Complex AG problem list

Below is a list of problems. You can pick one these to present, or you can make up your own problem (but check with me about that). See the last section for some vague suggestions.

0.1 Riemann surfaces

1. (a) Identify the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ with $\mathbb{P}^1_{\mathbb{C}}$ by $z \mapsto [z, 0]$ for $z \in \mathbb{C}$ and $\infty \mapsto [0, 1]$. Show that action of $GL_2(\mathbb{C})$ by Möbius or fractional linear transformations on the first space is compatible with the action of $[x, y] \mapsto [ax + by, cx + dy]$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$ on the second.

   (b) Let $SL_2(\mathbb{R})$ denote the group of 2×2 real matrices with det 1. Show that $SL_2(\mathbb{R})$ acts transitively on the upper half plane $\mathbb{H} = \{z \mid Im\, z > 0\}$ by Möbius or fractional linear transformations.

2. Continuing the notation from problem 1, let $\Gamma \subset SL_2(\mathbb{R})$ denote a discrete subgroup acting on $\mathbb{H}$ without fixed points ($g \neq 1 \Rightarrow gx \neq x$).

   (a) Show that $X = \mathbb{H}/\Gamma$ is a Riemann surface such that the quotient map $\mathbb{H} \to X$ is holomorphic,

   (b) Show that if α is a holomorphic 1-form on X then its pullback to $\mathbb{H}$ is given by $f(z)dz$, where $f : \mathbb{H} \to \mathbb{C}$ is a holomorphic function such that
   $$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 f(z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$
   $f(z)$ is called a weight 2 automorphic form.

3. Let X be a compact Riemann surface of genus g. A function on $X - S$, where $S \subset X$ is a finite set, is called meromorphic if it holomorphic and if the Laurent expansion with respect to any coordinate has a finite number of negative terms (i.e. it has no essential singularities). Let $\mathbb{C}(X)$ denote the field of meromorphic functions on X.

   (a) Show that elements of $\mathbb{C}(\mathbb{P}^1)$ are rational functions.

   (b) When X is an elliptic curve, show that $\mathbb{C}(X)$ is the field of elliptic functions.

(c) Given $x \in X$, let $ord_x(f) = n$, where n is the minimum exponent occurring in a Laurent expansion of f at x. Show that this is a discrete valuation on $\mathbb{C}(X)$ (see for example Atiyah-Macdonald for the definition of discrete valuation).

4. Continuing the set up from the previous problem, show that every discrete valuation on $\mathbb{C}(X)$ arrises as above in cases (a) and (b). (If you feel up to the challenge, you can try to prove this in general.)

5. (a) Let $f : X \to Y$ be a holomorphic map of compact Riemann surfaces of genus $g(X)$ and $g(Y)$ respectively. Suppose that there exists a finite set $R \subset Y$ such that the cardinality $|f^{-1}(y)| = 1$ for $y \in R$ and $|f^{-1}(y)| = d$ for $y \notin R$. One says that d is the degree of f, and f is totally ramified at points of R. Prove (a special case of) the *Riemann-Hurwitz formula* that

$$2g(X) - 2 = (2g(Y) - 2)d + (d - 1)|R|$$

Hint: One way to do this is to triangulate Y, and then apply Euler's formula (which you may assume) that the Euler characterisitic

$$2 - 2g(Y) = e(Y) = V - E + F$$

where V (resp. E, F) is the number vertices (resp. edges, triangles). Make sure to include R among the vertices of Y, now "lift" the triangulation to X, and apply Euler's formula again.

(b) Let $f : X \to X$ be an unramified map of a compact Riemann surface to itself of with degree at least 2. Then prove that X has genus 1 (use Riemann-Hurwitz). Give an example of such a map.

6. Let $n > 0$ be an integer. The Fermat curve of degree n is $X = \{[x, y, z] \in \mathbb{P}^2 \mid x^n + y^n + z^n = 0\}$. Check that X is nonsingular, and therefore that it defines a compact Riemann surface. Compute the genus of X by applying the (previous version) of the Riemann-Hurwitz formula to the projection $f : X \to \mathbb{P}^1$, $f([x, y, z]) = [x, y]$.

