form.

If α is $(1,0)$, then $d\alpha = \bar{\partial}\alpha$. This implies the first half (b). For the second half, use the identity

$$*dz = *(dx + idy) = dy - idx = -idz$$

Finally, note that the harmoncity condition is invariant under conjugation, so the (c) follows from (b).

For (d), we first observe that integration by parts (essentially Stokes' theorem) implies

$$\langle df, \alpha\rangle = \int_X df \wedge *\bar{\alpha} = \int d(f * \bar{\alpha}) - \int_X fd * \bar{\alpha} = -\int_X fd * \bar{\alpha}$$

If α is co-closed, then it follows that $\langle df, \alpha\rangle = 0$. Conversely, suppose that $\langle df, \alpha\rangle = 0$ for all $f \in C^\infty(X)$. Let $d * \alpha = g(x,y)dx \wedge dy$ in a coordinate disk D. If $g(p) \neq 0$, we can choose f with support in D such that $f(x,y)g(x,y) \geq 0$ everywhere and strictly positive at p. Therefore $\int_X fd * \alpha > 0$, so we can conclude that $d * \alpha = 0$. A similar argument using

$$\langle \alpha, *df\rangle = \int_X \alpha \wedge * * d\bar{f} = -\int_X d(\bar{f}\alpha) + \int_X \bar{f} d\alpha = \int_X \bar{f} d\alpha \qquad (2.1)$$

shows that $d\alpha = 0$ if and only if α is orthogonal to co-closed forms.

□

Here is the key fact. We will say more about this in later on.

Theorem 2.2.6 (Hodge theorem). *Every de Rham cohomology class has a unique harmonic representative.*

Remark 2.2.7. *This statement is actually due Weyl, which Hodge generalized to higher dimensions.*

Corollary 2.2.8 (Hodge decomposition). *We have*

$$H^1_{dR}(X, \mathbb{C}) \cong \Omega^1(X) \oplus \overline{\Omega^1(X)}$$

Therefore $\dim \Omega^1(X) = g$.

Proof. By proposition 2.2.5, a harmonic 1 can be uniquely decomposed as a sum of holomorphic 1-form and the complex conjugate of a holomorphic 1-form. □

For reasons that will be explained later, one normally denotes $\Omega^1(X)$ by $H^0(X, \Omega^1_X)$ and this notation will be used below.

2.3 Proof of the Hodge theorem

First we explain the connection between harmonic forms and harmonic functions. Recall that C^∞ function f on an open subset of $\mathbb{R}^2$ is harmonic if it satisfies the Laplace equation

$$\Delta f := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = 0$$

Lemma 2.3.1. *A* 1*-form on a disk is harmonic if and only if it is given by* df*, where* f *is a harmonic function.*

Proof. Since a disk D is contractible, a 1-form on D is closed if and only if equals df for some f. The form is also co-closed when

$$d{*}df = \Delta f dx \wedge dy = 0$$

□

Recall that Green's identity from calculus implies that if f and η are both C^∞ and η vanishes near the boundary ∂D, then

$$\int_D f \Delta\eta \, dxdy = \int_D (\Delta f)\eta \, dxdy$$

Therefore if f is harmonic, then the first integral vanishes. Weyl's lemma is a converse statement.

Theorem 2.3.2 ("Weyl's lemma"). *Let* $D \subset \mathbb{C}$ *be an open disk. Let* $f \in L^2(D)$ *be such that*

$$\int_D f \Delta\eta \, dxdy = 0$$

for every compactly supported C^∞ *function* η*, then* f *is a* C^∞ *harmonic function.*

Proof. The proof is not difficult but it takes a few pages, so we refer to [Farkas-Kra, Riemann surfaces]. □

The proof of the Hodge theorem we give uses the method of orthogonal projection. The idea is to use a generalization of a fact from basic linear algebra that if $S \subset V$ is a subspace of a finite dimensional inner product space, then

$$V = S \oplus S^{\perp}$$

When V is infinite dimensional, this is no longer true unless V is a Hilbert space and S is closed. Thus we first need to complete everything to a Hilbert space in order to apply this. Let us denote by $L^2\mathcal{E}^1(X)$ the Hilbert space completion with of this space. Let $\mathcal{E}^1_{ex}(X), \mathcal{E}_{cl}(X), \mathcal{E}^1_{co}(X) \subset \mathcal{E}^1(X)$ denote the space of exact, closed and co-exact 1-forms i.e. the forms $*df$. Since these spaces are orthogonal, we get that the closure

$$\overline{\mathcal{E}^1_{ex}(X) + \mathcal{E}^1_{co}(X)} = \overline{\mathcal{E}^1_{ex}(X)} \oplus \overline{\mathcal{E}^1_{co}(X)}$$

in $L^2\mathcal{E}^1(X)$. Let H denote the orthogonal complement of the above space. Then we have an orthogonal decomposition

$$L^2\mathcal{E}^1(X) = H \oplus \overline{\mathcal{E}^1_{ex}(X)} \oplus \overline{\mathcal{E}^1_{co}(X)} \tag{2.2}$$

Lemma 2.3.3. *H consists of the space of harmonic C^∞ 1-forms.*

Proof. Given a C^∞ form α the orthogonality conditions defining H imply that H is harmonic. Given an element of $\alpha \in H$, its restriction to a coordinate disk D can be viewed as a differential form $\alpha|_D = pdx + qdy$ with L^2 coefficients. Let η be a C^∞ function with compact support on D. The orthogonality conditions imply that

$$\langle pdx + qdy, d\eta_x - *d\eta_y \rangle = 0$$

Expanding the left side yields

$$\int_D p\Delta\eta \, dxdy = 0$$

This implies that p is harmonic by Weyl's lemma. Similarly q is harmonic. Therefore α is C^∞, and consequently harmonic.

□

Lemma 2.3.4. $\mathcal{E}^1_{cl}(X) \cap \overline{\mathcal{E}^1_{ex}(X)} = \mathcal{E}^1_{ex}(X)$

Proof. If $\alpha \in \mathcal{E}^1_{ex}(X)$, and $\beta \in \mathcal{E}^1_{cl}(X)$, then Stokes' theorem implies that $(\alpha, \beta) = \langle \alpha, *\bar{\beta} \rangle = 0$. By continuity, this continues to hold for $\alpha \in \overline{\mathcal{E}^1_{ex}(X)}$. Now suppose that $\alpha \in \mathcal{E}^1_{cl}(X) \cap \overline{\mathcal{E}^1_{ex}(X)}$. We just showed that the cohomology class of α satisfies $(\alpha, \beta) = 0$ for any class $\beta \in H^1_{dR}(X, \mathbb{C})$. Therefore by Poincaré duality $[\alpha] = 0$. This implies that α is exact. □

Proof of the Hodge theorem. Let $\alpha \in \mathcal{E}^1_{cl}(X)$. Then using (2.2), we may decompose $\alpha = \beta + \gamma + \delta$, with $\alpha \in H$ etc. We claim that $\|\delta\|^2 = 0$. By continuity, it is enough to assume that $\delta = *df$. Then the orthogonality conditions plus (essentially) (2.1) shows that

$$\|\delta\|^2 = \langle \alpha, *df \rangle = \pm \int_X d(f\alpha) = 0$$

Therefore $\alpha = \beta + \gamma$. By lemma 2.3.3, β is harmonic. Therefore $\gamma = \alpha - \beta$ is in $\mathcal{E}^1_{cl}(X)$. By lemma 2.3.4, γ is exact. □

We won't give a proof, but it is possible to get a stronger result that the space of 1-forms decomposes as below.

Theorem 2.3.5 (Hodge theorem II)**.** *We have a decomposition*

$$\mathcal{E}^1(X) = H \oplus \mathcal{E}^1_{ex}(X) \oplus \mathcal{E}^1_{co}(X)$$

where H, $\mathcal{E}^1_{ex}(X)$ and $\mathcal{E}^1_{co}(X)$ is the space of harmonic, exact and co-exact 1-*forms respectively.*

2.4 Background on sheaf cohomology

We want to give a somewhat different interpretation of the Hodge decomposition, which will rely on the machinery of sheaf cohomology. We will mostly treat this machinery as a black box, or perhaps a dark grey box. More thorough treatments can be found in the books on algebraic geometry by Griffiths-Harris, Hartshorne, Voisin, and myself. Let us start with some definitions. Given a topological space X, a presheaf of abelian groups is a contravariant functor from the category $Open(X)$ of open sets of X, where morphisms are inclusions, to the categorry of abelian groups Ab. More concretely a presheaf is a collection of abelian groups $\mathcal{F}(U), U \in Open(X)$, with restrictions $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$, when $V \subseteq U$, subject to appropriate compatibility conditions. Such a presheaf $\mathcal{F}$ is called a sheaf of abelian groups (henceforth just a sheaf) if for any open U with open cover $\{U_i\}$, any collection $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ is the restriction of a unique section $f \in \mathcal{F}(U)$. Let $Ab(X)$ denote the category of sheaves on X where a morphism is an additive natural transformation. This is an abelian category, so it comes with a natural notion of exact sequence. To spell it out, a sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is *exact* if for any $x \in X$, we can find an open nbhd U such that

$$0 \to \mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U)$$

is exact in Ab, and for every $\gamma \in \mathcal{C}(U)$, after shrinking U, γ lies in the image of $\mathcal{B}(U)$. For the last part, it would suffice to assume that $\mathcal{B}(U) \to \mathcal{C}(U)$ is surjective, although the condition is a bit weaker.

Example 2.4.1. *Let X be a Riemann surface. Let $\mathbb{Z}_X$ denote the sheaf of locally constant $\mathbb{Z}$ functions on X, $\mathcal{O}_X$ the sheaf of holomorphic functions, and $\mathcal{O}_X^*$ the sheaf of nowhere zero holomorphic functions viewed as a multiplicative group. We have an sequence*

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \stackrel{e}{\to} \mathcal{O}_X^* \to 1$$

where the first map is the obvious one, and second sends $f \to \exp(2\pi i f)$. If U is a coordinate disk, then

$$0 \to \mathbb{Z}(U) \to \mathcal{O}_X(U) \stackrel{e}{\to} \mathcal{O}_X^*(U) \to 1$$

is exact. Therefore the above sequence of sheaves is exact. This called the exponential sequence.

Example 2.4.2. *Let X be a Riemann surface once again. Let $\mathbb{Z}_X$ denote the sheaf of locally constant $\mathbb{Z}$ functions on X,. Then the sequence*

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \stackrel{d}{\to} \Omega_X^1 \to 0$$

is exact. This follows from the exactness of

$$0 \to \mathbb{C}_X(U) \to \mathcal{O}_X(U) \stackrel{d}{\to} \Omega_X^1(U) \to 0$$

for a coordinate disk U.

Example 2.4.3. *Again let X be a Riemann surface. Then we have a sequence*

$$0 \to \mathcal{O}_X \to C_X^\infty \stackrel{\bar\partial}{\to} \mathcal{E}_X^{01} \to 0$$

which we claim is exact. It suffices to check the exactness of

$$0 \to \mathcal{O}_X(U) \to C_X^\infty(U) \stackrel{\bar\partial}{\to} \mathcal{E}_X^{01}(U) \to 0$$

when U is a coordinate disk. The surjectivity of the last map follows from the $\bar\partial$-Poincaré lemma [Griffiths-Harris, p 5]. The injectivity of the first map is clear, and the exactness in the middle from the Cauchy-Riemann equations.

There is an obvious extension of exactness for a sequence of more than 3 sheaves.

Example 2.4.4. *Again X is a Riemann surface. Then*

$$0 \to \mathbb{C}_X \to \mathcal{E}_X^0 \stackrel{d}{\to} \mathcal{E}_X^1 \stackrel{d}{\to} \mathcal{E}_X^2 \to 0$$

is exact. This can be checked on the disk, where it follows from the usual Poincaré lemma [e.g. Spivak, Calculus on manifolds]. There are couple of variants worth mentioning. We can use real valued functions and forms and everything still works. X can be replace by an n-dimensional C^∞-manifold. We still get an exact sequence as above, except that it has length n.

Let us define a functor $\Gamma : Ab(X) \to Ab$ by $\Gamma(\mathcal{F}) = \mathcal{F}(X)$.

Lemma 2.4.5. *The functor Γ is left exact, i.e. given an exact sequence*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

we have an exact sequence

$$0 \to \Gamma(\mathcal{A}) \to \Gamma(\mathcal{B}) \to \Gamma(\mathcal{C})$$

It is generally not true that the last map above is surjective, and this not just a mere technicality:

Example 2.4.6. *Let $X = \mathbb{C}^*$, then the map*

$$e : \Gamma(\mathcal{O}_X) \to \Gamma(\mathcal{O}_X^*)$$

is not surjective because as is well known there is no way to define a holomorphic logarithm on $\mathbb{C}^*$

Following the usual pattern in homological algebra, we have

Theorem 2.4.7. *There exists a sequence of functors $H^i(X, -) : Ab(X) \to Ab$ such that*

$$H^0(X, \mathcal{F}) \cong \Gamma(\mathcal{F})$$

An exact sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

gives rise to a long exact sequence

$$0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to H^0(X, \mathcal{C}) \to H^1(X, \mathcal{A}) \to \dots$$

We need one more fact to make this useful. The following is special case of vanishing theorem for fine sheaves. We refer to the previous references for further information.

Theorem 2.4.8. *Let X be a C^∞-manifold, and let $\mathcal{F}$ be a sheaf of C_X^∞-modules (which means that each $\mathcal{F}(U)$ is a $C^\infty(U)$-module, restrictions respect the module structure), then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.*

2.5 Hodge theorem in terms of sheaf cohomology

With the previous results in hand, we can do some calculations.

Proposition 2.5.1 (Dolbeault)**.** *If X is a Riemann surface, then*

$$H^1(X, \mathcal{O}_X) \cong \frac{\mathcal{E}^{01}(X)}{\bar{\partial} C^\infty(X)}$$

$$H^i(X, \mathcal{O}_X) = 0, \text{ if } i \geq 2$$

Proof. This follows the exact sequence

$$0 \to \mathcal{O}_X \to C^\infty_X \xrightarrow{\bar\partial} \mathcal{E}^{01}_X \to 0$$

and theorems 2.4.7 and 2.4.8. □

Proposition 2.5.2 (de Rham)**.** *If X is a C^∞-manifold*

$$H^i(X, \mathbb{C}_X) \cong H^i_{dR}(X, \mathbb{C})$$

Proof. We just give the proof for $i = 1$ when X is a Riemann surface. The same method works in general. Break

$$0 \to \mathbb{C}_X \to \mathcal{E}^0_X \xrightarrow{d} \mathcal{E}^1_X \xrightarrow{d} \mathcal{E}^2_X \to 0$$

into exact sequences

$$0 \to \mathbb{C}_X \to \mathcal{E}^0_X \to \mathcal{E}^1_{X,cl} \to 0$$

$$0 \to \mathcal{E}^1_{X,cl} \to \mathcal{E}^1_X \xrightarrow{d} \mathcal{E}^2_X \to 0$$

Then by the above theorems

$$H^1(X, \mathbb{C}_X) = \mathit{coker}[H^0(X, \mathcal{E}^0_X) \to H^0(X, \mathcal{E}^1_{X,cl})]$$

$$= \frac{\ker H^0(X, \mathcal{E}^1_X) \xrightarrow{d} H^0(X, \mathcal{E}^1_X)]}{dH^0(X, \mathcal{E}^0_X)} = H^1_{dR}(X, \mathbb{C})$$

□

Theorem 2.5.3 (Hodge theorem for $\bar\partial$)**.** *If X is compact of genus g, then every element of*

$$\frac{\mathcal{E}^{01}(X)}{\bar\partial C^\infty(X)}$$

has a unique harmonic representative. Therefore

$$\dim H^1(X, \mathcal{O}_X) \cong \overline{H^0(X, \Omega^1_X)}$$

Proof. Observe that $\bar\partial$ is the $(0,1)$ part of d as well as $-i{*}d$ because

$$-i{*}df = -i({*}\partial f + {*}\bar\partial f) = -\partial f + \bar\partial f$$

Theorem 2.3.5 shows that

$$\mathcal{E}^1(X) = H \oplus dC^\infty(X) \oplus {*}dC^\infty(X)$$

where H is the space of harmonic 1-forms. Therefore the $(0,1)$-part of this decomposition yields

$$\mathcal{E}^{01}(X) = H^{01} \oplus \bar\partial C^\infty(X)$$

where H^{01} is the space of harmonic $(0,1)$-forms. This implies the theorem.

□

Corollary 2.5.4. *In the long exact sequence associated to*

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \to \Omega^1_X \to 0$$

we get an exact sequence

$$0 \to H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C}_X) \to H^1(X, \mathcal{O}_X) \to 0$$

Proof. Since $\dim H^0(X, \Omega^1_X) = \dim H^1(X, \mathcal{O}_X) = g$ and $\dim H^1(X, \mathbb{C}_X) = 2g$, the map ι below is injective, and p is surjective

$$H^0(X, \Omega^1_X) \xrightarrow{\iota} H^1(X, \mathbb{C}_X) \xrightarrow{p} H^1(X, \mathcal{O}_X)$$

□

Remark 2.5.5. *The isomorphism*

$$\dim H^1(X, \mathcal{O}_X) \cong \overline{H^0(X, \Omega^1_X)}$$

gives a natural splitting to above projection

$$H^1(X, \mathbb{C}_X) \to H^1(X, \mathcal{O}_X)$$

Corollary 2.5.6 (Serre duality)**.** *The pairing*

$$(\alpha, \beta) = \int_X \alpha \wedge \beta$$

on $H^1(X, \mathbb{C})$ induces an isomorphism

$$H^0(X, \Omega^1_X)^* \cong H^1(X, \mathcal{O}_X)$$

Furthermore,

$$H^1(X, \Omega^1_X) \cong \mathbb{C}$$

Proof. We showed earlier that that (,) is nondegenerate. This means that given a nonzero $\alpha \in H^1(X, \mathbb{C})$, we can find $\beta \in H^1(X, \mathbb{C})$ such $(\alpha, \beta) \neq 0$. Suppose that $\alpha \in H^0(X, \Omega^1_X)$, then $(\alpha, \beta) = 0$ because $\alpha \wedge \beta = 0$. Therefore we must be able to choose $\beta \in H^1(X, \mathcal{O}_X)$ (under the decomposition explained above). Therefore the pairing induces an injection

$$H^0(X, \Omega^1_X)^* \hookrightarrow H^1(X, \mathcal{O}_X)$$

This must be an isomorphism, because the spaces have the same dimension.

