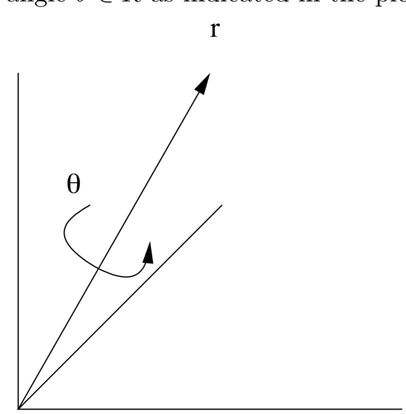
Chapter 13

The 3 dimensional rotation group

A rotation in space is a transformation $R : \mathbb{R}^3 \to \mathbb{R}^3$ determined by a unit vector $r \in \mathbb{R}^3$ and an angle $\theta \in R$ as indicated in the picture below.



A bit more precisely, the transformation $R = R(\theta, r)$ has the line through r as the axis, and the plane perpendicular to the line is rotated by the angle θ in the direction given by the right hand rule (the direction that the fingers of right hand point if the thumb points in the direction of r). R is a linear transformation, so it is represented by a matrix that we denote by the same symbol. It is invertible with inverse $R(-\theta, r)$. Therefore the set of rotations is a subset of $GL_3(\mathbb{R})$. We will show that it is a subgroup, and in particular that the product of two rotations is again a rotation. This is fairly obvious if the rotations share the same axis, but far from obvious in general. The trick is characterize the matrices that arise from rotations. Recall that a 3×3 matrix A is orthogonal if its columns are *orthonormal*, i.e. they unit vectors such that the dot product of any two is zero. This is equivalent to $A^T A = I$.

Lemma 13.1. *If A is orthogonal,* $\det A = \pm 1$.

Proof. From $A^TA = I$ we obtain $\det(A)^2 = \det(A^T)\det(A) = 1$. □

We already saw in the exercises to chapter 3 that the set of orthogonal matrices $O(3)$ forms a subgroup of $GL_3(\mathbb{R})$. Let $SO(3) = \{A \in O(3) \mid \det A = 1\}$.

Lemma 13.2. *$SO(3)$ is a subgroup of $O(3)$.*

Proof. If $A, B \in SO(3)$, then $\det(AB) = \det(A)\det(B) = 1$ and $\det(A^{-1}) = 1^{-1} = 1$. Also $\det(I) = 1$. □

Proposition 13.3. *Every rotation matrix lies in $SO(3)$.*

Proof. Given a unit vector $v_3 = r$ as above, fix $R = R(\theta, r)$. By Gram-Schmid we can find two more vectors, so v_1, v_2, v_3 is orthonormal. Therefore $A = [v_1v_2v_3]$ is an orthogonal matrix. After possibly switching v_1, v_2, we can assume that v_1, v_2, v_3 is right handed or equivalently that $\det A = 1$. Then

$$\begin{aligned}
R(v_1) &= \cos\theta v_1 + \sin\theta v_2 \\
R(v_2) &= -\sin\theta v_1 + \cos\theta v_2 \\
R(v_3) &= v_3
\end{aligned}$$

and therefore

$$RA = AM$$

where

$$M = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $M, A \in SO(3)$, it follows that $R = AMA^{-1} \in SO(3)$. □

In principle, the method of proof can be used to calculate $R(\theta, [a, b, c]^T)$ explicitly. In fact, I did find an expression with the help of a computer algebra package:

$$\begin{bmatrix} a^2 + \cos(\theta) - a^2\cos(\theta) & -c\sin(\theta) + ab - ab\cos(\theta) & ac - ac\cos(\theta) + b\sin(\theta) \\ ab - ab\cos(\theta) + c\sin(\theta) & b^2 + \cos(\theta) - b^2\cos(\theta) & -a\sin(\theta) + bc - bc\cos(\theta) \\ -b\sin(\theta) + ac - ac\cos(\theta) & bc - bc\cos(\theta) + a\sin(\theta) & -b^2 + b^2\cos(\theta) - a^2 + a^2\cos(\theta) + 1 \end{bmatrix}$$

However, the formula is pretty horrendous and essentially useless. We will see a better way to do calculations shortly (which is in fact what I used to calculate the previous matrix).

We want to prove that every matrix in $SO(3)$ is a rotation. We start by studying their eigenvalues. In general, a real matrix need not have any real eigenvalues. However, this will not be a problem in our case.

