Modules over a soft sheaf

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This is something that came up in class last week, and some of you were very confused about it yesterday after class.

Let $\mathcal{R}$ be a soft sheaf of rings on a metric space X.

Lemma 1. *A sheaf of $\mathcal{R}$-modules $\mathcal{M}$ is automatically soft.*

Proof. Let $S \subset X$ be closed, and let $m \in \mathcal{M}(S)$. Then m is defined on some open nbhd U of S. Since X is a metric space, one can find nbhds $S \subset U' \subset \overline{U'} \subset U$, with U' open. Let $T = X - U'$ and $V = X - \overline{U'}$. Since S, T are disjoint closed sets, and $\mathcal{R}$ is soft, there exists $\rho \in \mathcal{R}(X)$ such that $\rho|_S = 1$ and $\rho|_T = 0$. Now consider $\rho \cdot m \in \mathcal{M}(U)$. Since $U \cap V \subset T$, $\rho \cdot m|_{U \cap V} = 0$. Therefore, we can patch this with $0 \in \mathcal{M}(V)$ to get a global section $\mathcal{M}$, which restricts to m on S.

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