By the previous corollary and proposition 2.5.1, the long exact sequence associated to

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \to \Omega^1_X \to 0$$

gives an isomorphism

$$H^1(X, \Omega^1_X) \cong H^2(X, \mathbb{C}) \cong \mathbb{C}$$

□

2.6 Riemann's inequality

Let X be a compact Riemann surface of genus g. A function on $X - S$, where $S \subset X$ is a finite set, is called meromorphic if it holomorphic and if the Laurent expansion with respect to any coordinate has a finite number of negative terms (i.e. it has no essential singularities). Let $\mathbb{C}(X)$ denote the field of meromorphic functions on X. A basic fact, that we prove in this section, X always caries a nonconstant meromorphic function.

Given a finite set of distinct points $S = \{p_1, \ldots, p_n\}$, set $D = \sum p_i$ to the formal sum, and $\deg D = n$. If $S = \emptyset$, $D = 0$. We define a sheaf $\Omega^1_X(\log D)$ whose sections over U consist of holomorphic 1-forms on U with at worst simple poles at points of $U \cap S$.

Theorem 2.6.1 (Riemann's inequality)**.**

$$\dim H^0(X, \Omega^1_X(\log D)) \geq \deg D + g - 1$$

Proof. We define the *skyscraper* sheaf $\mathbb{C}_{p_i}$ to consist of

$$\mathbb{C}_{p_i} = \begin{cases} \mathbb{C} & \text{if } p_i \in U \\ 0 & \text{otherwise} \end{cases}$$

Then we have an exact sequence

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \xrightarrow{res} \bigoplus_{p_i \in S} \mathbb{C}_{p_i} \to 0$$

where the first map is the obvious inclusion, and the second sends a form ω to the sum of its residues (defined in the usual way) at points of S. This gives rise to an exact sequenec

$$0 \to H^0(X, \Omega^1_X) \to H^0(X, \Omega^1_X(\log D)) \to \mathbb{C}^{\deg D} \to H^1(X, \Omega^1_X)$$

Since we proved that the last space is one dimensional, the theorem follows immediately. □

Corollary 2.6.2. *X has a nontrivial meromorphic function.*

Proof. By the theorem we can find 2 elements $\omega_i \in H^0(X, \Omega^1_X(\log D))$ as soon as $\deg D + g - 1 \geq 2$. Locally $\omega_= f_i(z)dz$, and the ratio $\omega_1/\omega_2 = f_1/f_2$ can be seen to a globally well defined meromorphic function. □

Note that Riemann's inequality can be improved to a much sharper statement called the *Riemann-Roch* theorem. We will not give it, since we plan to go in a different direction.

Chapter 3

The Jacobian

3.1 The Jacobian

Fix a compact Riemann surface X of genus g. Then we have seen that

$$g = \dim H^0(X, \Omega^1_X) = \frac{1}{2} rank H_1(X, \mathbb{Z})$$

Define a map to the dual space

$$H_1(X, \mathbb{Z}) \to H^0(X, \Omega^1_X)^*$$

which sends a loop γ to the integral

$$\int_\gamma \in H^0(X, \Omega^1_X)^*$$

The symbol above is the functional

$$\omega \mapsto \int_\gamma \omega$$

Proposition 3.1.1. *The image of $H_1(X, \mathbb{Z})$ in $H^0(X, \Omega^1_X)^*$ is a lattice, i.e. a subgroup generated by an $\mathbb{R}$-basis.*

Proof. Recall that we have the Hodge decomposition

$$H^1(X, \mathbb{C}) \cong H^0(X, \Omega^1_X) \oplus \overline{H^0(X, \Omega^1_X)}$$

Consider the map

$$p : H^1(X, \mathbb{R}) \to H^0(X, \Omega^1_X)$$

given by inclusion and projection. Observe that both sides have the same real dimension $2g$. If $p(\alpha) = 0$, then $\alpha = p(\alpha) + \overline{p(\alpha)} = 0$. Therefore p is injective, consequently surjective. Dually, we find that

$$H_1(X, \mathbb{R}) \cong H^0(X, \Omega^1_X)^*$$

Since $H_1(X, \mathbb{Z})$ is a lattice in $H_1(X, \mathbb{R})$, the proposition follows. □

It follows that

$$J(X) = \frac{H^0(X, \Omega^1_X)^*}{H_1(X, \mathbb{Z})}$$

is g-dimensional compex torus called the *Jacobian* of X. Fix a basis $\omega_1, \ldots, \omega_g \in H^0(X, \Omega^1_X)$, then

$$J(X) \cong \mathbb{C}^g/L$$

where

$$L = \{(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g) \mid \gamma \in H_1(X, \mathbb{Z})\}$$

is called the period lattice.

Fix a base point $x_0 \in X$, the *Abel-Jacobi* map

$$\alpha_{x_0} : X \to J(X)$$

sends x to the vector $(\int_{x_0}^x \omega_i)$. Note that we have to choose a path from x_0 to x to make sense of this. However, α_{x_0} is independent of the path, because we work mod L. It does depend on the base point, but in a fairly easy to understand way:

$$\alpha_{x_1}(x) = \alpha_{x_0}(x) - \alpha_{x_0}(x_1)$$

We usually will suppress the base point, and simply write α.

Proposition 3.1.2. *The map α is holomorphic.*

Proof. Choose a coordinate disk $U \subset X$. Since ω_i is closed, its restriction to U is df_i, for some holomorphic function f_i. Then, up to translation, $\alpha|_U = (f_1, \ldots, f_g)$ is holomorphic. □

We define

$$X^n = X \times X \times \ldots X \quad (n \text{ times})$$

and extend α to a holomorphic map $\alpha : X^n \to J(X)$

$$\alpha(x_1, \ldots, x_n) = \alpha(x_1) + \ldots \alpha(x_n)$$

Theorem 3.1.3 (Jacobi)**.** *The map $\alpha : X^g \to J(X)$ is surjective.*

Proof. The one nontrivial fact we need from several complex variables is that $\alpha(X^g)$ is an analytic subvariety [Gunning-Rossi, Analytic functions of several complex variables, p 162]. Suppose that $\alpha(X^g)$ is a proper subvariety of $J(X)$. Then it is not difficult to see that this would imply the rank of the derivative of α is strictly less that g at all points of X^g. We will derive a contradiction.

Fix a basis ω_i of $H^0(X, \Omega^1_X)$ and choose a point $(x_1, \ldots, x_g) \in X^g$. These choices will get modified as the proof proceeds. Choose coordinate disks around each x_i, and assume that $\omega_j = df_j$ in each of these disks. Then

$$\alpha(x_1, \ldots, x_g) = (f_1(x_1) + \ldots + f_1(x_g), \ldots f_g(x_1) + \ldots + f_g(x_g))$$

The derivative of α is represented by the (entirely different kind of) Jacobian

$$\left(\frac{\partial}{\partial x_j}(f_i(x_1)+\ldots)\right) = \begin{pmatrix} f_1'(x_1) & \ldots & f_1'(x_g) \\ & \ldots & \\ f_g'(x_1) & \ldots & f_g'(x_g) \end{pmatrix}$$

Since $\omega_1 \neq 0$, we can choose x_1 so that $f_1'(x_1) \neq 0$. After replacing $\omega_2, \ldots$ by $\omega_2 - c_2\omega_1$, for suitable constants c_1, we can assume that $f_2'(x_1) = f_3'(x_1)\ldots = 0$. Similarly, we can assume that $f_2'(x_2) \neq 0$. Then after a similar change of basis, we can arrange $f_3'(x_2) = \ldots 0$. Continuing this way, we can assume that the above matrix is upper triangular with nonzero entries on the diagonal. This implies that the derivative has rank g, which is the desired contradiction. □

It is possible to prove a finer statement by taking the quotient X^g/S_g, where S_g is the symmetric group. Although this space might appear to have singularities, one can show that it is a complex manfold. Then α induces a surjective holomorphic map $X^g/S_g \to J(X)$. Then on can show that

Theorem 3.1.4 (Jacobi, part II)**.** *The map $X^g/S_g \to J(X)$ is generically one to one, i.e. there exists a dense open set $U \subset J(X)$ such that the map is an isomorphism over U.*

Example 3.1.5. *If X has genus one, then the Jacobi theorem(s) imply that $X \cong J(X)$ as complex manifolds. We earlier defined an elliptic curve to be a one dimensional complex torus. An alternative definition is that an elliptic curve is a genus one curve with a base point. The torus structure then follows from the above isomorphism.*

3.2 Divisor class group

Let X be compact Riemann surface as before. The group $Div(X)$ of divisors on X is the free abelian group. The elements, called *divisors* are finite formal sums $D = \sum_i n_i p_i$ with $p_i \in X$. The degree $\deg D = \sum_i n_i$. Set $Div^0(X) = \ker : Div(X) \xrightarrow{\deg} \mathbb{Z}$. Natural examples arise from meromorphic functions. Let $\mathbb{C}(X)$ denote the set of meromorphic functions on X. This is a field. Given $x \in X$, choose coordinate z at x. Given $f \in \mathbb{C}(X)$, we can expand it as Laurent series

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$$

If $a_n \neq 0$, set $ord_x(f) = n$. Define $ord_x(0) = \infty$. Then

Proposition 3.2.1. *The function $ord_x : \mathbb{C}(X) \to \mathbb{Z} \cup \{\infty\}$ is a discrete valuation, i.e.*

1. *If $f, g \in \mathbb{C}(X)^*$, $ord_x(fg) = ord_x(f) + ord_x(g)$*
2. *$ord_x(f+g) \geq ord_x(f) + ord_x(g)$*

Proof. Elementary. □

Given $f \in \mathbb{C}(X)^*$, set

$$div(f) = \sum_{x \in X} ord_x(f)x$$

Since the zeros and poles of f are isolated and X is compact, this sum is finite, so it defines a divisor. A divisor of this form is called *principal*.

Lemma 3.2.2. *The map $div : \mathbb{C}(X)^* \to Div(X)$ is a homomorphism. Therefore the image $Princ(X)$ is a subgroup.*

Proof. This is immediate from the previous proposition. □

The *divisor class group*

$$Cl(X) = \frac{Div(X)}{Princ(X)}$$

Theorem 3.2.3. *The degree of a principal divisor is* 0.

Proof. Given $f \in \mathbb{C}(X)^*$, choose coordinate disks D_j around each zero or pole of f. Then by basic complex analysis

$$\deg div(f) = \sum_j \frac{1}{2\pi i} \int_{\partial \overline{D}_j} \frac{f'(z)}{f(z)} dz$$

By the Stokes' theorem, the integral on the right equals

$$\frac{1}{2\pi i} \int_{X - \cup D_j} d\left(\frac{f'(z)}{f(z)} dz\right) = 0$$

□

Therefore $Princ(X) \subset Div^0(X)$, and we define

$$Cl^0(X) = \frac{Div^0(X)}{Princ(X)}$$

It should be clear that there is an exact sequence

$$0 \to Cl^0(X) \to Cl(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$

so that $Cl^0(X)$ is the interesting part of the class group.

Exercise 3.2.4. *Check that $X = \mathbb{P}^1$, any degree zero divisor is the divisor of a rational function. It follows that $Cl^0(X) = 0$.*

The next theorem explains a fundamental property of the Jacobian.

Theorem 3.2.5 (Abel-Jacobi). *If X is a compact Riemann surface, then there is an isomorphism*

$$Cl^0(X) \cong J(X)$$

as abstract groups.

We will give a proof later on, but we want to make some comments about it now. We define homomorphism

$$\alpha : Div(X) \to J(X)$$

by

$$\alpha(\sum_i n_i x_i) = \sum_i n_i \alpha(x_i)$$

Jacobi's theorem implies that the restriction $Div^0(X) \to J(X)$ is surjective. The other half of the above theorem is

Theorem 3.2.6 (Abel). *The kernel of $\alpha : Div^0(X) \to J(X)$ is $Princ(X)$.*

Example 3.2.7. *If $X = \mathbb{P}^1$, then we recover that $Cl^0(X) = J(X) = 0$.*

Example 3.2.8. *If X is an elliptic curve then $Cl^0(X) \cong X$.*

3.3 Dual description of the Jacobian

Let X be compact Riemann surface of genus g. The long exact sequence associated the exponential sequence is

$$0 \to H^0(X,\mathbb{Z}) \to H^0(X,\mathcal{O}_X) \to H^0(X,\mathcal{O}_X^*) \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z})$$

The first few terms are simply

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{e} \mathbb{C}^*$$

Since the last map is surjective, the next few terms of the previous sequence becomes

$$0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to J^\vee(X) \to 0$$

where we define

$$J^\vee(X) = \ker : H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z})$$

Theorem 3.3.1. *We have an isomorphism*

$$J(X) \cong J^\vee(X)$$

Sketch. By the Poincaré and Serre duality theorems, we have isomorphisms labelled P and S below

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(X,\mathbb{Z}) & \longrightarrow & H^0(X,\Omega^1_X)^* & \longrightarrow & J(X) & \longrightarrow & 0 \\ & & \downarrow P & & \downarrow S & & \downarrow \cong & & \\ 0 & \longrightarrow & H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathcal{O}_X) & \longrightarrow & J^\vee(X) & \longrightarrow & 0 \end{array}$$

Therefore the desired isomorphism follows. □

We will say more about this isomorphism when we discuss abelian varieties.

3.4 Čech cohomology and Abel's theorem

Before giving the proof of Abel's theorem, we need to make a digression into homological algebra. Given an open cover $\mathcal{U} = \{U_i\}$ of a space X, let $U_{ij} = U_i \cap U_j$ etc. A Čech 1-*cocycle* with coefficients in a sheaf $\mathcal{F}$ with respect to $\mathcal{U}$ is a collection $f_{ij} \in \mathcal{F}(U_{ij})$ such that

$$f_{ij} = -f_{ji}$$

and the restrictions to U_{ijk} satisfy

$$f_{ij} + f_{jk} + f_{ki} = 0$$

Let $Z^1(\mathcal{U},\mathcal{F})$ denote the group of 1-cocycles. We define f_{ij} to be a 1-*coboundary* if there exists $\phi_i \in \mathcal{F}(U_i)$ such that

$$f_{ij} = \phi_i - \phi_j$$

Let $B^1(\mathcal{U},\mathcal{F})$ denote the subgroup of 1-coboundaries . Define the Čech cohomology groups

$$\check{H}^1(\mathcal{U},\mathcal{F}) = \frac{Z^1(\mathcal{U},\mathcal{F})}{B^1(\mathcal{U},\mathcal{F})}$$

$$\check{H}^1(X,\mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U},\mathcal{F})$$

Theorem 3.4.1. *We have an isomorphism*

$$H^1(X,\mathcal{F}) \cong \check{H}^1(X,\mathcal{F})$$

If $\mathcal{U}$ is an open cover such that $H^i(U_{j_1\ldots j_k},\mathcal{F}) = 0$ for all $i > 0$ and all indices $j_1,\ldots,$ in which the cover is called Leray, then

$$H^i(X,\mathcal{F}) \cong \check{H}^1(\mathcal{U},\mathcal{F})$$

Let us return to the setting where X is a compact Riemann surface of genus g. Suppose that $D = n_1x_1+n_2x_2\ldots$ is a nontrivial divisor. Construct and open cover $\mathcal{U} = \{U_0, U_1\ldots\}$ where $U_0 = X - \{x_1, x_2, \ldots\}$ and $U_1, \ldots$ are coordinate disks around $x_1, \ldots$ chosen small enough that none of disks intersect. Let $z_1, \ldots$ denote coordinates. If $j > 0$ set $\phi_{0j} = z_j^{n_j} \in \mathcal{O}(U_0 \cap U_j)$, and $\phi_{j0} = z_j^{-n_j}$, since there are no triple intersections, we can see that this This forms a 1-cocycle on $\mathcal{U}$ with values in $\mathcal{O}_X^*$. Let $[D]$ denote the class of this cocycle in $H^1(X, \mathcal{O}_X^*)$. If D is trivial, then we simply define $[D] = 0$.

Theorem 3.4.2. *If* $\deg D = 0$*, then* $[D] \in J^\vee(X)$*. Under the previous isomorphism, this coincides up to sign with* $\alpha(D)$*.*

Abel's theorem now becomes the following statement.

Theorem 3.4.3 (Abel)**.** *If* $D \in Div^0(X)$*, then* $[D] = 0$ *if and only of* $D \in Princ(X)$*.*

Proof. Let $n_j, \{U_j\}, \phi_{ij}$ be as above. Suppose that $D = div(f)$. Set $\phi_0 = f|_{U_0}$ and $\phi_j = (f|_{U_j}/z_j^{n_j})$ for $j > 0$. Then

$$\phi_{ij} = \phi_i\phi_j^{-1}$$

This means that ϕ_{ij} is a coboundary. Therefore $[D] = 0$.

Suppose $[D] = 0$. We can assume that $D \neq 0$. Then the cover $\{U_j\}$ consists of noncompact open sets. The only nontrivial fact we assume without proof is that the open cover is a Leray cover[1] for $\mathcal{O}_X^*$. Therefore $H^1(X, \mathcal{O}_X^*) = \check{H}^1(\{U_j\}, \mathcal{O}_X^*)$. It follows that ϕ_{ij} is a coboundary so that

$$\phi_{ij} = \phi_i\phi_j^{-1}$$

for some appropriate functions on the right. Set $f = \phi_0$, and we can see that $D = div(f)$.