Lemma 13.4. *A 3×3 real matrix has a real eigenvalue.*

Proof. The characteristic polynomial $p(\lambda) = \lambda^3 + a_2\lambda^2 + \ldots$ has real coefficients. Since λ^3 grows faster than the other terms, $p(\lambda) > 0$ when $\lambda \gg 0$, and $p(\lambda) < 0$ when $\lambda \ll 0$. Therefore the graph of $y = p(x)$ must cross the x-axis somewhere, and this would give a real root of p. (This intuitive argument is justified by the intermediate value theorem from analysis.)

□

Lemma 13.5. *If $A \in O(3)$, 1 or -1 is an eigenvalue.*

Proof. By the previous lemma, there exists a nonzero vector $v = [x, y, z]^T \in \mathbb{R}^3$ and real number λ such that $Av = \lambda v$. Since a multiple of v will satisfy the same conditions, we can assume that the square of the length $v^T v = x^2 + y^2 + z^2 = 1$. It follows that

$$\lambda^2 = (\lambda v)^T(\lambda v) = (Av)^T(Av) = v^T A^T A v = v^T v = 1$$

□

Theorem 13.6. *A matrix in $SO(3)$ is a rotation.*

Proof. Let $R \in SO(3)$. By the previous lemma, ± 1 is an eigenvalue.

We divide the proof into two cases. First suppose that 1 is eigenvalue. Let v_3 be an eigenvector with eigenvalue 1. We can assume that v_3 is a unit vector. We can complete this to an orthonormal set v_1, v_2, v_3. The vectors v_1 and v_2 form a basis for the plane $v_3^\perp$ perpendicular to v_3. The matrix $A = [v_1, v_2, v_3]$ is orthogonal, and we can assume that it is in $SO(3)$ by switching v_1 and v_2 if necessary. It follows that

$$RA = [Rv_1, Rv_2, Rv_3] = [Rv_1, Rv_2, v_3]$$

remains orthogonal. Therefore Rv_1, Rv_2 lie in $v_3^\perp$. Thus we can write

$$\begin{aligned} R(v_1) &= av_1 + bv_2 \\ R(v_2) &= cv_1 + dv_2 \\ R(v_3) &= v_3 \end{aligned}$$

The matrix

$$A^{-1}RA = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

lies in $SO(3)$. It follows that the block $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ lies in $SO(2)$, which means that it is a plane rotation matrix $R(\theta)$. It follows that $R = R(\theta, v_3)$.

Now suppose that -1 is an eigenvalue and let v_3 be an eigenvector. Defining A as above, we can see that

$$A^{-1}RA = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

This time the upper 2×2 is block lies $O(2)$ with determinant -1. This implies that it is a reflection. This means that there is a nonzero vector v in the plane $v_3^\perp$ such $Rv = v$. Therefore R also $+1$ as an eigenvalue, and we have already shown that R is a rotation. □

From the proof, we extract the following useful fact.

Corollary 13.7. *Every matrix in $SO(3)$ has $+1$ as an eigenvalue. If the matrix is not the identity then the corresponding eigenvector is the axis of rotation.*

We excluded the identity above, because everything would be an axis of rotation for it. Let us summarize everything we've proved in one statement.

Theorem 13.8. *The set of rotations in $\mathbb{R}^3$ can be identified with $SO(3)$, and this forms a group.*

13.9 Exercises

1. Check that unlike $SO(2)$, $SO(3)$ is not abelian. (This could get messy, so choose the matrices with care.)

2. Given two rotations $R_i = R(\theta_i, v_i)$, show that the axis of $R_2 R_1 R_2^{-1}$ is $R_2 v_1$. Conclude that a normal subgroup of $SO(3)$, different from $\{I\}$, is infinite.

3. Check that
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
has $1, e^{\pm i\theta}$ as complex eigenvalues. With the help of the previous exercise show that this holds for any rotation $R(\theta, v)$.

4. Show the map $f : O(2) \to SO(3)$ defined by
$$f(A) = \begin{bmatrix} A & 0 \\ 0 & \det(A) \end{bmatrix}$$
is a one to one homomorphism. Therefore we can view $O(2)$ as a subgroup of $SO(3)$. Show that this subgroup is the subgroup $\{g \in SO(3) \mid gr = \pm r\}$, where $r = [0, 0, 1]^T$.

5. Two subgroups $H_i \subseteq G$ of a group are *conjugate* if for some $g \in G$, $H_2 = gH_1g^{-1} := \{ghg^{-1} \mid h \in H_1\}$. Prove that $H_1 \cong H_2$ if they are conjugate. Is the converse true?

6. Prove that for any nonzero vector $v \in \mathbb{R}^3$, the subgroup $\{g \in SO(3) \mid gv = \pm v\}$ (respectively $\{g \in SO(3) \mid gv = v\}$) is conjugate, and therefore isomorphic, to $O(2)$ (respectively $SO(2)$). (Hint: use the previous exercises.)