7. Let G be a finite group of automorphisms of compact Riemann surface X of genus g. Suppose that the action of G is fixed point free. Show that $Y = X/G$ can be made into a compact Riemann surface such that the quotient map $X \to Y$ is holomorphic. Conclude by Riemann-Hurwitz that the cardinality $|G|$ divides $g - 1$.

8. Given a compact Riemann surface X of genus $g > 0$, choose a basis $\omega_1, \ldots, \omega_g \in H^0(X, \Omega^1_X)$. Assume that for every $x \in X$, some ω_i is nonzero at x (in fact this is always true, so we don't need to assume it). Given a local coordinate z on a coordinate disk U, we can write $\omega_i = f_i(z)dz$. Show that the map $U \to \mathbb{P}^{g-1}$ defined by $z \mapsto [f_1(z), \ldots, f_g(z)]$ is independent of z. Therefore it defines a holomorphic map $X \to \mathbb{P}^{g-1}$ called the canonical map.

0.2 Abelian varieties and Jacobians

1. Let $A = \mathbb{C}^n/L$ and $B = \mathbb{C}^m/M$ be complex tori. A homomorphism $f : A \to B$ is holomorphic map induced by a linear map $F : \mathbb{C}^n \to \mathbb{C}^m$ such that $F(L) \subseteq M$. Given a homomorphism $f : A \to B$, show that the image and cokernl are both naturally complex tori. What about the kernel? (Consider multiplication by 2 from A to itself.)

2. Given an abelian vareity X, let $End(A)$ denote the ring of endomorphisms (homomorphisms from A to itself). For instance multiplication by an integer is an endomorphism, therefore $End(A) \supseteq \mathbb{Z}$. When $A = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ is an elliptic curve with $Im\,\tau > 0$, show that $End(A) = \mathbb{Z}$ unless τ is a quadratic algebraic number (i.e. it satisfies a quadratic equation with rational coefficients). Show the converse as well. Hint: an endomorphism is given a complex number such that

$$\alpha = a + b\tau, \quad \alpha = c + d\tau$$

   for integers a, b, c, d. Now eliminate α.

3. Let X be a compact Riemann surface. Show that any (holomorphic) automorphism of X will act on its Jacobian $A = J(X)$. Let X be the degree n Fermat curve defined above, conclude that $End(J(X)) \supseteq \mathbb{Z}[\zeta]$, where ζ is a primitive nth root of unity.

4. Let $f : X \to Y$ be a nonconstant holomorphic map of compact Riemann surfaces. Choose a base point $x_0 \in X$ and $y_0 = f(x_0)$. Show that there exists a homorphism $F : J(X) \to J(Y)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ J(X) & \xrightarrow{F} & J(Y) \end{array}$$

   where α, β are the Abel-Jacobi maps with respect to the base points.

5. Somewhat surprisingly the Jacobian is both covariant and contravariant. Let $f : X \to Y$ be a nonconstant holomorphic map of compact Riemann surfaces, and suppose for simplicity it is unramified. Given a divisor $D = \sum_i n_i y_i \in Div(Y)$, let $f^*D = \sum_i n_i \sum_{x \in f^{-1}(y_i)} x$. Show that f^* takes a principal divisors to principal divisors, and therefore it induces a homomorphism $f^* : Cl(Y) \to Cl(X)$. Show that there exists a homomorphism $f^* : J(Y) \to J(X)$ which is compatible with the map just defined on class groups.

0.3 Other topics

- Elliptic curves. We only really scratched the surface. There is a lot more to say. See for example books by Silverman.

- Algebraic curves. There is a lot that we haven't talked about, such as Riemann-Roch etc. See for example chap 4 of Hartshorne.
- Abelian varieties. Mumford, in his book, does the complex theory in about 50 pages, and redoes everything algebraically...
- Hodge theory, there is much more to the story that what we covered. See books of Griffiths-Harris or Voisin, or...