□

3.5 Riemann bilinear relations

Let X be a compact Riemann surface of genus g. Recall

$$(\alpha, \beta) = \int_X \alpha \wedge \beta$$

defines a nondegenerate skew symmetric blinear form on $H^1_{dR}(X, \mathbb{C})$. This corresponds to the interesction form on $H_1(X, \mathbb{Z})$ under the de Rham isomorphism

$$H^1_{dR}(X, \mathbb{C}) = Hom(H_1(X, \mathbb{Z}), \mathbb{C})$$

[1] This follows from Poincaré duality which implies that $H^2(U_{j_1,\ldots}, \mathbb{Z}) = 0$ and theorem of Benke-Stein that $U_{j_1,\ldots}$ are Stein which implies $H^1(U_{j_1,\ldots}, \mathcal{O}_X) = 0$

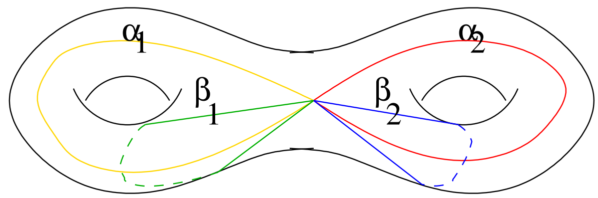
A basis $\alpha_1, \ldots, \alpha_g, \beta_1 \ldots \beta_g \in H_1(X, \mathbb{Z})$ is called symplectic if

$$(\alpha_i, \alpha_j) = (\beta_i, \beta_j) = 0$$

and

$$(\alpha_i, \beta_j) = \delta_{ij}$$

Such bases exist, for example a standard choice is



If $\alpha_1^*, \ldots \beta_g^* \in H^1(X, \mathbb{Z})$ denotes the dual basis, then they will also satisfy the same symplectic relations.

Theorem 3.5.1 (Riemann bilinear relations)**.** *Choose $\omega, \eta \in H^0(X, \Omega^1_X)$ and a symplectic basis $\alpha_1, \ldots, \alpha_g, \beta_1 \ldots \beta_g \in H_1(X, \mathbb{Z})$. Then*

(a)

$$(\omega, \eta) = 0$$

(b)

$$\sum_k \left(\int_{\alpha_k} \omega \int_{\beta_k} \eta - \int_{\beta_k} \omega \int_{\alpha_k} \eta \right) = 0$$

(c) If $\omega \neq 0$, then

$$i(\omega, \bar{\omega}) > 0$$

(d) If $\omega \neq 0$, then

$$Im \left(\sum_k \overline{\int_{\alpha_k} \omega} \int_{\beta_k} \omega \right) > 0$$

Proof. For (a) observe that $\omega \wedge \eta = 0$. Item (b) follows from (a) by expanding

$$\omega = \sum_k \left(\int_{\alpha_k} \omega \, \alpha_k^* + \int_{\beta_k} \omega \, \beta_k^* \right)$$

$$\eta = \sum_k \left(\int_{\alpha_k} \eta \, \alpha_k^* + \int_{\beta_k} \eta \, \beta_k^* \right)$$

and using the symplectic relations. For (c) write $\omega = f(z)dz$, then

$$(\omega, \bar{\omega}) = 2 \int_X |f(z)|^2 dx \wedge dy > 0$$

Item (d) follows from (c) by expanding ω in $\alpha_1^*, \ldots, \beta_g^*$ as above. □

Corollary 3.5.2. *A basis* $\omega_1, \ldots, \omega_g \in H^0(X, \Omega^1_X)$ *can be chosen so that the* $g \times g$ *matrix*

$$\left(\int_{\alpha_j} \omega_i\right) = I$$

and

$$\Omega = \left(\int_{\beta_j} \omega_i\right)$$

is symmetric with positive definite imaginary part.

Proof. Choose any basis ω_i and let

$$A = \left(\int_{\alpha_j} \omega_i\right), \quad B = \left(\int_{\beta_j} \omega_i\right)$$

Then (b) implies that AB^T is symmetric, and (d) implies that AB^T has positive definite imaginary part. Therefore AB^T is invertible, so the same is true for A. Consequently, we can change basis so that $(A, B) \mapsto (A^{-1}A, A^{-1}B) = (I, \Omega)$. Repeating the previous argument shows that Ω is symmetric with positive definite imaginary part. □

We say that the basis ω_i, or period matrix, is normalized if it satisfies the conditions of the previous corollary. We define the $g \times g$ *Siegel upper half plane* to be

$$\mathbb{H}_g = \{\Omega \in Mat_{g\times g} \mid \Omega^T = \Omega, Im\,\Omega > 0\}$$

To summarize what we did in this section, X plus a choice of symplectic basis for $H_1(X, \mathbb{Z})$ determines a point of $\mathbb{H}_g$.

Chapter 4

Abelian varieties

4.1 Abelian varieties

An abelian variety is a higher dimensional version of an elliptic curve. Here is the definition we use: a (complex) abelian variety is a complex torus, i.e. a quotient $X = \mathbb{C}^g/L$ where L is a lattice, that embeds into some complex projective space as a complex manifold. The significance of the last condition stems from Chow's theorem.

Theorem 4.1.1 (Chow)**.** *A complex submanifold of $\mathbb{P}^n_{\mathbb{C}}$ is a nonsingular projective algebraic variety i.e. it is the zero set of a collection of homogeneous polynomials.*

Corollary 4.1.2. *An abelian variety X is a projective variety. Furthermore the group operations $+ : X \times X \to X$ and $- : X \to X$ are morphisms of algebraic varieties*

For the last statement, we apply Chow's theorem to the graphs of $+$ and $-$. Abelian varieties can be defined over arbitrary fields. In this case, the statement in the corollary is taken as the definition.

Although we saw that every elliptic curve is projective, it is not true for arbitrary tori. To formulate sufficient conditions, we modify what we did before, but now we replace the element τ in the upper half, with matrix $\Omega \in \mathbb{H}_g$. Given such a matrix, we can define the lattice

$$L_\Omega = \mathbb{Z}^g + \Omega\mathbb{Z}^g$$

The following is basically due to Riemann and Lefschetz

Theorem 4.1.3 (Riemann-Lefschetz)**.** *If $L \subseteq L_\Omega$ is a sublattice, then $A = \mathbb{C}^g/L$ is an abelian variety.*

Corollary 4.1.4. *A Jacobian is an abelian variety.*

A sketch of a proof will occupy the rest of this section. For simplicity, we treat the case where $L = L_\Omega$

As a first step, we define the Riemann theta function on $\mathbb{C}^g$ by

$$\theta(z) = \sum_{n\in\mathbb{Z}^g} \exp(\pi i n^t \Omega n + 2\pi i n^t z)$$

This is a generalization of the Jacobi function. Since $Im\,\Omega$ has positive eigenvalues, $\|\exp(\pi i n^t \Omega n)\| \to 0$ rapidly as $\|n\| \to \infty$. So the series converges to a holomorphic function. It is quasi-periodic in the sense that

$$\theta(z+n) = \theta(z)$$

$$\theta(z+\Omega n) = \exp(-\pi i n^t \Omega n - 2\pi i z^t n)\theta(z)$$

for $n \in \mathbb{Z}^g$. See [Mumford, Lectures on theta I] for details.

For each natural number ℓ, define V_ℓ to be the space of holomorphic functions satisfying

$$f(z+n) = f(z)$$

$$f(z+\Omega n) = \exp(-\pi i \ell n^t \Omega n - 2\pi i \ell z^t n) f(z)$$

Of course V_1 contains θ. Given $a, b \in \mathbb{C}^g$, let

$$\theta_{ab}(z) = \theta(z+a+b)\theta(z-a)\theta(z-b)$$

One sees that $\theta_{ab}(z) \in V_3$.

Lemma 4.1.5. $\dim V_\ell = \ell^g$

Sketch. When $\ell = 1$, $f(z) \in V_1$ satisfies $f(z+n) = f(z)$. Therefore it can be expanded in Fourier series

$$f_n(z) = \sum a_n \exp(2\pi i n^t z)$$

The second functional equation

$$f(z+\Omega n) = \exp(-\pi i n^t \Omega n - 2\pi i z^t n) f(z)$$

implies $f(z) = a_0\theta(z)$. For general ℓ, the analogue of the last equation yields a recurrence condition on the Fourier coefficients with ℓ^g free choices. □

Lemma 4.1.6. *Given $z_0 \in \mathbb{C}^g$, there exists $f(z) \in V_3$ such that $f(z_0) \neq 0$.*

Proof. Since $\theta \neq 0$, we can find a, b such that $\theta_{a,b}(z_0) \neq 0$. □

Choose a basis $f_1, \ldots, f_{3^g} \in V_3$, then the previous lemma shows that the map $\iota : A \to \mathbb{P}^{3^g-1}$ given by

$$\iota(z) = [f_1(z), \ldots, f_{3^g}(z)]$$

defined everywhere. To finish the proof of theorem 4.1.3, we will show that this is an embedding.

It is convenient to switch to more geometric language. A *divisor* on A is a formal integer linear combination of hypersurfaces in it. Given a nonzero theta function f, which is irreducible in the sense that it can't be factored, its zero set $D = Z(f)$ will be invariant under L, so it defines a divisor in A. In particular, $\Theta = Z(\theta) \subset A$ is called the theta divisor. (We omit the proof of irreducibility, but it's true.)

Given $a \in A$, let $t_a : A \to A$ denote translation by a. Let t_a^*D denote the preimage of D under this operation. A little thought, shows that $t_a^*D = D - a$.

Lemma 4.1.7. *$b \in t_a^*D$ if and only $a \in t_b^*D$.*

Proof. Both sides are equivalent to $a + b \in D$. □

Lemma 4.1.8. *If $a \neq 0$, then $t_a^*\Theta \neq \Theta$.*

Then the divisor of $\theta_{a,b}$ is

$$\Theta_{a,b} = t_{a+b}^*\Theta + t_{-a}^*\Theta + t_{-b}^*\Theta$$

Given an irreducible divisor $D \subset A$ defined by f, let $D_{smooth} \subset D$ denote the set of smooth points (the set where ∇f is nonzero). To each $x \in D_{smooth}$, define the *Gauss map* by $G(x) = [\nabla f(x)] \in \mathbb{P}^{g-1}$. In other words, $G(x)$ is the tangent space to D at x viewed as a subspace of $\mathbb{C}^g$.

Lemma 4.1.9. *The Gauss map of Θ is nonconstant.*

Proof. Intuitively, this comes to the fact that the Gauss map of a hypersurface is nonconstant unless it's "linear", but Θ clearly isn't. We refer to pp 81-82 [Birkenhake-Lange, Complex Abelian Varietes] for a detailed proof. □

Corollary 4.1.10. *Given a nonzero vector $v \in \mathbb{C}^g$, there exists a point $x \in \Theta_{smooth}$ such that v is not tangent to Θ at x.*

Proof of theorem 4.1.3. We need to show that ι is injective. It suffices to show that given points $x \neq y$, we can find $\Theta_{a,b}$ containing x but not y. Since $x - y \neq 0$, $t_{x-y}^*\Theta \neq \Theta$ by lemma 4.1.8. Therefore $t_x^*\Theta \neq t_y^*\Theta$. Consequently, there exists $a \in A$ with $-a \in t_x^*\Theta$ and $-a \notin t_y^*\Theta$. This implies that $x \in t_{-a}^*\Theta$ and $y \notin t_{-a}^*\Theta$ by lemma 4.1.7. A similar argument shows that there exists b such that $y \in t_{-b}^*\Theta \cup t_{a+b}^*\Theta$. Therefore $\Theta_{a,b}$ contains x but not y.

To finish the proof, we need to show that the derivative of ι is injective at all points. In the language of divisors, it is enough to prove that given $x \in A$ and a tangent vector v to A at x, there exists a, b such that $x \in \Theta_{a,b}$ is a nonsingular point and v is not tangent to $\Theta_{a,b}$ at x. This follows from corollary 4.1.10. □

It is worth noting that in the language of algebraic geometry, the above proof shows that 3Θ is a very ample divisor, and therefore that Θ is ample.

Corollary 4.1.11. *A Jacobian is an abelian variety.*

4.2 Riemann forms

We want to characterize lattices of the form $L \subseteq L_\Omega = \mathbb{Z}^g + \Omega\mathbb{Z}^g$, with $\Omega \in \mathbb{H}_g$, in coordinate free language.

Definition 4.2.1. *A* polarization *or* Riemann form *(the terms are interchangeable) on a lattice $L \subset V$, in a finite dimensional complex vector space, is a nondegenerate real skew-symmetric bilinear form $E : V \times V \to \mathbb{R}$ such that*

(a) $E(u, v) \in \mathbb{Z}$, when $u, v \in L$.

(b) There exist a positive definite hermitian form H on $\mathbb{C}^g$, such that $E = Im\, H$.

This is called a principal polarization if in addition $\det E = 1$.

Lemma 4.2.2. *If E is a polarization, then H above is uniquely determined.*

Proof. We leave it as linear algebra exercise to show that

$$H(x, y) = E(ix, y) + iE(x, y)$$

□

Lemma 4.2.3. *Suppose $L = L_\Omega = \mathbb{Z}^g + \Omega\mathbb{Z}^g$ is a lattice with $\Omega \in \mathbb{H}_g$. Let $e_1, \ldots e_g$, be the standard basis of $\mathbb{Z}^g$. We can extend this to basis of L, by taking e_{g+j} to be the jth column of Ω. This can also be viewed as a* real *basis of $\mathbb{C}^g$. Let $E : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{R}$ be the real bilinear form with matrix*

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{4.1}$$

with respect to this basis. Then E is a principal polarization on L

Proof. The form E is clearly integer valued on L with $\det E = 1$. The form

$$H(u, v) = u^t (Im\Omega)^{-1} \bar{v}$$

is positive hermitian because the matrix $(Im\Omega)^{-1}$ is positive definite symmetric. We can check that $E = Im\, H$ by calculation: for $j, k \le g$,

$$Im\, H(e_j, e_k) = Im[e_j^t (Im\Omega^{-1}) e_k] = 0$$

$$Im\, H(e_j, e_{g+k}) = Im[(e_j^t (Im\,\Omega^{-1})(Re\,\Omega e_k - \sqrt{-1} Im\,\Omega e_k)] = -\delta_{jk}$$

Finally, by the spectral theorem, we can assume that $Im\,\Omega$ is the diagonal matrix $diag(\tau_1, \ldots, \tau_g)$. Then if $j \neq k$

$$Im\, H(e_{g+j}, e_{g+k}) = 0$$

and

$$Im\, H(e_{g+j}, e_{g+j}) = Im[\tau_j e_j^t (\tau_j^{-1}) \bar{\tau}_j e_j] = 0$$

□

Corollary 4.2.4. *A sublattice of L_Ω carries a not necessarily principal polarization.*

We omit the proof, but the previous lemma has converse.

Lemma 4.2.5. *If L has a polarization, then after choosing bases of $\mathbb{C}^g$ and L, we have $L \subseteq L_\Omega$ for some $\Omega \in \mathbb{H}_g$. If the polarization is principal, we can choose a basis such that $L = L_\Omega$.*

Theorem 4.2.6 (Riemann-Lefschetz)**.** *$\mathbb{C}^g/L$ is an abelian variety if and only if L possesses a polarization.*

The key fact, we need is

Theorem 4.2.7. *There is an isomorphism $H^2(\mathbb{P}^N_C, \mathbb{Z}) \cong \mathbb{Z}$. Under the natural embedding $H^2(\mathbb{P}^N_\mathbb{C}, \mathbb{Z}) \subset H^2_{dR}(\mathbb{P}^N_\mathbb{C}, \mathbb{C})$, a generator is represented by the differential form*

$$\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log(|\zeta_0|^2 + \ldots + |\zeta_N|^2)$$

where $\zeta_0, \ldots$ are homogeneous coordinates.

We want to make a few comments about the formula.

1. The definition of $\partial, \bar\partial$ are similar to what we did what we did in dimension one. Given f, collect the terms of df involving $d\zeta_0, \ldots$ (resp. $d\bar\zeta_0, \ldots$) to get ∂f (resp. $\bar\partial f$).

2. Homogeneous coordinates are not true coordinates. To get those, we take ratios ζ_i/ζ_j on the sets $U_j = \{\zeta_j \neq 0\}$. We can rewrite

$$\omega|_{U_j} = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log(|\zeta_0/\zeta_j|^2 + \ldots + |\zeta_N/\zeta_j|^2)$$

   in terms of the true coordinates.

3. The significance of the normalization is that ω can be rewritten as a real differential form such that

$$\int_{\mathbb{P}^1} \omega = 1$$

   where $\mathbb{P}^1$ is the line defined by $z_2 = \ldots z_N = 0$. This is why the class of ω generates $H^2(\mathbb{P}^N, \mathbb{Z})$, and in fact it gives the preferred generator.

4. ω is a *positive* $(1,1)$ form. This means that

$$\omega = \sqrt{-1}\sum_{jk} h_{jk} d(\zeta_j/\zeta_0) \wedge \overline{d(\zeta_k/\zeta_0)}$$

   where h_{jk} is a positive definite Hermitian matrix. To see this observes that ω is invariant under the action of the unitary group $U(N+1)$, and after some calculation that at $[1, 0, \ldots, 0]$ $h_{jk} = (1/2\pi)\delta_{jk}$.

5. Finally one could ask where ω comes from. Without defining what it means, the answer is that ω is the Kähler form for the Fubini-Study metric.

