Chapter 14

Finite subgroups of the rotation group

At this point, it should it should come as no surprise that finite subgroups of the $O(2)$ are groups of a symmetries of a regular polygon. We prove a slightly more precise statement.

Theorem 14.1. *A finite subgroup of $SO(2)$ is cyclic, and a finite subgroup of $O(2)$ not contained in $SO(2)$ is dihedral.*

Recall that the dihedral group D_n is defined by generators and relations $R^n = I, F^2 = I$ and $FRF = R^{-1}$. We include the “degenerate” cases $D_1 \cong \mathbb{Z}_2$, and $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (see exercises).

Proof. First suppose that $G \subset SO(2)$ is finite. The elements of G are of course rotations through some angle $\theta \in [0, 2\pi)$. Let $R \in G - \{I\}$ be the rotation with the smallest possible θ. Let $S \in G - \{I\}$ be another element with angle ϕ. Since $\phi \geq \theta$, we can write $\phi = n\theta + \psi$, where $n \geq 0$ is an integer and $\psi \geq 0$. By choosing n as large as large as possible, we can assume that $\psi < \theta$. Since ψ is the angle of SR^{-n}, we must have $\psi = 0$. This proves that $S = R^n$. So G is generated by R, and therefore cyclic.

Now suppose that $G \subset O(2)$ is finite but not contained in $SO(2)$. Then there exists $F \in G$ with $\det F = -1$. This is necessarily a reflection so that $F^2 = I$. $G \cap SO(2)$ is cyclic with generator R by the previous paragraph. Let us suppose that R has order n. We have that $\det FR = -1$, so it is also a reflection. This means that $FRFR = I$ or $FRF = R^{-1}$. Together with the relations $F^2 = I$ and R^n, we see that $G \cong D_n$. □

Let us now turn to finite subgroups of $SO(3)$. Since $O(2) \subset SO(3)$, we have the above examples. We also have symmetry groups of a regular tetrahedron, cube or dodecahedron. Remarkably, the converse is also true. We will be content to prove a weaker statement.

Theorem 14.2. *Let $G \subset SO(3)$ be a finite subgroup. Then either G is cyclic, dihedral or else it has order $12, 24$ or 60.*

The proof will be broken down into a series of lemmas. Let us suppose that $G \subset SO(3)$ is a nontrivial finite subgroup. Then G acts on the sphere S of radius one centered at the origin. We define a point of S to be a *pole* of G if it is fixed by at least one $g \in G$ with $g \neq I$. Let P be the set of poles. For $g \neq I$, there are exactly two poles $\pm p$, where the axis of g meets S. It follows that P is a finite set with even cardinality. We will see in an exercise that G acts on P. So, we can partition P into a finite number, say n, of orbits. Choose one point p_i, in each orbit.

Lemma 14.3.

$$2\left(1-\frac{1}{|G|}\right) = \sum_{i=1}^{n}\left(1-\frac{1}{|\operatorname{Stab}(p_i)|}\right) \tag{14.1}$$

Proof. By Burnside's formula

$$n = \frac{1}{|G|}\sum_{g\in G} |\operatorname{Fix}(g)|$$

As noted above $|\operatorname{Fix}(g)| = 2$, when $g \neq I$. Therefore, with the help of the orbit-stabilizer theorem

$$\begin{aligned} n &= \frac{1}{|G|}\left(2(|G|-1)+|P|\right) \\ &= \frac{1}{|G|}\left(2(|G|-1)+\sum_{1}^{n}|\operatorname{Orb}(p_i)|\right) \\ &= \frac{1}{|G|}\left(2(|G|-1)+\sum_{1}^{n}\frac{|G|}{|\operatorname{Stab}(p_i)|}\right) \end{aligned}$$

This can be rearranged to get

$$2\left(1-\frac{1}{|G|}\right) = \sum_{1}^{n}\left(1-\frac{1}{|\operatorname{Stab}(p_i)|}\right)$$

□

Lemma 14.4. *With above notation, if $G \neq \{I\}$ then either $n = 2$ or 3 in (14.1).*

Proof. Since $|G| \geq 2$ and $|\operatorname{Stab}(p_i)| \geq 2$, we must have

$$1 \leq 2\left(1-\frac{1}{|G|}\right) < 2$$

and

$$\frac{n}{2} \leq \sum\left(1-\frac{1}{|\operatorname{Stab}(p_i)|}\right) < n$$

The only way for (14.1) to hold is for $n = 2, 3$. □

Lemma 14.5. *If $n = 2$, G is cyclic.*

Proof. Since $\mathrm{Stab}(p_i) \subseteq G$, we have

$$\left(1 - \frac{1}{|\,\mathrm{Stab}(p_i)|}\right) \leq \left(1 - \frac{1}{|G|}\right) \tag{14.2}$$

