Chapter 11

The sign of a permutation

Theorem 11.1. *Suppose $n \geq 2$.*

(a) Every permutation in S_n is a product of transpositions.

(b) If the identity $I = \tau_1 \dots \tau_r$ in S_n is expressed as product of transpositions, r must be even.

Before giving the proof, we need the following lemmas.

Lemma 11.2. *Suppose $a, b, c, d \in \{1, \dots, n\}$ are mutually distinct elements. We have the following identities among transpositions*

$$(ab) = (ba)$$

$$(ab)(ab) = I \tag{11.1}$$

$$(ac)(ab) = (ab)(bc) \tag{11.2}$$

$$(bc)(ab) = (ac)(cb) \tag{11.3}$$

$$(cd)(ab) = (ab)(cd) \tag{11.4}$$

Proof. The first couple are obvious, the rest will be left as an exercise. □

Lemma 11.3. *Any product of transpositions $\tau_1\tau_2 \dots \tau_r$, in S_n, is equal to another product of transpositions $\tau'_1 \dots \tau'_{r'}$, such that r and r' have the same parity (in other words, they are either both even or both odd) and n occurs at most once among the τ'_i.*

Proof. Rather than giving a formal proof, we explain the strategy. Use (11.3) and (11.4) to move transpositions containing n next to each other. Then apply (11.1) and (11.2) to eliminate one of the n's. In each of these moves, either r stays the same or drops by 2. Now repeat.

Here are a couple of examples when $n = 4$,

$$(43)(41)(24) = (41)(13)(24) = (41)(24)(13) = (24)(12)(13)$$

$$(34)(12)(34) = (34)(34)(12) = (12)$$

□

Proof of theorem 11.1. We prove both statements by induction on n. The base case $n = 2$ of (a) is clear, the only permutations are (12) and (12)(12). Now suppose that (a) holds for S_n. Let $f \in S_{n+1}$. If $f(n+1) = n+1$, then $f \in \mathrm{Stab}(n+1)$ which can be identified with S_n. So by induction, f is a product of transpositions. Now suppose that $j = f(n+1) \neq n+1$. Then the product $g = (n+1\, j)f$ sends $n+1$ to $n+1$. This implies that g is a product of transpositions $\tau_1\tau_2\ldots$ by the previous case. Therefore $f = (n+1\, j)\tau_1\tau_2\ldots$.

Statement (b) holds when $n = 2$, because $I = (12)^r$ if and only if r is even. Suppose that (b) holds for S_n. Let

$$I = \tau_1\tau_2\ldots\tau_r \tag{11.5}$$

in S_{n+1}. By using these lemma 11.3, we can get a new equation

$$I = \tau_1'\ldots\tau_{r'}' \tag{11.6}$$

where at most one of the τ_i''s contains $n+1$, and r' has the same parity as r. If exactly one of the τ_i''s contains $n+1$, then $\tau_1'\ldots\tau_{r'}'$ will send $n+1$ to a number other than $n+1$. This can't be the identity contradicting (11.6). Therefore none of the τ_i''s contains $n+1$. This means that (11.6) can be viewed as an equation in S_n. So by induction, we can conclude that r' is even.

□

Corollary 11.4. *If a permutation σ is expressible as a product of an even (respectively odd) number of transpositions, then* **any** *decomposition of σ as a product of transpositions has an even (respectively odd) number of transpositions.*

Proof. Write

$$\sigma = \tau_1\ldots\tau_r = \tau_1'\ldots\tau_{r'}'$$

where τ_i, τ_j' are transpositions. Therefore

$$I = \tau_r^{-1}\ldots\tau_1^{-1}\tau_1'\ldots\tau_{r'}' = \tau_r\ldots\tau_1\tau_1'\ldots\tau_{r'}'$$

which implies that $r + r'$ is even. This is possible only if r and r' have the same parity. □

Definition 11.5. *A permutation is called even (respectively odd) if it is a product of an even (respectively odd) number of transpositions. Define*

$$\mathrm{sign}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

Lemma 11.6. *The map* $\mathrm{sign} : S_n \to \{1, -1\}$ *is a homomorphism.*

Proof. Clearly $\mathrm{sign}(I) = 1$ and $\mathrm{sign}(\sigma\tau) = \mathrm{sign}(\sigma)\,\mathrm{sign}(\tau)$. □

Definition 11.7. *The alternating group $A_n \subset S_n$ is the subgroup of even permutations.*

Observe that A_n is a subgroup, and in fact a normal subgroup, because it equals ker(sign). We can identify S_n/A_n with $\{1, -1\}$. Therefore

Lemma 11.8. $|A_n| = \frac{1}{2}n!$.

Earlier as an exercise, we found that the symmetry group of dodecahedron had order 60, which is coincidentally the order of A_5. A more precise analysis, which we omit, shows that these groups are in fact isomorphic.

Let us apply these ideas to study functions of several variables. A function $f : X^n \to \mathbb{R}$ is called *symmetric* if

$$f(x_1, \ldots, x_i, \ldots, x_j, \ldots x_n) = f(x_1, \ldots, x_j, \ldots, x_i \ldots x_n)$$

and *antisymmetric* if

$$f(x_1, \ldots, x_i, \ldots, x_j, \ldots x_n) = -f(x_1, \ldots, x_j, \ldots, x_i, \ldots x_n)$$

for all $i \neq j$. For example, when $X = \mathbb{R}$

$$x_1 + x_2 + x_3$$

is symmetric, and

$$(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

is antisymmetric. Clearly when f is antisymmetric,

$$f(x_1, \ldots, x_n) = \text{sign}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

holds for any permutation. A similar equation holds for symmetric functions, with sign(σ) omitted. We define the symmetrization and antisymmetrization operators by

$$\text{Sym}(f) = \frac{1}{n!} \sum_{\sigma \in S_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

$$\text{Asym}(f) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sign}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

We'll see in the exercises that these operators produce (anti)symmetric functions.

11.9 Exercises

1. Check the identities in lemma 11.2.
2. Prove that if $\sigma \in S_n$ is odd, then so is σ^{-1}.
3. Prove that a cycle of length r is even if and only if r is odd.

4. Prove that if $G \subseteq S_n$ is a subgroup of odd order, then $G \subseteq A_n$.

5. Prove that $\mathrm{Sym}(f)$ (respectively $\mathrm{Asym}(f)$) is symmetric (respectively antisymmetric), and furthermore that $f = \mathrm{Sym}(f)$ ($f = \mathrm{Asym}(f)$) if and only if f symmetric (antisymmetric).

6. If f is symmetric, prove that $\mathrm{Asym}(f) = 0$.

7. Prove that
$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$
holds for all $\sigma \in A_n$ if and only if f is a sum of a symmetric and antisymmetric function.