Let $A = \mathbb{C}^g/L$. Choose a basis $e_1, \ldots, e_{2g}$ for L, the dual basis e_i^* for L^*. Then using the Künneth formula, we can easily compute cohomology,

Lemma 4.2.8. *We isomorphisms*

$$H^1(A, \mathbb{Z}) \cong L^* = \bigoplus \mathbb{Z}e_i^*$$

$$H^2(A, \mathbb{Z}) \cong \wedge^2 L^* = \bigoplus \mathbb{Z}e_i^* \wedge e_j^*$$

$$\ldots$$

$$H^1_{dR}(A, \mathbb{C}) = \bigoplus \mathbb{C}e_i^*$$

$$H^2_{dR}(A, \mathbb{C}) = \bigoplus \mathbb{C}e_i^* \wedge e_j^*$$

$$\ldots$$

The last set of isomorphisms imply that any de Rham cohomology class is represented by a unique differential form with constant coefficients. If ω is closed p-form, then its cohomology class is represented by the p-form with constant coefficients given by

$$I(\omega) = \sum_{j_1 < \ldots < j_p} \left(\int_{e_{j_1} \times \ldots \times e_{j_p}} \omega \right) e_{j_1}^* \wedge \ldots \wedge e_{j_p}^*$$

where $e_{j_i} \times \ldots$ is understood as the subtorus of A given by the projection of the product of line segments $[0, e_{j_1}] \times \ldots \subset \mathbb{C}^g$.

Sketch of proof of theorem 4.2.6. The "if" direction was theorem 4.1.3 above. Let us briefly explain the converse. By definition we have an embedding $A \subset \mathbb{P}^N_{\mathbb{C}}$. The class $[\omega] \in H^2(\mathbb{P}^N, \mathbb{Z})$ restricts to a class $E \in H^2(A, \mathbb{Z})$ which can be viewed as an integer valued skew symmetric form on L using the isomorphism $H^2(A, \mathbb{Z}) \cong \wedge^2 L^*$. To verify the remaining axiom, under the embedding $H^2(A, \mathbb{Z}) \subset H^2_{dR}(A, \mathbb{C})$, E can be represented by the differential given as the restriction of ω on $\mathbb{P}^N$ above. The form $\omega|_A$ will continue to be a positive $(1,1)$ form. A bit of thought using the previous formula shows that $I(\omega|_A)$ will remain a positive $(1,1)$ form. In other words

$$I(\omega|_A) = \sqrt{-1} \sum_{jk} h_{jk} dz_j \wedge d\bar{z}_j$$

where $H = (h_{jk})$ is positive definite hermitian matrix with constant coefficients. This will satisfy $E = ImH$. $\square$

4.3 Examples of abelian varieties and tori

4.3.1 Non abelian tori

Let $g > 1$. A lattice in $\mathbb{C}^g$ is given $2g$ $\mathbb{R}$-linearly independent vectors $v_1, \ldots v_{2g}$. Two tori are isomorphic if the underlying lattice of one of them can be taken to the other by a linear automorphism. Therefore we can assume that $(v_1, \ldots v_g)$ is the identity. Without trying to be too rigorous, this shows that a g dimensional torus depends on g free parameters $v_{g+1}, \ldots, v_{2g}$. On the other hand a g dimensional abelian variety depends on a choice of $\Omega \in \mathbb{H}_g$, which involves $g(g+1)/2$ parameters. Thus there should (and does !) exist g-dimensional tori which are not abelian varieties.

4.3.2 Dual tori

Let $L \subset V$ be a lattice in a finite dimensional complex vector space, then $A = V/L$ is a complex torus. A function $f : V \to \mathbb{C}$ is antilinear if it is additive, and $f(av) = \bar{a}f(v)$ for $a \in \mathbb{C}$, $v \in V$. Let

$$\hat{V} = \{f : V \to \mathbb{C} \mid f \text{ is antilinear }\}$$

This is naturally a complex vector space of the same dimension as V called the antilinear dual. It is easy to see that

$$\hat{L} = \{f \in \hat{V} \mid Im\, f(\hat{L}) \subseteq \mathbb{Z}\}$$

is a lattice. We define the *dual torus* by

$$\hat{A} = \hat{V}/\hat{L}$$

Suppose that E is a polarization with associated hermitian form H, then we can define an isomorphism

$$\phi_E : V \cong \hat{V}, \quad \phi_E(v) = H(v, -)$$

This satisfies $\phi(L) \subseteq \hat{L}$ with equality if E is principal. In general, we can always find an integer $N > 0$ such that $N\hat{L} \subset \phi(L)$. Restricting the polarization given by E to $\hat{L}$ under this embedding yields a polarization on $\hat{L}$. Therefore we can conclude that:

Proposition 4.3.3. *If A is an abelian variety, then so is $\hat{A}$. If A is principally polarized then $A \cong \hat{A}$.*

4.3.4 Albanese varieties

We have see that Jacobians are principally polarized abelian varieties. The construction can be generalized as follows. Let X be a smooth projective variety (over $\mathbb{C}$) of dimension n. The Hodge theorem in higher dimensions shows that

$H_1(X,\mathbb{Z})/torsion$ embeds as a lattice in $H^0(X,\Omega^1_X)^*$ by sending $\gamma \to \int_\gamma$ as before. The Albanese torus

$$Alb(X) = \frac{H^0(X,\Omega^1_X)^*}{H_1(X,\mathbb{Z})/torsion}$$

We claim that this is an abelian variety. To prove this, we need to construct a polarization. Fix an embedding $X \subset \mathbb{P}^N$. Choose $n-1$ hyperplanes $H_1,\ldots,H_{n-1}$ in general position. Bertini shows that $C = X \cap H_1 \cap \ldots H_{n-1}$ is a smooth curve. We can now obtain a (possibly nonprincipal) polarization E on $Alb(X)$ by restricting to C. At the level of differential forms

$$E(\alpha,\beta) = \int_C \alpha|_C \wedge \beta|_C$$

Therefore $Alb(X)$ is an abelian variety.

When $x_0 \in X$, one can define a holomorphic map $\alpha : X \to Alb(X)$ by

$$\alpha(x) = \int_{x_0}^{x} \mod H_1(X,\mathbb{Z})$$

exactly as in the one dimensional case. This is a really important construction in algebraic geometry, since it allows us to partially reduce questions about X to abelian varieties which are easier to understand. When X is an abelian variety, α gives an isomorphism $X \cong Alb(X)$. So in particular, every abelian variety is an Albanese.

4.3.5 Picard varieties

The construction $J^\vee(X)$ also generalizes to higher dimensions Let X be a smooth projective variety. Using the exponential sequence as before, we get an exact sequence

$$\ldots H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z})$$

Again using the Hodge theorem for X, one can see that

$$Pic^0(X) := \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})} = \ker[H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z})]$$

is a torus call the Picard torus. The previous argument for Riemann surfaces generalizes to show that

$$Pic^0(X) = \widehat{Alb(X)}$$

Therefore this is also an abelian variety. Unlike the one dimensional case $Alb(X)$ and $Pic^0(X)$ are generally not isomorphic. If X is an abelian variety, $Pic^0(X) \cong \hat{X}$.

4.4 Further comments about Jacobians

As we have seen, given a compact Riemann surface X of genus g, we can associate an abelian variety $J(X)$ with a canonical polarization E. What is equivalent, and more geometric, is to consider the theta divisor $\Theta \subset J(X)$. Under Poincaré duality, the class $E \in H^2(J(X), \mathbb{Z}) \cong H_{2g-2}(J(X), \mathbb{Z})$ is the homology class $[\Theta]$. The divisor has a very nice geometric interpretation:

Theorem 4.4.1 (Riemann)**.** *Up to translation Θ is the image $\alpha(X^{g-1})$ (which is traditionally denoted by W_{g-1}). In particular, $[\Theta] = [W_{g-1}]$*

We won't give the proof. It can be found in Griffiths and Harris. The polarized Jacobian is a complete invariant:

Theorem 4.4.2 (Torelli)**.** *$J(X)$ with its canonical polarization determines X.*

In the simplest case, where the genus $g = 1$, $X \cong J(X)$, so the polarization is not needed. But otherwise, it is. There exists examples of nonisomorphic curves with isomorphic Jacobians (but the isomorphism won't respect polarizations). When $g = 2$, Torelli's theorem follows immediately from Riemann's theorem, because it implies $X \cong \Theta$. In general, X is also reconstructed using Θ, but the proof is more involved. Again we refer to Griffiths and Harris for the details.

Finally, we might wonder about the relationship between Jacobians and arbitrary abelian varieties. In fact, is not true is that every abelian variety is a Jacobian. But what is true is given an abelian variety A, we can always find a surjective homomorphism $J(X) \to A$ from a Jacobian. This trick was used often enough that Mumford, in the introduction to his book on abelian varieties says, "Rather stubbornly I wanted to prove that the theory of abelian varieties could be developed without the crutch of 'reduction to Jacobians' ".

Chapter 5

Modular Curves

5.1 Moduli of elliptic curves

Lemma 5.1.1. *$SL_2(\mathbb{R})$ acts transitively on the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ by fractional linear transformations. The stabilizer of i is $SO(2)$. Therefore, we can identify $\mathbb{H} = SL_2(\mathbb{R})/SO(2)$.*

I will omit the proof, which is not hard, and hopefully presented by one of you. We can view $\mathbb{H}$ as the upper hemisphere of the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$. The action of $SL_2(\mathbb{R})$ extends to the boundary $\partial\mathbb{H} = \mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. In order to better visualize the action, it useful to note that $\mathbb{H}$ has a Riemannian metric, called the hyperbolic or Poincaré metric, where the geodesics are lines or circles meeting $\partial\mathbb{H}$ at right angles. The action of $SL_2(\mathbb{R})$ preserves this metric, so it takes a geodesic to another geodesic.

Recall that an elliptic curve can be written as a quotient $E_\tau = \mathbb{C}/L_\tau$ where $L_\tau = \mathbb{Z} + \mathbb{Z}\tau$ with $\tau \in \mathbb{H}$. The origin $0 \in E_\tau$ is a distinguished point, which is part of the structure. In particular, two elliptic curves E_τ and $E_{\tau'}$ are called isomorphic if there is there a holomorphic isomorphism $f : E_\tau \to E_{\tau'}$ taking the origin of the first curve to the origin of the second.

Theorem 5.1.2. *E_τ and $E_{\tau'}$ are isomorphic if and only of there exists a matrix*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

such that

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

Proof. The isomorphism $f : E_\tau \to E_{\tau'}$ is induced by a holomorphic map $F : \mathbb{C} \to \mathbb{C}$ satisfying

$$F(z + \lambda) = F(z) + \Lambda(z, \lambda)$$

for all $\lambda \in L_\tau$ and some function $\Lambda : \mathbb{C} \times L_\tau \to L_{\tau'}$. Since $\Lambda(-, \lambda)$ is necessarily continuous, it is constant. Differentiating the previous identify shows that F' is doubly periodic, and therefore $F'(z) = \phi$ is constant. Therefore we can assume that $F(z) = \phi z$ since $f(0) = 0$. We must have $\phi L_\tau = L_{\tau'}$. Since $1, \tau'$ (resp. $1, \tau$) is a positively oriented basis of $L_{\tau'}$ (resp. ϕL_τ), we can find $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that

$$\begin{aligned} \tau' &= a\phi\tau + b\phi \\ 1 &= c\phi\tau + d\phi \end{aligned} \tag{5.1}$$

Therefore

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

Conversely, suppose the last equation is true. Set $\phi = (c\tau + d)^{-1}$. Then (5.1) holds. Therefore $\phi L_\tau = L_{\tau'}$. So that multiplication by ϕ gives an isomorphism $E_\tau \cong E_{\tau'}$. □

Corollary 5.1.3. *There is a natural bijection between the set of isomorphism classes of elliptic curves and the quotient space*

$$\mathcal{A}_1 := SL_2(\mathbb{Z})\backslash\mathbb{H} = SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO(2)$$

At the moment, $\mathcal{A}_1$ is just a set. In order to give more structure, we need to analyze the action more carefully. First observe that $-I$ acts trivially on $\mathbb{H}$, so the action factors through $\Gamma = PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm I\}$. Consider the closed region $F \subset \mathbb{H}$ lying above the unit circle and between the lines $\operatorname{Im} z = \pm 1/2$ depicted below.

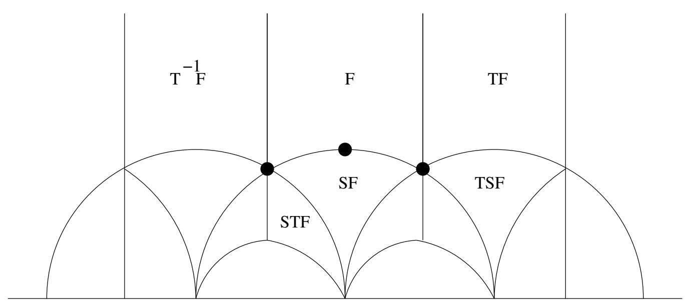


Figure 5.1: Fundamental domain

Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. These act by $z \mapsto -1/z$ and $z \mapsto z + 1$ respectively. S is a reflection about i which interchanges the regions $|z| \geq 1$ and $|z| \leq 1$. They generate a subgroup $G \subseteq \Gamma$.

Theorem 5.1.4.

(a) *The union of translates gF, $g \in G$, covers $\mathbb{H}$.*

(b) *An interior point of F does not lie in any other translate of F under Γ.*

(c) *The isotropy group of $z \in F$ is trivial unless it is one of the points $\{i, e^{\pi i/3}, e^{2\pi i/3}\}$ marked in the diagram. The isotropy group is $\langle S\rangle, \langle ST\rangle, \langle TS\rangle$ respectively.*

Proof. The intuition behind this can be understood from the picture. Repeatedly applying S and $T^{\pm 1}$ to F gives a *tiling* of $\mathbb{H}$ by hyperbolic triangles. Choose $\tau \in \mathbb{H}$, we want to find $A' \in SL_2(\mathbb{Z})$ and $\tau' \in F$ such that $A' \cdot \tau' = \tau$. Using (**??**), we can see that $\{\operatorname{Im} A \cdot \tau \mid A \in SL_2(\mathbb{Z})\}$ has a maximum M. Choose an A which realizes this maximum. Choose an integer n so that $\tau' = T^n A\tau$ has real part in $[-1/2, 1/2]$. Observe that $\operatorname{Im} \tau' = M$. If $|\tau'| < 1$ then $-1/\tau'$ would have imaginary bigger than M which is impossible. It follows that $\tau' \in F$, and τ lies in its orbit. This proves (a). For the remaining parts, see page 79 of [Serre, A Course in Arithmetic]

□

The set F is called a *fundamental domain* for the action of G. We can draw a number of useful conclusions.

Corollary 5.1.5. *$G = PSL_2(\mathbb{Z})$, i.e. S and T generate $PSL_2(\mathbb{Z})$.*

Proof. Let $z \in F$ be an interior point, and $h \in \Gamma$. Then $hz = gz$ for some $g \in G$. Since $z \in h^{-1}gF$, we must have $h^{-1}g = I$. □

Corollary 5.1.6. *The nontrivial elements of finite order in $PSL_2(\mathbb{Z})$ (resp. $SL_2(\mathbb{Z})$) are conjugate to S or $(ST)^{\pm 1}$ (resp. $-I, \pm S, \pm(ST)^{\pm 1}$).*

Proof. It is enough to prove the statement for $PSL_2(\mathbb{Z})$. A nontrivial element of $PSL_2(\mathbb{Z})$ of finite order must lie in the isotropy group of some point in $\mathbb{H}$. The points in the plane with nontrivial isotropy groups must be a translate of i or $e^{2\pi i/3}$. Their isotropy groups must be conjugate to the isotropy groups of one these two points. □

Corollary 5.1.7. *The action of $PSL_2(\mathbb{Z})$ is properly discontinuous, which means that for every point $p \in \mathbb{H}$, there is a neighbourhood U such that $gU \cap U = \emptyset$ for all but finitely many g.*

We can give $\mathcal{A}_1$ the quotient topology where $U \subseteq \mathcal{A}_1$ is open if and only its pullback to $\mathbb{H}$, under the projection $\pi : \mathbb{H} \to \mathcal{A}_1$ is open.

Proposition 5.1.8. *The topology on $\mathcal{A}_1$ is Hausdorff. In fact, it is homeomorphic to $\mathbb{C}$*

Proof. The first statement follows immediately from the last corollary. Using the above results, one can see that A_1 is obtained by gluing the two bounding lines of F and folding the circular boundary in half. This is easily seen to be homeomorphic to the sphere minus the north pole. □

$\mathcal{A}_1$ has a natural compactification $\bar{\mathcal{A}}_1$ given by adding single point at infinity to make it a sphere. We will follow the convention of the automorphic form literature and call it a *cusp.* It is important to keep in mind that this clashes with the usual terminology in algebraic geometry, that a cusp is a singularity of the form $y^2 = x^3$. We will refer the last thing as cuspidal singularity in order to avoid confusion. We can construct this a quotient as follows. Let $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\} \subset \mathbb{P}^1$. The action of Γ on $\mathbb{P}^1$ stabilizes $\mathbb{H}^*$. On $\mathbb{H}$ it coincides with the standard action, and on $\mathbb{Q} \cup \{\infty\}$ it consists of a single orbit. Thus $\Gamma\backslash\mathbb{H}^* = \bar{\mathcal{A}}_1$ as a set. In order to get the correct topology on the quotient, one needs a somewhat exotic topology of $\mathbb{H}^*$. On $\mathbb{H}$ it's the usual one, but on $\partial\mathbb{H}^*$ a fundamental system of punctured neighbourhoods of (a translate of) ∞ are (translates of) strips $\operatorname{Im} z > n$, $n \in \mathbb{N}$. These can be visualized as interiors of circles tangent to the boundary circle $\partial\mathbb{H}$.