But (14.1) implies

$$2\left(1 - \frac{1}{|G|}\right) = \left(1 - \frac{1}{|\,\mathrm{Stab}(p_1)|}\right) + \left(1 - \frac{1}{|\,\mathrm{Stab}(p_2|}\right)$$

and this forces equality in (14.2) for both $i = 1, 2$. This implies that $G = \mathrm{Stab}(p_1) = \mathrm{Stab}(p_2)$. This means that $g \in G$ is a rotation with axis the line L connecting p_1 to 0 (or p_2 to 0, which would have to be the same). It follows that g would have to be a rotation in the plane perpendicular to L. So that G can be viewed as subgroup of $SO(2)$. Therefore it is cyclic by theorem 14.1. ☐

We now turn to the case $n = 3$. Let us set $n_i = |\,\mathrm{Stab}(p_i)|$ and arrange them in order $2 \leq n_1 \leq n_2 \leq n_3$. (14.1) becomes

$$2\left(1 - \frac{1}{|G|}\right) = \left(1 - \frac{1}{n_1}\right) + \left(1 - \frac{1}{n_2}\right) + \left(1 - \frac{1}{n_3}\right)$$

or

$$1 + \frac{2}{|G|} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}$$

The left side is greater than one, so we have a natural constraint.

Lemma 14.6. *The only integer solutions to the inequalities*

$$2 \leq n_1 \leq n_2 \leq n_3$$

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} > 1$$

are as listed together with the corresponding orders of G.

(a) $(2, 2, n_3)$ *and* $|G| = 2n_3$.

(b) $(2, 3, 3)$ *and* $|G| = 12$.

(c) $(2, 3, 4)$ *and* $|G| = 24$.

(d) $(2, 3, 5)$ *and* $|G| = 60$.

To complete the proof of theorem 14.2, we need the following

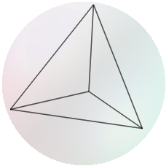
Lemma 14.7. *A subgroup $G \subset SO(3)$ corresponding to the triple $(2, 2, n)$ is isomorphic to D_n.*

Proof. We will deal with $n = 2$ in the exercises, so let us assume that $n > 2$. Let $H = \text{Stab}(p_3)$. This has order n. Note that $\text{Stab}(-p_3) = H$. We must have an element $F \in G$ which takes p_3 to $-p_3$, because otherwise we would have two orbits with stabilizers of order $n > 2$ contradicting our assumptions. Let $K \subseteq G$ be the subgroup generated by F and the elements of H. Let $k = |K|$. Then $k = qn$ with $q > 1$ because $H \subsetneqq K$. This forces $k = 2n$. Therefore $G = K$. This implies $G \subset \{g \in SO(3) \mid gp_3 = \pm p_3\} \cong O(2)$ (by a previous exercise). Theorem 14.1 implies that G is dihedral.

□

14.8 Exercises

1. Let D_2 be generated by F, R with relations $F^2 = R^2 = I$ and $FRF = R^{-1}$. Prove that this is abelian, and that the map $f : \mathbb{Z}_2 \times \mathbb{Z}_2 \to D_2$ given by $f(1, 0) = F$ and $f(0, 1) = R$ gives an isomorphism. This group is usually called the Klein four group.
2. Suppose that (G, e) is a group of order 4 such that every element satisfies $g^2 = e$. Prove that $D_2 \cong G$.
3. Let $G \subset SO(3)$ be a finite group and P be the set of poles. Show that if $p \in P$, and $g \in G$, then $gp \in P$.
4. Prove lemma 14.6.
5. Let $G \subset SO(3)$ be a subgroup corresponding to the triple $(2, 2, 2)$ in the sense of lemma 14.6. Prove that $G \cong D_2$.
6. Consider a regular tetrahedron inscribed in the unit sphere S.



Show that the set of poles P of the symmetry group T of the tetrahedron consists of the vertices, midpoints of edges and midpoints of faces extended to S. Show that the T action on P has three orbits, where one of them has a stabilizer of order 2 and the remaining two have stabilizers of order 3.

7. Determine the poles of the symmetry group of the cube, and determine the orbits and stabilizers as in the previous exercise.