5.2 Modular forms

Since $\mathcal{A}_1$ has a topology, we can talk about continuous functions on it. We can see that $f : A_1 \to \mathbb{C}$ is continuous if and only if it's pullback $\pi^* f := f \circ \pi$ is continuous. Let us also declare that a function on an open subset of $\mathcal{A}_1$ is holomorphic or meromorphic if its pullback to $\mathbb{H}$ has the same property. This means that such functions correspond to Γ-invariant functions on $\mathbb{H}$. Before constructing nontrivial examples, we want to relax the condition. We say that f is automorphic, with automorphy factor $\phi_\gamma(z)$, if it satisfies the functional equation

$$f(\gamma z) = \phi_\gamma(z) f(z)$$

This is very similar to what we did with theta functions. If we have two such functions with the same factor, their ratio would be invariant. Note that for this to work, we need to impose a consistency condition

$$\phi_{\gamma\xi}(z) f(z) = f(\gamma\xi z)$$

$$= \phi_\gamma(\xi z) f(\xi z) = \phi_\gamma(\xi z)\phi_\xi(z) f(z)$$

Cancelling f, leads to a so called cocycle condition on the automorphy factor

$$\phi_{\gamma\xi}(z) = \phi_\gamma(\xi z)\phi_\xi(z)$$

As the terminology suggests, ϕ_γ does give an element of a certain cohomology group. Rather than pursuing this direction, let us look for natural automorphic forms/factors in nature. Given a meromorphic differential form $\omega = f(z)dz$ on $\mathbb{H}$, let us see how it transforms under $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. We can see that

$$\omega \mapsto f(\gamma \cdot z) d\left(\frac{az+b}{cz+d}\right) = (cz+d)^{-2} f(\gamma \cdot z) dz$$

We say that $f(z)$ is a weakly modular form of weight 2, with respect to Γ, if $f(z)dz$ is invariant. We say that f is *weakly modular of weight* $2k$ if it

$$f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right) \tag{5.2}$$

This means that the tensor $f(z)dz^{\otimes k}$ is invariant. More generally, it makes sense to consider weakly modular forms of arbitrary integer weight ℓ, satisfying

$$f(z) = (cz+d)^{-\ell} f\left(\frac{az+b}{cz+d}\right)$$

However, when ℓ is odd, taking $\gamma = -I$, shows that $f = -f$, so it's zero! Natural nonzero examples do exist for other groups however, as we shall see shortly.

To drop the "weakly", we impose holomorphy conditions on $\mathbb{H}$ but also at infinity. To understand what the last part means, we first note that by using S and T, (5.2) is equivalent to

$$\begin{aligned} f(z+1) &= f(z) \\ f(-1/z) &= z^{2k} f(z) \end{aligned} \tag{5.3}$$

The first condition means that we have a Fourier expansion

$$f(z) = \sum_{-\infty}^{\infty} a_n e^{2\pi i n z} = \sum_{-\infty}^{\infty} a_n q^n$$

where $q = e^{2\pi i z}$. Note that as $z \to i\infty$, $q \to 0$. So we want to think of q as the local parameter at infinity. Then the Fourier series becomes the Laurent series in q. f is a *modular form of weight* $2k$ if it is holomorphic in $\mathbb{H}$, (5.2) holds, and the Fourier coefficients $a_n = 0$ for $n < 0$. It is called a *cusp form* of weight $2k$ if in addition $a_0 = 0$.

Theorem 5.2.1. *The Eisenstein series*

$$G_{2k}(z) = \sum_{\mathbb{Z}^2 - 0} \frac{1}{(mz+n)^{2k}}$$

is a modular form of weight $2k$*, when* $k \geq 2$*.*

$$\Delta(z) = (60G_4(z))^3 - 27(140G_6(z))^2$$

is a cusp form of weight 12*.*

Proof. The sum can be seen to converge uniformly on compact sets, so it must converge to a holomorphic function on $\mathbb{H}$. One has

$$G_{2k}\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} \sum \frac{1}{(ma+ndc)z + (mb+nd))^{2k}}$$

The vectors $(ma + ndc, mb + nd)$ can be seen to run over $\mathbb{Z}^2 - 0$. So the right side can be rewritten as

$$(cz + d)^{2k} G_{2k}(z)$$

as required.

We have to check holomorphicity at infinity. By uniform convergence, we can evaluate the limit as $z \to \infty$ term by term. When $m \neq 0$, have $(mz+n)^{-2k} \to 0$ as $z \to \infty$. Therefore

$$\lim_{z\to\infty} G_{2k}(z) = 2\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = 2\zeta(2k)$$

where ζ is the Riemann zeta function. Euler gave explicit formulas for the values

$$\zeta(4) = \frac{\pi^4}{90}$$

$$\zeta(6) = \frac{\pi^6}{945}$$

This allows us to evaluate $\lim_{z\to\infty} \Delta(z)$ and check that it's zero. □

Corollary 5.2.2.

$$j(z) = 1728 \frac{(60G_4(z))^3}{\Delta}$$

is weakly modular of weight 0*.*

The strange normalization is explained by the next result.

Proposition 5.2.3. *The function $j(z)$ is holomorphic on $\mathbb{H}$, and it has q-expansion*

$$j(z) = \frac{1}{q} + 744 + \ldots$$

where the series has integer coefficients.

See Serre [A Course in Arithmetic] for the proof.

5.3 Modular curves

We want to make $\bar{\mathcal{A}}_1$ into a Riemann surface in a natural way. We already have a topology on it. We just have to say what holomorphic functions are. Let $\Gamma(1) = SL_2(\mathbb{Z})$. Given $U \subset \mathcal{A}_1$, let $\tilde{U} \subset \mathbb{H}$ denote its preimage with projection $\pi : \tilde{U} \to U$. Let us say that $f : U \to \mathbb{C}$ is holomorphic if and only if $f \circ \pi : U \to \mathbb{C}$ is holomorphic in the usual sense. We define $q = e^{2\pi i}$ to be the coordinate at ∞. So function is holomorphic at ∞ if it can be expanded as a power series in q, or equivalently, the Fourier expansion of the pullback to (an open subset of) $\mathbb{H}$, has no negative coefficients.

Proposition 5.3.1. *$\bar{\mathcal{A}}_1$ is a Riemann surface.*

Sketch. The key point is to show that any point $x \in \bar{\mathcal{A}}_1$ has a neighbourhood D with a homeomorphism z, called a local coordinate or parameter, to a disk in $\mathbb{C}$, such that holomorphic functions on both disks coincide. There are three cases: $x = \infty$, x is an image of one of the fixed points $i, e^{2\pi i/3}$, or x is any other point. The first case was done above. The third case is straight forward. The map $\pi : \mathbb{H} \to X(1)$ is unramified over x A local coordinate z at a point $y \in \mathbb{H}$ lying over x will give a local coordinate at x. The map π is ramified at i and $e^{2\pi i/3}$ with ramification index $r = 2$ and 3 respectively. z^r will give a local coordinate at the image. □

The importance of $j(z)$ stems from the following.

Corollary 5.3.2. *Two elliptic curves E_τ and $E_{\tau'}$ are isomorphic if and only if $j(\tau) = j(\tau')$.*

Proof. By proposition 5.2.3 j factors through a holomorphic map $\mathcal{A}_1 \to \mathbb{C}$, with pole of order 1 at ∞. Therefore j induces a holomorphic map $\bar{\mathcal{A}}_1 \to \mathbb{P}^1$ of degree 1, which is necessarily bijective. Since $\mathcal{A}_1$ is the set equivalence classes of elliptic curves, the corollary follows.

□

While it's intuitively clear what it means that $\mathcal{A}_1$ parameterizes elliptic curves, the actual statement requires a bit more precision. Let us define an analytic family of (compact) complex manifolds to be a (proper) holomorphic submersion of complex manifolds $f : E \to B$. We recall that a submersion is map such that derivative is surjective on tangent spaces. This implies that fibres $E_b = f^{-1}(b)$ are complex submanifolds. By an analytic family of elliptic curves we mean an analytic family of compact complex manifolds $f : E \to B$ with a holomorphic section $s : B \to E$ such that each fibre E_b is a compact Riemann surface of genus one. We can regard E_b as an elliptic curve with origin $s(b)$. Given an elliptic curve $E = E_\tau$, set $j(E) = j(\tau)$.

Theorem 5.3.3. *$\mathcal{A}_1$ has the following properties:*

(a) The map $E \mapsto j(E)$ gives a bijection between the set of isomorphism classes of elliptic curves over $\mathbb{C}$ and points of $\mathcal{A}_1$.

(b) Given an analytic family elliptic curves $E \to B$, the map $B \to \mathcal{A}_1$, called the classifying map, given by $b \mapsto j(E_b)$ is holomorphic.

One might hope for stronger property:

Question 5.3.4. *Does there exists a universal family of elliptic curves over $\mathcal{A}_1$, i.e. a family of elliptic curves such that any other family is obtained by pulling it back with respect to j?*

It would be very desirable, for a number of reasons, to have an affirmative answer (in which case $\mathcal{A}_1$ would be called a *fine moduli space*). Unfortunately, the answer is no, as shown by the following example:

Example 5.3.5. *If $\mathcal{A}_1$ were a fine moduli space, then any family of elliptic curves with constant j-invariant would be trivial. However, let E be either E_i or $E_{\exp(2\pi i/3)}$ Either curve has a nontrivial automorphism group G, which is cyclic in both cases. Choose a manifold $\tilde{B}$ on which G acts freely, e.g. $\mathbb{C}^*$. The quotient $(E \times \tilde{B})/G \to \tilde{B}/G$ is a nontrivial family with constant j-invariant.*

There are a number of things we could do at this point:

1. State the precise universal property that $\mathcal{A}_1$ satisfies. This is weaker than the existence of a universal family, but stronger than what the last theorem says.
2. Explain what *stack* is. This addresses the lack of fineness in a different way.
3. Show how to add additional structure to get a universal family.

We will follow the 3rd option, which the easiest to explain. The reason why example 5.3.5 works is because the elliptic curves E_i or $E_{\exp(2\pi i/3)}$ have extra automorphisms. This is closely related to the fact that $SL_2(\mathbb{Z})$ has torsion and that it does not act freely on $\mathbb{H}$. The solution to both problems is to pass to subgroup. Given an integer $N > 0$, the *principal congruence subgroup of level* N of $\Gamma(1) = SL_2(\mathbb{Z})$ is

$$\Gamma(N) = \ker[\Gamma(1) \to SL_2(\mathbb{Z}/N)] = \{M \in \Gamma(1) \mid M \equiv I \mod N\}$$

Lemma 5.3.6. *If $N \geq 3$, $\Gamma(N)$ is torsion free and it acts freely on $\mathbb{H}$.*

Proof. This follows easily from corollary 5.1.6 □

Given such a group, it will act on $\mathbb{H}^*$, let $Y(N) = \Gamma(N)\backslash\mathbb{H}$ and let $X(N) = \Gamma'\backslash\mathbb{H}^*$. $X(N)$ can be made into a compact Riemann surface the same way as we did for $\bar{\mathcal{A}}_1$, and $Y(N) \subset X(N)$ is an open subset, and the points of $X(N) - Y(N)$ are called cusps. Given an elliptic curve E, the group of N-torsion points is isomorphic to $(\mathbb{Z}/N)^2$. A *level* N-structure for E is a basis for this group, i.e. a pair of N-torsion points which generates it.

Theorem 5.3.7. *If $N \geq 3$, $Y(N)$ is a fine moduli space parameterizing pairs (E, L), where E is an elliptic curve with a level N-structure.*

Sketch. Let G be the group of holomorphic automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by $\Gamma(N)$ acting on the first factor and the translations

$$(\tau, z) \mapsto (\tau, z + 1)$$

$$(\tau, z) \mapsto (\tau, z + \tau)$$

As an abstract group G is the semidirect product $\Gamma(N) \rtimes \mathbb{Z}^2$. Using lemma 5.3.6, we can see that G acts freely on $\mathbb{H} \times \mathbb{C}$. Therefore $\mathcal{E}(N) = G\backslash(\mathbb{H} \times \mathbb{C})$ is a complex manifold. The projection on the first factor gives a holomorphic map $\mathcal{E}(N) \to Y(N)$ which can be seen to a family of elliptic curves. The maps $\tau \mapsto 1/N$ and $\tau \mapsto \tau/N$ give sections $\sigma_i : Y(N) \to \mathcal{E}(N)$ which give level N-structures on each fibre. The family $(\mathcal{E}(N), \sigma_1, \sigma_2)$ can be checked to be the universal family of elliptic curves with level N-structure. $\square$

Theorem 5.3.8. *When $N \geq 3$, the genus of $X(N)$ is*

$$g = 1 + \frac{d(N-6)}{12N}$$

where

$$d = \frac{1}{2}|SL_2(\mathbb{Z}/N\mathbb{Z})|$$

(There are standard formulas for computing the order, e.g. if $N = p$ is prime, then $|SL_2(\mathbb{Z}/p\mathbb{Z})| = p(p^2-1)$.)

Proof. We have a holomorphic map

$$\pi : X(N) \to X(1) = \mathbb{P}^1$$

induced by inclusion $\Gamma(N) \subset \Gamma(1) = SL_2(\mathbb{Z})$. This is a branched covering. So we can compute the genus using the Riemann-Hurwitz formula, which says that if $Y \to X$ is a degree d branched covering of compact Riemann surfaces of genus $g(Y)$ and $g(X)$, then

$$2g(Y) - 2 = (2g(X) - 2)d + \underbrace{\sum_{y \in Y} (e_y - 1)}_{R}$$

where e_y is the ramification index which counts the number of sheets which "come together" at y.

The covering $\pi : X(N) \to X(1)$ is Galois with group $G = PSL_2(\mathbb{Z})/\operatorname{im}\Gamma(N)$. Let $\bar{S}, \bar{T} \in G$ denote the images of S and T. The degree of this covering $|G| = d$, when $N \geq 3$. Let p_2 and p_3 represent the images of i and $e^{2\pi i/3}$ in $X(1)$. Then p_2, p_3, ∞ are the ramification points. Given one of these points p, and $q \in \pi^{-1}(p)$, e_q is the order of the isotropy group $G_q = \{g \in G \; gq = q\}$. This is independent of q, because all the isotropy groups are conjugate. For $p = p_2$ and suitable q, $G_q = \langle \bar{S} \rangle$ so that $e_q = 2$. We can calculate the other values in a similar way to get

$$p = p_2, G_q = \langle \bar{S} \rangle, e_q = 2$$
$$p = p_3, G_q = \langle \bar{S}\bar{T} \rangle, e_q = 3$$
$$p = \infty, G_q = \langle \bar{T} \rangle, e_q = N$$

We also have $|\pi^{-1}(p)| = d/|G_q|$. This allows to calculate the ramification term above to get

$$R = \frac{d}{2} + \frac{2d}{3} + \frac{d(N-1)}{N}$$

Putting this into Riemann-Hurwitz and simplifying proves the theorem.

$\square$

Corollary 5.3.9. *There are nonzero modular forms of weight* 2 *for* $\Gamma(N)$ *as soon as* $N \geq 6$.

If X is a compact Riemann surface of genus $g \geq 2$, then Hurwitz showed that the automorphism group satisfies

$$|Aut(X)| \leq 84(g-1)$$

The next example shows that this bound is sharp.

Example 5.3.10. *The group* $PSL_2(\mathbb{Z}/7\mathbb{Z})$ *has cardinality* 168 *and it acts on* $X(7)$. *By the above formula, it has genus* 3.

It is useful to consider a generalization as follows. Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a finite index subgroup. Then Γ will act on $\mathbb{H}$ and $\mathbb{H}^*$ as before, and the quotient $\Gamma\backslash\mathbb{H}$ (resp. $\Gamma\backslash\mathbb{H}^*$) can be made into a (compact) Riemann surface. This construction is particularly interesting to number theorists, when Γ is a *congruence* group, which means that it contains some $\Gamma(N)$. In this case, the quotients are called modular curves. These curves $\Gamma\backslash\mathbb{H}$ will parameterize elliptic curves with extra structure. For example, for

$$\Gamma_1(N) = \{M \in \Gamma(1) \mid M \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N\}$$

the extra structure is the choice of a point of order N. For more information about these curves, and their applications to number theory, see [Diamond, Shurman, A First Course in Modular Forms]. One of the more dramatic applications was the proof of Fermat's Last Theorem by Wiles about 30 years ago. Some, but by no means all, of the ingredients are explained in that book.

5.4 Belyi's theorem

Given a subfield $K \subset \mathbb{C}$, we say that complex algebraic variety is definable over K, if we can choose coefficients of the defining equations to lie in K. This becomes particularly interesting when $K = \mathbb{Q}$ or $\bar{\mathbb{Q}}$.

Example 5.4.1. *There exist elliptic curves which are not defined over* $\bar{\mathbb{Q}}$, *and in fact most are not. This comes down to the simple observation that* $\mathcal{A}_1$ *has uncountably many points but only countably many could correspond to elliptic curves defined over* $\bar{\mathbb{Q}}$.

There is a surprisingly simple criterion for when a curve can be defined over $\bar{\mathbb{Q}}$, which was discovered not that long ago.

Theorem 5.4.2 (Belyi, 1979)**.** *Let* X *be a smooth projective curve over* $\mathbb{C}$. *The following are equivalent.*

(1) X *is definable over* $\bar{\mathbb{Q}}$.

(2) *There exists a finite index subgroup* $\Gamma \subseteq SL_2(\mathbb{Z})$ *such that* $X \cong \Gamma\backslash\mathbb{H}^*$.

(3) *There exists a finite holomorphic map* $\pi : X \to \mathbb{P}^1$ *unramified outside of* $\{0, 1, \infty\}$.

Proof. We note that (2) $\Rightarrow$ (1) is not at all obvious, and it follows from a deep result of Grothendieck[1] that if Y is a smooth curve over an algebraically closed field K of characteristic 0, then the category of étale covers of Y is equivalent to the category of étale covers of $Y \times_{Spec\, K} Spec\, L$ for any algebraically closed field $L \supset K$. For the implication, one applies this with $Y = \mathbb{P}^1 - \{0, 1, \infty\}$, $K = \bar{\mathbb{Q}}$ and $L = \mathbb{C}$. We just outline the equivalence of (2) and (3); for the remaining step , (1) $\Rightarrow$ (3), see [Serre, Lectures on the Mordell-Weil theorem].

For (2) $\Rightarrow$ (3), the inclusion $\Gamma \subseteq SL_2(\mathbb{Z})$ induces a map $X = \Gamma\backslash\mathbb{H}^* \to SL_2(\mathbb{Z})\backslash\mathbb{H}^* = \mathbb{P}^1$. This is unramified outside of 3 points corresponding to the images $i, e^{2\pi i/3}, \infty$. After composing with an automorphism of the line, we can move these points to $0, 1, \infty$

For (3) $\Rightarrow$ (2), we need to know that we can find subgroup $\Gamma_2 \subset SL_2(\mathbb{Z})$ of finite index, which acts freely on $\mathbb{H}$, such that $\Gamma_2\backslash\mathbb{H} \cong \mathbb{P}^1 - \{0, 1, \infty\}$. Suppose that $\pi : X \to \mathbb{P}^1$ unramified outside of $\{0, 1, \infty\}$ Then it follows that $Y = \pi^{-1}(\mathbb{P}^1 - \{0, 1, \infty\})$ is of the form $Y = \Gamma\backslash\mathbb{H}$ for some $\Gamma \subseteq \Gamma_2$ of finite index. Since $\Gamma\backslash\mathbb{H}^*$ is the unique Riemann surface compactification of Y, $X = \Gamma\backslash\mathbb{H}^*$. □

[1]SGA1: Revêtement étales et groupe fondamental

Chapter 6

Moduli space of abelian varieties

6.1 The action of the symplectic group

We want to generalize the constructions given earlier for elliptic curves to higher dimensions. The first step is to replace the usual upper half plane with the Siegel upper half plane. Recall that this

$$\mathbb{H}_g = \{\Omega \in Mat_{g\times g}(\mathbb{C}) \mid \Omega = \Omega^T, \mathrm{Im}\,(\Omega) > 0\}$$

This is an open subset of the space of symmetric matrices. So its dimension is $g(g+1)/2$. Given $\Omega \in H_g$ we can construct a lattice $L_\Omega = \mathbb{Z}^g + \Omega\mathbb{Z}^g$ and an abelian variety $A_\Omega = \mathbb{C}^g/L_\Omega$. This carries a principal polarization E_Ω given by the matrix

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{6.1}$$

with respect to the basis of L_Ω obtained by combining the standard basis of $\mathbb{Z}^g$ with the columns of Ω. This was explained before in detail.

For now the key point, is that $\Omega \in \mathbb{H}_g$ gives a pair (A_Ω, H_Ω) consisting of an abelian variety with a principal polarization. This is a key difference from the one dimensional theory, and that is that in order to get a good moduli theory, we need to keep track of a polarization. Given two such pairs $(V_i/L_i, E_i)$, by an isomorphism we mean a linear isomorphism $f : V_1 \xrightarrow{\sim} V_2$ such that $\phi(L_1) = L_2$ and $E_1(v,w) = E_2(\phi(v), \phi(w))$.

Lemma 6.1.1. *Given any g dimensional principally polarized abelian variety (A, E), there exists $\Omega \in H_g$ and an isomorphism $(A, E) \cong (A_\Omega, E_\Omega)$.*

Sketch. We just outline the argument, because it is essentially the same as what we did to obtain the normalized period matrix for a Riemann surface. Write

$A = V/L$. One of the conditions for E to be principal polarization guarantee we can choose a basis $\alpha_1, \dots \beta_g$ for L satisfying

$$E(\alpha_i, \alpha_j) = E(\beta_i, \beta_j) = 0, E(\alpha_i, \beta_j) = \delta_{ij}$$

Then observe that the remaining conditions for a polarization guarantee that we can choose a basis $\omega_1, \dots, \omega_g$ for L such that the change of basis matrix from the second to the first basis is (I, Ω) with $\Omega \in \mathbb{H}_g$.

□

Given (A, E) as above, there are many Ω's which satisfy $(A, E) \cong (A_\Omega, E_\Omega)$. The next problem is to deal with the nonuniqueness. A point of $\mathbb{H}_g$ gives rise to a polarized abelian variety with a preferred basis for the lattice, given the columns of (Ω, I). We need to mod out the choice of basis. It is important to restrict to change of bases which are compatible with the polarization. For any commutative ring R (e.g. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$) we define the symplectic group

$$Sp_{2g}(R) = \left\{ M \in GL_{2g}(R) \mid M^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \right\}$$

In other words, this is the group of matrices with preserves the symplectic form E.

Lemma 6.1.2. *Given $\Omega \in \mathbb{H}_g$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{R})$*

$$M \cdot \Omega := (A\Omega + B)(C\Omega + D)^{-1} \in \mathbb{H}_g$$

This defines an action of $Sp_{2g}(\mathbb{R})$ on $\mathbb{H}_g$.

Proof. For M as above, one checks the following identities: $A^T C$ and $B^T D$ are symmetric, and $A^T D - C^T B = I$. Let $M(\Omega) = (A\Omega + B)(C\Omega + D)^{-1}$. After expanding, using the above identities, and canceling, we obtain

$$(C\Omega + D)^T (M(\Omega) - M(\Omega)^T)(C\Omega + D) = \Omega - \Omega^T = 0$$

Therefore $M(\Omega)$ is symmetric. Similarly

$$(C\Omega + D)^T \operatorname{Im} M(\Omega)(C\Omega + D) = \operatorname{Im} \Omega > 0$$

which implies that $M(\Omega)$ is positive definite. □

One can put the Siegel space in the more general framework of symmetric spaces using the following:

Lemma 6.1.3. *The action of $Sp_{2g}(\mathbb{R})$ on $\mathbb{H}_g$ is transitive and the stabilizer of iI is*

$$\left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid AB^T = BA^T, AA^T + BB^T = I \right\} \cong U_g(\mathbb{R})$$

where the isomorphism is given by sending

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + iB$$

Proof. Let $\Omega = X + iY \in H_g$. Since Y is symmetric and positive definite, we can find an $A \in GL_g(\mathbb{R})$ so that $Y = AA^T$. Then $M = \begin{pmatrix} A & X(A^T)^{-1} \\ 0 & (A^T)^{-1} \end{pmatrix}$ sends iI to Ω. The formula for the stabilizer can be checked by calculation. □

Corollary 6.1.4. *Thus* $\mathbb{H}_g \cong Sp_{2g}(\mathbb{R})/U_g(\mathbb{R})$.

6.2 The moduli space of abelian varieties

We define

$$\mathcal{A}_g = Sp_{2g}(\mathbb{Z})\backslash\mathbb{H}_g = Sp_{2g}(\mathbb{Z})\backslash Sp_{2g}(\mathbb{R})/U_g(\mathbb{R})$$

Theorem 6.2.1. *Given* $\Omega, \Omega' \in \mathbb{H}_g$*, the pairs* $(A_\Omega, E_\Omega), (A_{\Omega'}, E_{\Omega'})$ *are isomorphic if and only if there exists* $M \in Sp_{2g}(\mathbb{Z})$ *with* $\Omega' = M \cdot \Omega$*. In particular,* $\mathcal{A}_g$ *can be identified with the set of isomorphism classes of* g*-dimensional principally polarized abelian varieties.*

Proof. An isomorphism $(A_\Omega, E_\Omega) \cong (A_{\Omega'}, E_{\Omega'})$ is induced by an invertible $g \times g$ complex matrix Φ such that (a) $\Phi L_\Omega = L_{\Omega'}$ and (b) Φ takes E_Ω to $E_{\Omega'}$. Condition (a) implies that there exists integer $g \times g$ matrices A, B, C, D such that

$$\begin{aligned} \Omega' &= \Phi A\Omega + \Phi B \\ I &= \Phi C\Omega + \Phi D \end{aligned} \tag{6.2}$$

Condition (b) implies that $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z})$. We can solve (6.2) to obtain $\Omega' = M \cdot \Omega$. Conversely, if $\Omega' = M \cdot \Omega$, with M as above, then (6.2) holds with $\Phi = (C\Omega + D)^{-1}$. Therefore Φ gives an isomorphism $(A_\Omega, E_\Omega) \cong (A_{\Omega'}, E_{\Omega'})$.

□

We will give a more conceptual proof later. Our next goal is to understand the geometry of $\mathcal{A}_g$ better.

Lemma 6.2.2. *The action of* $Sp_{2g}(\mathbb{Z})$ *on* $\mathbb{H}_g$ *is properly discontinuous. Therefore the quotient is a Hausdorff space.*

Proof. Given compact sets $K_1, K_2 \subset H_g$, we have to show that $S = \{M \in Sp_{2g}(\mathbb{Z}) \mid M(K_1) \cap K_2 \neq \emptyset\}$ is finite. Let us identify $H_g = Sp_{2g}(\mathbb{R})/U_g(\mathbb{R})$ as above. Note that the group $U_g(\mathbb{R})$ is compact, so that the projection $p : Sp_{2g}(\mathbb{R}) \to H_g$ is proper. $M \in Sp_{2g}(\mathbb{Z})$ lies in S if and only if $Mp^{-1}K_1 \cap p^{-1}K_2 \neq \emptyset$ if and only if $M \in T = (p^{-1}(K_1))^{-1} \cap p^{-1}(K_2)$. Now T is compact because it is the image of $K_1 \times K_2$ under $(M_1, M_2) \mapsto M_1^{-1}M_2$. Therefore S is the intersection of a compact set with a discrete set, so it's finite. □

The action of $Sp_{2g}(\mathbb{Z})$ on $\mathbb{H}_g$ is not free, so $\mathcal{A}_g$ will have singularities. The solution as before is to pass to a suitable subgroup. Given an integer $n > 0$, define the principal congruence group by

$$\Gamma(n) = \ker[Sp_{2g}(\mathbb{Z}) \to Sp_{2g}(\mathbb{Z}/n\mathbb{Z})]$$

Proposition 6.2.3. *Let $n \geq 3$. Suppose that γ is an automorphism of a principally polarized abelian variety (A, E) which acts trivially on the lattice mod n. Then $\gamma = 1$. The action for $\Gamma(n)$ on $\mathbb{H}_g$ is free.*

Proof. We assume that $\gamma \neq 1$. Then it has finite order, which we can assume is a prime p, by replacing γ a power. Then by assumption, $1 - \gamma = n\phi$ where $\phi \in End(A)$. Let ζ be a nontrivial eigenvalue of γ, and let η be the corresponding eigenvalue of ϕ. ζ is a primitive pth root of unity and η is an algebraic integer in the cyclotomic field $\mathbb{Q}(\zeta)$. We have a relation $n\eta = 1 - \zeta$. Taking the norm with respect to $\mathbb{Q}(\zeta)/\mathbb{Q}$ yields an equality of integers

$$n^{p-1}N(\eta) = (1 - \zeta)(1 - \zeta^2)\dots(1 - \zeta^{p-1}) = p$$

But this impossible because p is prime and $n \geq 3$. This proves the first part.

Suppose that $\gamma \in \Gamma(n)$ fixes a point of $\Omega \in \mathbb{H}_g$. Then γ will be an automorphism of (A_Ω, E_Ω) which is trivial mod n. Therevfore $\gamma = 1$. □

A level n-structure on an abelian variety $A = \mathbb{C}^g/L$ is a choice of symplectic basis of

$$H^1(A, \mathbb{Z}/n\mathbb{Z}) \cong Hom(L, \mathbb{Z}/n\mathbb{Z})$$

Let

$$\mathcal{A}_{g,n} = \Gamma(n)\backslash\mathbb{H}_g$$

Theorem 6.2.4. *Suppose $n \geq 3$. Then $\mathcal{A}_{g,n}$ is a complex manifold. The semidirect product $\Gamma(n) \ltimes \mathbb{Z}^{2g}$ acts naturally on $\mathbb{H}_g \times \mathbb{C}^g$ and the quotient can be equipped with a family of sections and polarizartions so that it becomes the universal family of principally polarized g-dimensional abelian varieties with level n-structure. In particular $\mathcal{A}_{g,n}$ is a fine moduli space.*

Outline. Since $\Gamma(n)$ acts freely on $\mathbb{H}_g$, $A_{g,n}$ is a manifold. The remaining statements are similar to case of $g = 1$ discussed earlier. □

The moduli space allows us to study the behaviour of a typical abelian variety. Recall that a set in a complete metric space is called meagre if it is a countable union of nowhere dense sets. Such sets should be viewed as very small, and in particular the Baire category theorem says that a meagre set has a nonempty complement.

Proposition 6.2.5. *There exists a set $U \subset \mathcal{A}_g$ which is the complement of meagre set and such that for any $(A, E) \in U$*

(a) (A, E) is not a product of polarized abelian varieties of smaller dimension

(b) $End(A) = \mathbb{Z}$.

Proof. We just explain the proof for (a). Let

$$R' = \bigcup_{g>h>0} \mathbb{H}_h \times \mathbb{H}_{g-h} \subset \mathbb{H}_g$$

and let R denote the union of all translates of R under $Sp_{2g}(\mathbb{Z})$, Then R is meagre in $\mathbb{H}_g$, and the same is true for its image $\bar{R} \subset \mathcal{A}_g$. Let U be the complement of $\bar{R}$. □

6.3 $\mathcal{A}_g$ is an algebraic variety

At the moment, we have an analytic construction of $\mathcal{A}_g$, which shows that it is almost a complex manifold; more precisely a quotient of a manifold by a finite group. In fact, it turns out to be a quasi-projective algebraic variety. The first step goes back to Satake, who constructed an explicit analytic compactification $\mathcal{A}_g^* \supset \mathcal{A}_g$ now called the Satake compactification. It is also called the Bailey-Borel compactifcation because these authors gave a more general construction a few years later. Before saying more, it is helpful to recall what happened when $g = 1$. Then we found that $\mathcal{A}_1 \cong \mathbb{C}$. This can be compactified by adding a point to get $\mathcal{A}_1^* = \mathbb{P}_1$. Before taking the quotient, we should add ∞, and all its translates to $\mathbb{H}$. This amounts to taking $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. In order to get $SL_2(\mathbb{Z})\backslash\mathbb{H}^* = \mathbb{P}^1$ as a topological space, we need to put a somewhat strange topology on $\mathbb{H}^*$. The basic neighbourhoods of ∞ are the strips $\operatorname{Im} z > c$, $c \in \mathbb{R}_+$, and for other points on the boundary we take translates of these.

Now let us suppose that $g > 1$. We describe $\mathcal{A}_g^*$ as a set. As a first step, we switch to the disk model.

Lemma 6.3.1. $\mathbb{H}_g$ *is isomorphic to*

$$D_g = \{Z \in Mat_{g\times g}(\mathbb{C}) \mid Z^T = Z, I - \bar{Z}Z > 0\}$$

by sending

$$\Omega \mapsto (\Omega - \sqrt{-1}I)(\Omega + \sqrt{-1}I)^{-1}$$

Let $\overline{D}_g$ denote the closure of D_g in the space of symmetric matrices. For $r < g$, we can embed $D_r \hookrightarrow \overline{D}_g$ by identifying it with

$$\left\{\begin{pmatrix} Z & 0 \\ 0 & I \end{pmatrix} \mid Z \in D_r\right\}$$

Let D_g^* denote D_g union the of images of D_r under $Sp_{2g}(\mathbb{Z})$ for all $r < g$. Then define

$$\mathcal{A}_g^* := Sp_{2g}(\mathbb{Z})\backslash D_g^*$$

The action of $Sp_{2g}(\mathbb{Z})$ on each boundary component D_r factors through the action of $Sp_{2r}(\mathbb{Z})$. Thus

$$\mathcal{A}_g^* = \mathcal{A}_g \amalg \mathcal{A}_{g-1} \amalg A_{g-2} \ldots$$

which for the moment is only a set. However, it has more structure. To formulate the result, we need to say what an analytic space is. A reduced analytic space is roughly speaking a compact manifold with singularities. The basic example is a pair $(Z, \mathcal{O}_Z)$ consisting of the zero set Z of a collection of holomorphic functions $f_1, \ldots, f_r$ on the ball $B \subset \mathbb{C}^n$, and $\mathcal{O}_Z$ the sheaf of holomorphic functions on B restricted to this. In general, a reduced analytic space is a ringed space $(X, \mathcal{O}_X)$ consisting of a metrizable space X and a sheaf of complex valued functions $\mathcal{O}_X$ which is locally isomorphic, at each point, to one of the basic examples.

Theorem 6.3.2 (Satake, 1957)**.** *The set D_g^* carries an $Sp_{2g}(\mathbb{Z})$-invariant topology, such that the quotient topology on $\mathcal{A}_g^*$ is compact Hausdorff. Furthermore, when $g > 1$ we have a sheaf of functions $\mathcal{O}_{\mathcal{A}_g^*}$, such that $f \in \mathcal{O}_{\mathcal{A}_g^*}(U)$ if and only if its pullback to the preimage of U is $\mathbb{H}_g$ is holomorphic. The pair $(\mathcal{A}_g^*, \mathcal{O}_{\mathcal{A}_g^*})$ is a reduced analytic space*

To finish the story define a *Siegel modular form* of weight k to be a holomorphic function $f : \mathbb{H}_g \to \mathbb{C}$ such that

$$\forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z}),\ f((A\Omega + B)(C\Omega + D)^{-1}) = \det(C\Omega + D)^k f(\Omega)$$

This generalizes the notion of modular form discussed earlier when $g = 1$. However, unlike that case, when $g > 1$ there are no conditions at infinity.

Theorem 6.3.3 (Bailey, 1962)**.** *For some (in fact infinitely many) k, there exists a set $f_0, \ldots f_N$ of Siegel modular forms of weight k such that there is an embedding $\mathcal{A}_g \hookrightarrow \mathbb{P}^N$ given by $\Omega \mapsto [f_0(\Omega), \ldots, f_N(\Omega)]$. Furthermore, the closure of $\mathcal{A}_g$ is isomorphic to $\mathcal{A}_g^*$.*

By applying Chow's theorem, we get

Corollary 6.3.4. *$\mathcal{A}_g^*$ is a projective variety and $\mathcal{A}_g$ is a quasi-projective variety.*

Note that $\mathcal{A}_g^*$ is very singular when $g > 1$. So it is not a nice compactification from the view point of geometry. In the 1970's, Mumford and his collaborators constructed a family of compactifications called toroidal compactifications. These are often nonsingular, but are somewhat complicated to describe since they involve both representation theory and combinatorics.

Mumford gave a completely different proof of quasi-projectivity in the mid 1960's which showed much more:

Theorem 6.3.5 (Mumford, 1965)**.** *There exists a quasiprojective scheme $\mathcal{A}_{g,\mathbb{Z}}$ over $\operatorname{Spec}\mathbb{Z}$, such that for any field k,*

$$Hom_{schemes}(\operatorname{Spec} k, \mathcal{A}_{g,\mathbb{Z}}) = \textit{the set of iso. classes of } g\textit{-dim ppav over } k$$

where ppav = principally polarized abelian varieties.

This is not an easy theorem. In fact, Mumford got the Fields medal partly for this work. The fact that this is defined over $\mathbb{Z}$ is essential for many number theoretic applications. For example, Faltings used it in an essential way to prove:

Theorem 6.3.6 (Faltings, 1983). *If X is a smooth projective curve of genus at least 2 defined over a number field K, then X has only finitely many K-rational points.*

Corollary 6.3.7. *Suppose that $f(x, y, z) \in K[x, y, z]$ has degree 4 or more and that it defines a nonsingular curve in $\mathbb{P}^2_{\bar{K}}$. Then $f = 0$ has only finitely many solutions in $\mathbb{P}^2_K$.*

Finally in 1990, Faltings and Chai gave an arithmetic construction of the various compactifications discussed above. So these are also defined over $\mathbb{Z}$.

6.4 Hodge structures

Although the previous constructions were very explicit, it is easy to get lost in the matrix computations, and lose sight of what is really happening. We want to give an alternate basis free description of $\mathbb{H}_g$ and $\mathcal{A}_g$. It is helpful to backtrack to where we first encountered this, namely from the cohomology of a compact Riemann surface X of genus g. To X we associated a cohomology group

$$H^1(X, \mathbb{C}) \cong H^1(X, \mathbb{Z}) \otimes \mathbb{C}$$

with an intersection pairing

$$E : H^1(X, \mathbb{Z}) \otimes H^1(X, \mathbb{Z}) \to \mathbb{Z}$$

and a distinguished g dimensional subspace

$$H^{10} = H^0(X, \Omega^1) \subset H^1(X, \mathbb{C})$$

All of this data satisfies the Hodge decomposition

$$H^{10} \oplus \overline{H}^{10} = H^1(X, \mathbb{C})$$

and the Riemann bilinear relations. This leads to the following definitions.

Definition 6.4.1. *A Hodge structure of type $\{(1, 0), (0, 1)\}$ consists of a finitely generated free abelian group $H_\mathbb{Z}$, with subspace $H^{10} \subset H := H_\mathbb{Z} \otimes \mathbb{C}$ satisfying the Hodge decomposition as above, A (principal) polarization on this is a non-degenerate (unimodular) skew-symmetric pairing $E : H_\mathbb{Z} \otimes H_\mathbb{Z} \to \mathbb{Z}$ satisfying the 2 Riemann relations*

$$\forall v, w \in H^{10}, E(v, w) = 0 \tag{6.3}$$

$$\forall v \in H^{10} - \{0\}, -iE(v, \bar{v}) > 0 \tag{6.4}$$

To shorten things, we will usually drop the phrase "of type $\{(1, 0), (0, 1)\}$". We note that the axioms imply that there is an integer g such that $\dim H^{10} = g$ and $rank H_\mathbb{Z} = 2g$. Finally, observe that there is some redundancy in the above axioms. If $H^{10} \subset H$ is a g-dimensional subspace satisfying the Riemann bilinear relations, then $H^{10} \cap \overline{H}^{10} = 0$ so the Hodge decomposition follows. Hodge structures form category where a morphism is $\mathbb{Z}$-linear map of lattices $H_{\mathbb{Z}} \to G'_{\mathbb{Z}}$ which takes $H^{10} \to G^{10}$. These completely captures the linear algebra aspects of abelian varieties because:

Lemma 6.4.2. *There is an equivalence of categories between the category of polarizable Hodge structures and the category of abelian varieties.*

Proof. In one direction, given a Hodge structure $H_{\mathbb{Z}}$, $H/(H_{\mathbb{Z}}+H^{10})$ is an abelian variety. □

In view of the last lemma, $\mathcal{A}_g$ can be identified with the set of isomorphism classes of $2g$-dimensional principally polarizable Hodge structures. We give a more explicit description below.

Lemma 6.4.3. *Let $H_{\mathbb{Z}} = \mathbb{Z}^{2g}$ with the standard skew-symmetric form E_{std} given by the matrix (6.1). Let $\mathbb{H}'_g$ denote the set of Hodge structures on $H_{\mathbb{Z}}$ polarized by E_{std} (i.e. the set of subspaces $H^{10} \subset \mathbb{C}^{2g} = H_{\mathbb{Z}} \otimes \mathbb{C}$ satisfying the previous conditions). Then there is a bijection between $\mathbb{H}_g$ and $\mathbb{H}'_g$ given by $\Omega \mapsto \{(v, -\Omega v) \mid v \in \mathbb{C}^g\}$*

This is really a lemma in linear algebra. The proof is not hard, and we will skip it. Instead we will refine this result. Since $\mathbb{H}_g$ is an open subset of the space of $g \times g$ complex matrices, it has the structure of a complex manifold. This is also true for $\mathbb{H}'_g$, although in a less obvious way. The set of the g-dimensional subspaces $H^{01} \subset \mathbb{C}^{2g}$ satisfying the first Riemann relation (6.3) is parameterized by a projective algebraic variety $LGr(g)$ sometimes called the Lagrangian Grassmanian. $LGr(g)$ is nonsingular becomes the symplectic group $Sp_{2g}(\mathbb{C})$ acts on it by translating subspaces, and this action can be seen to be transitive. Now observe that $\mathbb{H}'_g$ is an open subset of $LGr(g)$, because the second Riemann relation (6.4) is an open condition. When $g = 1$, $LGr(1) = \mathbb{P}^1$, and the embedding $\mathbb{H}_1 \subset \mathbb{P}^1$ is the usual one. The subgroup $Sp_{2g}(\mathbb{R}) \subset Sp_{2g}(\mathbb{C})$ preserves $\mathbb{H}'_g$ because it preserve both Riemman relations. A sharper version of the previous lemma is that:

Lemma 6.4.4. *We have an isomorphism $\mathbb{H}_g \cong \mathbb{H}'_g$ as complex manifolds, and this is compatible with $Sp_{2g}(\mathbb{R})$-actions.*

We can regard an element of $\mathbb{H}'_g$ as a Hodge structure with a preferred symplectic isomorphism $H_{\mathbb{Z}} \cong \mathbb{Z}^{2g}$. We can view

$$\mathcal{A}_g = Sp_{2g}(\mathbb{Z})\backslash\mathbb{H}'_g$$

and the projection $\mathbb{H}'_g \to \mathcal{A}_g$ forgets the isomorphism.

Finally, let us say a few words about the nonprincipally polarized case.

Theorem 6.4.5 (Frobenius)**.** *Given a nondegenerate skew-symmetric form $E : \mathbb{Z}^{2g} \otimes \mathbb{Z}^{2g} \to \mathbb{Z}$, we can find positive integers $d_1|d_2|\ldots$ and a basis such that E*

is represented by

$$E_{std}(d_1, d_2, \ldots) = \begin{pmatrix} 0 & 0 & d_1 & 0 \\ 0 & 0 & 0 & d_2 \\ & \ddots & & \ddots \\ -d_1 & 0 & 0 & 0 \\ 0 & -d_2 & 0 & 0 \\ & \ddots & & \ddots \end{pmatrix}$$

The set of integers $(d_1, \ldots)$ is called the type of the polarization. Fix the type, and modify $\mathbb{H}'_g$ to denote the set of Hodge structures on $H_{\mathbb{Z}} = \mathbb{Z}^{2g}$ polarized by $E_{std} = E_{std}(d_1, \ldots)$. Define the symplectic group

$$\Gamma = Sp(\mathbb{Z}^{2g}, E_{std}) = \{A \in GL_{2g}(\mathbb{Z}) \mid A^T E_{std} A = E_{std}\}$$

Then the moduli space of abelian varieties of type d_i is

$$\mathcal{A}_g(d_1, d_2, \ldots) = \Gamma \backslash \mathbb{H}'_g$$

Chapter 7

Low dimensional examples

7.1 Genus 2 curves and abelian surfaces

The Siegel upper half plane $\mathbb{H}_2$ is an open subset of the space of 2×2 symmetric matrices. It follows that this, and therefore $\mathcal{A}_2$ is three dimensional. Igusa studied this and a fairly explicit description of this space. We will describe his results, but first, we need the following basic result.

Theorem 7.1.1 (Weil)**.** *A two dimensional principally polarized variety is isomorphic to either a product of two elliptic curves, or the Jacobian of a smooth projective genus 2 curve.*

Sketch. Let (A, Θ) be a 2 dimensional ppav. One checks using some standard facts about algebraic surfaces (Riemann-Roch and the adjunction formula) that the curve Θ has arithmetic genus 2. If Θ is nonsingular, then it is a genus 2 curve. In this case, one can see that the induced map $J(C) \cong A$ is an isomorphism. If Θ is singular, then further analysis shows that it is a union of 2 elliptic curves $E_1 \cup E_2$. Then one finds that $A \cong E_1 \times E_2$. $\square$

In principle, this leads to the following description

$$\mathcal{A}_2 = \{\text{set parametrizing genus 2 curves}\} \cup \{\text{set parameterizing products of elliptic curves}\}$$

In some sense we understand the second set on the right. So we focus on the first. Given a degree 6 polynomial $f(x)$, with distinct roots, the smooth projective curve determined by

$$z^2 = f(x)$$

has genus 2. This follows from the Riemann-Hurwitz formula. Conversely, any genus 2 can be realized this way (see for example [Hartshorne, Algebraic Geometry p. 304]), but the choice of f is far from unique. The nonuniqueness can be understood. First replace f by the homogenous polynomial $F(x, y)$, then $SL_2(\mathbb{C})$ acts on the space V of such polynomials through the standard action on

$\begin{pmatrix} x \\ y \end{pmatrix}$. We are interested in invariant polynomials $V \to \mathbb{C}$ of even degree. These form a ring whose generators were known since the 19th century by Clebsch and Bolza. To describe them, let us factor

$$f(x) = u_0 x^6 + u_1 x^5 + \dots u_6 = u_0 \prod (x - r_i)$$

or equivalently

$$F(x, y) = u_0 x^6 + u_1 x^5 y + \dots u_6 y^6 = \prod (p_i x - q_i y)$$

The standard action of $SL_2(\mathbb{C})$ on the $\begin{pmatrix} p_i \\ q_i \end{pmatrix}$ is compatible with the action on F. The expressions

$$R_{ij} = \det \begin{pmatrix} p_i & p_j \\ q_i & q_j \end{pmatrix}^2$$

are clearly invariant under $SL_2(\mathbb{C})$. We also introduce

$$A(u) = R_{12}R_{34}R_{56} + \sum_{\sigma \in S_6 - \{1\}} R_{\sigma(1)\sigma(2)}R_{\sigma(3)\sigma(4)}R_{\sigma(5)\sigma(6)}$$

$$B(u) = R_{12}R_{23}R_{31}R_{45}R_{56}R_{64} + \dots$$

$$C(u) = R_{12}R_{23}R_{31}R_{45}R_{56}R_{64}R_{14}R_{25}R_{36} + \dots$$

$$D(u) = \prod_{i<j} R_{ij}$$

where the last two sums are symmetrized in the same way. Note that by the theorem on elementary symmetric functions, A, B, C, D are polynomials in the u's. These are also invariant, because the expressions R_{ij} are invariant. The expressions $A(u), B(u), C(u), D(u)$ generate the ring of invariants. Note that

$$D(u) = u_0^{10} \prod_{i<j} (r_i - r_j)^2$$

is just the discriminant, so that $D(u) \neq 0$ if and only if f has distinct roots. A genus 2 curve corresponds to a $PGL_2(\mathbb{C})$ orbit of the point $[u] \in \mathbb{P}(V)$ with $D(u) \neq 0$. This is determined by the ratios of the values of the fundamental invariants. Here is the precise statement:

Theorem 7.1.2 (Igusa)**.** *The moduli space $\mathcal{M}_2$ of genus 2 curves is the complement of the divisor $D = 0$ in the projective variety* $\operatorname{Proj} \mathbb{C}[A, B, C, D]$*, i.e. it is affine variety whose coordinate ring is the degree* 0 *part of* $\mathbb{C}[A, B, C, D^{\pm 1}]$.

To analyze further, define new invariants by

$$J_2 = \frac{1}{8}A$$
$$J_4 = \frac{1}{96}(4J_2^2 - B)$$
$$J_6 = \frac{1}{576}(8J_2^3 - 160J_2J_4 - C)$$
$$J_{10} = 2^{-12}D$$

Let μ_5 be the group of 5th roots of unity. Then Igusa showed that the degree 0 part of the graded ring $\mathbb{C}[A, B, C, D^{\pm 1}] = \mathbb{C}[J_2 \ldots, J_{10}^{\pm 1}]$ is isomorphic to $\mathbb{C}[t_1, t_2, t_3]^{\mu_5}$, via

$$J_2^{e_1} J_4^{e_2} J_6^{e_3} J_{10}^{-e_5} \mapsto t_1^{e_1} t_2^{e_2} t_4^{e_3}$$

where

$$e_1 + 2e_2 + 3e_3 = 5e_5$$

Therefore, we get the following explicit description:

Theorem 7.1.3. *$\mathcal{M}_2$ is isomorphic to $\mathbb{C}^3/\mu_5$, where μ_5 acts by $(t_1, t_2, t_3) \mapsto (\zeta t_1, \zeta^2 t_2, \zeta^3 t_3)$.*

We have an injective map $\mathbb{H}^2 \to \mathbb{H}_2$ given by

$$(\tau_1, \tau_2) \mapsto \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$$

Let $\Delta \subset \mathcal{A}_2$ denote the image of $\mathbb{H}^2$. This is a divisor parameterizing products of pairs of elliptic curves. From theorem 7.1.1, we see that the complement consists of Jacobians of genus 2 curves. Given a curve, $J(C)$ with its canonical principal polarization, the curve can be recovered simply as the theta divisor Θ. Therefore, we have proved:

Theorem 7.1.4. *$\mathcal{A}_2$ contains a divisor Δ parameterizing products of pairs of elliptic curves. The complement of $A_2 - \Delta \cong M_2$.*

A Siegel modular form of weight k is a holomorphic function f on $\mathbb{H}_2$ which transforms in the "expected way" under $Sp_4(\mathbb{Z})$:

$$f((\alpha\Omega + \beta)(\gamma\Omega + \delta)^{-1}) = \det(\gamma\Omega + \delta)^k f(\Omega), \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sp_4(\mathbb{Z})$$

One important class of examples are Eisenstein series, which are sums of the form

$$E_k(\Omega) = \sum \det(\gamma\Omega + \delta)^{-k}$$

where sum runs over suitable set of pairs of matrices (γ, δ). The sum of the spaces of modular forms over all weights forms a graded algebra. As an application of the last theorem, Igusa goes on to determine an explicit set of generators and relations for this algebra.

Theorem 7.1.5 (Igusa)**.** *The algebra of Siegel modular forms of even weight is generated by the Eisenstein series E_4, E_6, E_{10} and E_{12}.*

7.2 Hilbert modular surfaces.

We recall that a real quadratic field is a field $K = \mathbb{Q}(\sqrt{d}) = \{a+b\sqrt{d} \mid a, b \in \mathbb{Q}\}$, where $d > 1$ is square free integer. We have two distinct field homomorphisms $\sigma_1, \sigma_2 : K \to \mathbb{R}$, where $\sigma_1(a + b(\sqrt{d}) = a + b\sqrt{d}$ and $\sigma_2(a + b\sqrt{d}) = a - b\sqrt{d}$. Let $\mathcal{O}_K \subset K$ denote the ring of integers. This is the integral closure of $\mathbb{Z}$ in K, and it is described explicitly by

$$\mathcal{O}_K = \begin{cases} \mathbb{Z} + \frac{1+\sqrt{d}}{2}\mathbb{Z} & \text{if } d \equiv 1 \mod 4 \\ \mathbb{Z} + \sqrt{d}\mathbb{Z} & \text{if } d \not\equiv 4 \mod 4 \end{cases}$$

Lemma 7.2.1. *If we embed $\mathcal{O}_K \hookrightarrow \mathbb{R}^2$ by $x \mapsto (\sigma_1(x), \sigma_2(x))$, then the image is discrete.*

This is false with only a single σ_i.

Proof. This is elementary. For example, when $d \equiv 2, 3 \mod 4$, this follows from the inequality

$$(a + b\sqrt{d})^2 + (a - b\sqrt{d})^2 = a^2 + b^2 d \geq 1$$

for a nonzero integer $a + b\sqrt{d}$. □

The *Hilbert modular group* (for a given K) is $\Gamma_K = SL_2(\mathcal{O}_K)$. We have an embedding $\iota : \Gamma_K \to SL_2(\mathbb{R})^2$ by $M \mapsto (\sigma_1(M), \sigma_2(M))$. From the previous lemma, we easily deduce:

Corollary 7.2.2. *The image of the Hilbert modular group in $SL_2(\mathbb{R})^2$ is discrete with respect to the usual topology.*

The group Γ_K acts on $\mathbb{H}^2 = \mathbb{H} \times \mathbb{H}$ via ι. Note that $-I$ acts trivially, so the action factors through $PSL_2(\mathcal{O}_K) = SL_2(\mathcal{O}_K)/\{\pm I\}$. It is not difficult to deduce from the previous corollary that

Proposition 7.2.3. *This action is properly discontinuous. Therefore $\Gamma_K \backslash \mathbb{H}^2$ is Hausdorff.*

Definition 7.2.4. *The quotient $\Gamma_K \backslash \mathbb{H}^2$ is called a Hilbert modular surface. The term is also applied to various related examples given below. Also it is important to keep in mind that "surface" here means 2 complex dimensions, in contrast to the term "Riemman surface".*

The action Γ_K is not free, so this surface will have singularities. We fix the problem as we did earlier by passing to suitable subgroup. Given a nonzero ideal $I \subset \mathcal{O}_K$, we can define the corresponding principal congruence group as

$$\Gamma_K(I) = \ker[SL_2(\mathcal{O}_K) \to SL_2(\mathcal{O}_K/I)]$$

This applies, in particular, to an ideal of the form (N), N where N is a nonzero integer.

Lemma 7.2.5. $\Gamma_K(N)$ *is torsion free when* $N \geq 3$. *Therefore the action on* $\mathbb{H}^2$ *is free.*

Proof. A proof can be found in [Frietag, Hilbert Modular forms]. $\square$

We will refer to a subgroup $\Gamma \subseteq \Gamma_K$ containing some $\Gamma(N)$ as a congruence group. If Γ is such a group, then $Y(\Gamma) = \Gamma \backslash \mathbb{H}^2$ is noncompact analytic surface. It can be compactified by adding a finite number of points at infinity called cusps. We embed $\mathbb{P}^1(K) \to \mathbb{P}(\mathbb{R})^2$ by $\sigma_1 \times \sigma_2$. Using this, we can regard points of $\mathbb{P}(K)$ as lying on the boundary of $\mathbb{H}^2$ via

$$\mathbb{P}^1(\mathbb{R})^2 \subset \mathbb{P}^1(\mathbb{C})^2 \supset \mathbb{H}^2$$

A Γ-orbit of point of $\mathbb{P}^1(K)$ is called a *cusp* with respect to Γ. Given a point $[a, b] \in \mathbb{P}^1(K)$, we let (a, b) denote the fractional ideal generated by these elements. Although this ideal is not well defined, but its class in the class group

$$Cl(\mathcal{O}_K) = \frac{\{\text{fractional ideals}\}}{\{\text{principal frac. ideals}\}}$$

is.

Proposition 7.2.6. *The above map gives a bijection between the set of cusps for* Γ_K *and* $Cl(\mathcal{O}_K)$.

Proof. Let $\phi : \mathbb{P}^1(K) \to Cl(\mathcal{O}_K)$ denote the above map. It is known that any fractional ideal of $\mathcal{O}_K$ is generated by two elements. Therefore ϕ is surjective. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\phi(A \cdot [x, y]) = (ax + by, cx + dy) \subseteq (x, y)$$

The same argument, using A^{-1}, gives the opposite inclusion. Therefore ϕ factors through a map $\bar{\phi} : \Gamma_K \backslash \mathbb{P}^1(K) \to CL(\mathcal{O}_K)$.

It remains to prove that $\bar{\phi}$ is injective. We assume two points of $\mathbb{P}^1(K)$ have the same image under ϕ. For simplicity, we treat the case where on the points is $\infty = [1, 0]$. Denote the other by $[x, y]$. We can assume that both $x, y \in \mathcal{O}_K$. Since $(x, y) = \phi(\infty) = (1)$, we must have $ax + by = 1$ for some $a, b \in \mathcal{O}_K$. Then

$$A = \begin{pmatrix} a & b \\ -y & x \end{pmatrix}$$

lies Γ_K and it maps $[x, y]$ to ∞. Therefore they lie in the same Γ_K-orbit. $\square$

Corollary 7.2.7. Γ_K *has* $h = |Cl(\mathcal{O}_K)|$ *cusps. A congruence subgroup* $\Gamma \subset \Gamma_K$ *has a finite number of cusps.*

Set

$$(\mathbb{H}^2)^* = \mathbb{H} \cup \mathbb{P}^1(K)$$

Theorem 7.2.8 (Baily-Borel)**.** *For any congruence group* Γ*, the set* $X(\Gamma) = \Gamma\backslash(\mathbb{H}^2)^*$ *can be given the structure of a normal projective surface compactifying* $Y(\Gamma)$*. In particular,* $Y(\Gamma)$ *is quasiprojective.*

When $\Gamma = \Gamma(N), N \geq 3$, then $Y(\Gamma)$ is nonsingular. However, $X(\Gamma)$ will always be singular. Hirzebruch found procedure to resolve the singularities of $X(\Gamma)$ in an explicit way. This is described in Van der Geer's book on Hilbert modular surfaces.

The surface $Y(\Gamma_K)$ is a moduli space, which we now describe. A 2-dimensional abelian variety A is said to have *real multiplication* if $End(A) \otimes \mathbb{Q}$ contains a real quadratic field. We would like describe all abelian varieties with real multiplication by a given real quadratic field K. More precisely, we will, describe all principally polarized 2 dimensional abelian varieties A with $End(A) \supseteq \mathcal{O}_K$. As a $\mathbb{Z}$-module, $\mathcal{O}_K \cong \mathbb{Z}^2$. For each vector $\tau = (\tau_1, \tau_2) \in \mathbb{H}^2$ define $L_\tau \subset \mathbb{C}^g$ to be the image of $\mathcal{O}_K^2$ under the map

$$\iota_\tau(\alpha, \beta) = (\sigma_1(\alpha)\tau_1 + \sigma_1(\beta), \sigma_2(\alpha)\tau_2 + \sigma_2(\beta))$$

Proposition 7.2.9. L_τ *is a lattice, and the quotient* $A_\tau = \mathbb{C}^2/L_\tau$ *is an abelian variety with* $\mathcal{O}_K \subseteq End(A_\tau)$*.*

Proof. To see that $L_\tau \subset \mathbb{C}^2$ is lattice, it is enough to see that it is freely generated by an $\mathbb{R}$-basis, and this is easy to check. For example if $d \not\equiv 1 \mod 4$, then a $\mathbb{Z}$-basis for L_τ is $(\tau_1, \tau_2), \sqrt{d}(\tau_1, -\tau_2), (1, 1), \sqrt{d}(1, -1)$, and these are $\mathbb{R}$-linearly independent. Since L_τ is a lattice, A_τ is a torus. Consider the Hermitian form

$$H(u, v) = \sum \frac{u_j \bar{v}_j}{\operatorname{Im} \tau_j}$$

We claim that this is a polarization. It is clearly positive definite. It remains to show that the imaginary part $\operatorname{Im} H$ is integer valued on the lattice. Let $E_{std} : \mathcal{O}^2 \times \mathcal{O}^2 \to \mathbb{Z}$ be the pairing defined by

$$E_{std}(\alpha_1, \alpha_2; \beta_1, \beta_2) = \operatorname{trace}(\alpha_1\beta_2 - \alpha_2\beta_1) \tag{7.1}$$

One can check that if $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathcal{O}_K^2$, then

$$\operatorname{Im} H(\iota_\tau(\alpha_1, \alpha_2), \iota_\tau(\beta_1, \beta_2)) = E_{std}(\alpha_1, \alpha_2; \beta_1, \beta_2)$$

So it is a polarization as claimed. Therefore A_τ is an abelian variety. Furthermore, one can check that the determinant of E_{std} is 1, so this is a principal polarization.

Finally, we have an embedding $\mathcal{O}_K \subset M_{2\times 2}(\mathbb{C})$ which sends α to the diagonal matrix with entries $\sigma_j(\alpha)$. L_τ is stable under the resulting $\mathcal{O}_K$-action. Therefore $\mathcal{O}_K \subset End(A_\tau)$. □

Theorem 7.2.10. *The points of* $Y(\Gamma_K) = \Gamma_K\backslash\mathbb{H}^2$ *are in one to one correspondence with isomorphism classes of* 2*-dimensional polarized abelian varieties* (A, E)*, such that there is inclusion* $\mathcal{O}_K \subseteq End(A)$ *for which there is* $\mathcal{O}_K$*-module isomorphism* $H_1(A, \mathbb{Z}) \cong \mathcal{O}_K^2$ *taking* E *to* E_{std}*.*

Proof. We just describe one direction. We note that by construction, A_τ carries an inclusion $\mathcal{O}_K \subseteq End(A_\tau)$ and an isomorphism

$$H_1(A_\tau, \mathbb{Z}) \cong L_\tau \cong \mathcal{O}_K^2$$

such that E_{std} polarizes A_τ.

If $\tau, \tau' \in \mathbb{H}^2$ lie in the same Γ_K-orbit, Then there is matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$ such that

$$\tau'_j = \frac{a\tau_j + b}{c\tau_j + d}$$

Then multiplication by the diagonal matrix with entries $\sigma_j(c)\tau_j + \sigma_j(d)$ induces and isomorphism $A_{\tau'} \cong A_\tau$.

□

7.3 Riemann-Roch and applications

Given a divisor $D = \sum n_p p$ on a compact Riemann surface of genus g, set

$$L(D) = \{f \in \mathbb{C}(X)^* \mid div(f) + D \geq 0\} \cup \{0\}$$

where

$$div(f) = \sum ord_p(f)p$$

and $D \geq 0$ means that the coefficients satisfy this inequality. The above condition says that $ord_p(f) \geq -n_p$. Therefore $L(D)$ should be understood as the space of functions with prescribed zeros and bounds on the poles. Riemann's inequality says

$$\dim L(D) \geq \deg D + 1 - g$$

If define a sheaf

$$\mathcal{O}_X(D)(U) = \{f \text{ meromorphic on } U \mid div(f) + D|_U \geq 0\} \cup \{0\}$$

then $L(D) = H^0(X, \mathcal{O}_X(D))$. Roch added a correction term to Riemann's inequality, which in modern language says

Theorem 7.3.1 (Riemann-Roch)**.**

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)) = \deg D + 1 - g$$

The left side above is the Euler characteristic $\chi(\mathcal{O}_X(D))$. The higher dimensional version of this is due Hirzebruch. Given a coherent sheaf $\mathcal{F}$ on a smooth projective variety X, the theorem gives formula for

$$\chi(\mathcal{F}) = \sum_0^{\dim X} (-1)^i \dim H^i(X, \mathcal{F})$$

in terms of topological data (Chern classes) of $\mathcal{F}$ and X. We just explain what it looks like when $\dim X = 2$, and $\mathcal{F} = \mathcal{O}_X(D)$. In this case, D is a finite (formal) sum $D = \sum n_i D_i$, where $n_i \in \mathbb{Z}$ and $D_i \subset X$ are irreducible closed curves on X. To each D_i, we can attach a discrete valuation $ord_{D_i} : \mathbb{C}(X) \to \mathbb{Z} \cup \{\infty\}$, and we can define $\mathcal{O}_X(D)$ as above. It turns out that an line bundle (or more preciselyt rank one locally free sheaf) L is isomorphic to $\mathcal{O}_X(D)$, for some divisor D, and further it is unique as an element of the divisor class group $Cl(X)$. In particular, this applies to the bundle of 2-forms $\Omega^2_X = \mathcal{O}_X(K)$; K is called the canonical divisor class.

Suppose that C and D are distinct irreducible curves on X. Then $C \cap D$ is finite. If $p \in C \cap D$, define the intersection multiplicity at p by

$$(C \cdot D)_p = \dim \mathcal{O}_{S,p}/(f,g) = \dim \hat{\mathcal{O}}_{S,p}/(f,g)$$

where $\mathcal{O}_{X,p}$ is the local ring of the surface at p, and f, g are local equations of C and D in this ring. For example, this number is 1 if C and D are nonsingular and meet transversely at p, because f and g generate the maximal ideal of the completion $\hat{\mathcal{O}}_{X,p}$. Define the intersection number

$$C \cdot D = \sum_{p \in C \cap D} (C \cdot D)_p \tag{7.2}$$

Theorem 7.3.2. *There exists a bilinear form $Cl(S) \times Cl(S) \to \mathbb{Z}$ which agrees with the above intersection number whenever it is defined.*

Proof. See chap 5 of Hartshorne. □

Theorem 7.3.3 (Riemann-Roch for surfaces)**.** *For any divisor*

$$\chi(\mathcal{O}(D)) = \frac{1}{2} D \cdot (D - K) + \chi(\mathcal{O}_S)$$

and

$$\chi(\mathcal{O}_S) = \frac{K^2 + e(S)}{12}$$

where $K^2 = K \cdot K$ and $e(S)$ is the topological Euler characteristic.

Proof. For the first formula, see chap 5 of Hartshorne. The second is special case of the Hirzebruch-Riemann-Roch theorem; see [Barth, Hulek, Peters, Van de Ven, Complex surfaces]. □

In practice, we usually want to compute $\dim H^0(X, \mathcal{O}_X(D))$. So we need an additional tool to get rid of the higher cohomologies. The first such result is

Theorem 7.3.4 (Kodaira Vanishing)**.** *If E is an ample divisor on a smooth projective variety X of any dimension, then for $i > 0$ we have*

$$H^i(X, \mathcal{O}_X(K + E)) = 0$$

Amplness should be understood as a positivity condition. Indeed when $\dim X = 1$, E is ample if and only if $\deg E > 0$ In this case, the theorem follows easily from Serre duality, so it is true over any field. When $\dim X > 0$, this is known to be false in positive characteristic without further assumptions. As a corollary, we find

Corollary 7.3.5. *If X is a surface and $D = K + E$, with E ample,*

$$\dim H^0(\mathcal{O}_X(D)) = \frac{1}{2} D \cdot (D - K) + \chi(\mathcal{O}_S)$$

Let turn to applications to modular forms. Suppose $Y = Y(\Gamma) = \Gamma\backslash\mathbb{H}$ be a modular curve associated to group Γ acting freely (e.g $\Gamma(N), N \geq 3$). We can compactify to a smooth projective curve $X = X(\Gamma)$. Let $D = X - Y$ be the set points at infinity called cusps. We view this as a divisor with coefficients equal to 1.

Theorem 7.3.6. *The space weight $2k$ modular forms (resp. cusp forms) $M_{2k}(\Gamma)$ (resp. $S_{2k}(\Gamma)$) is isomorphic to $\Gamma(X, \mathcal{O}(kK + kD))$ (resp. $\Gamma(X, \mathcal{O}(kK + (k-1)D))$. In particular, $S_2(\Gamma) \cong \Gamma(X, \Omega^1_X)$*

Proof. Let $f(z) \in M_{2k}(\Gamma)$. Then $f(z)(dz)^{\otimes k}$ is a Γ-invariant holomorphic section of $(\Omega^1_{\mathbb{H}})^{\otimes k}$, so it descends to a holomorphic section of $(\Omega^1_{Y(\Gamma')})^{\otimes k}$. We have to check what happens near a cusp. We have a local coordinate $q = e^{2\pi i z/n}$. By assumption f can be expanded as $\sum_0^\infty a_m q^m$, with $a_0 = 0$ for a cusp form. We have $dz = (n/2\pi i) dq/q$. So

$$f(z)(dz)^{\otimes k} = \left(\frac{n}{2\pi i}\right)^k (a_0 q^{-k} + a_1 q^{1-k} + \ldots) dq^{\otimes k}$$

So the theorem follows.

□

Corollary 7.3.7. *Suppose that X has genus g with m cusps, then*

$$\dim S_{2k}(\Gamma) = \begin{cases} g & \text{if } k = 1 \\ (2k-1)(g-1) + (k-1)m & \text{if } k > 1 \end{cases}$$

Proof. The first case is an immediate consequence of the theorem. For the second, we use Riemann-Roch.

$$\begin{aligned} h^0(\mathcal{O}(kK + (k-1)D)) &= h^0(\mathcal{O}(kK + (k-1)D)) - h^0(\mathcal{O}((1-k)K - (k-1)D) \\ &= \deg(kK + (k-1)D) + 1 - g \end{aligned}$$

□

Now choose a real quadratic field K, and let $\Gamma \subset \Gamma_K = SL_2(\mathcal{O}_K)$ be torsion free congruence group. Let $Y = Y(\Gamma) = \Gamma\backslash\mathbb{H}^2$ be the associated Hilbert modular surface, and let $X = X(\Gamma)$ be the compactification given by adding cusps. Recall that X is singular. We can blow it up to get nonsingular surface $\tilde{X}$. The union of preimages of the cusps in X forms a divisor D. A Hilbert modular form of weight k is a holomorphic function $f : \mathbb{H}^2 \to \mathbb{C}$ satisfying

$$f\left(\frac{\sigma_1(a)z_1+\sigma_1(b)}{\sigma_1(c)z_1+\sigma_1(d)}, \frac{\sigma_2(a)z_2+\sigma_2(b)}{\sigma_2(c)z_1+\sigma_2(d)}\right) = (\sigma_1(c)z_1+\sigma_1(d))^k(\sigma_2(c)z_2+\sigma_2(d))^k f(z_1,z_2)$$

for every element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Applying this for $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ shows that f is periodic, and therefore it admits a Fourier expansion. One has similar expansions for each conjugate of P. The form f is a *cusp form* if the zeroth coefficients in each of these expansions is zero. In more geometric language, f can be interpreted as a section of line bundle over $\tilde{X}$. It is a cusp form if the corresponding section vanishes at the preimage of the cusps D. The precise statement is as follows:

Theorem 7.3.8. *The divisor $F = K + D$ is ample, and for any $m > 0$, we have an isomorphism of the space of Hilbert modular cusp forms of weight $2m$*

$$S_{2m}(\Gamma) \cong H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(mF - D)) = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(K + (m-1)F))$$

Details can be found in [Van der Geer, Hilbert Modular Surfaces].

From Kodaira vanishing plus Riemann-Roch, we obtain:

Corollary 7.3.9.

$$\dim S_{2m}(\Gamma) = \frac{m-1}{2}(K + (m-1)F) \cdot F + \chi(\mathcal{O}_{\tilde{X}})$$

The right side can be simplified further, see [Van der Geer].