Chapter 3

Rotations and reflections in the plane

We want another important source of nonabelian groups, which is one that most people should already be familiar. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \dots \end{bmatrix}$$

be an $n \times n$ matrix with entries in $\mathbb{R}$. If B is another $n \times n$ matrix, we can form their product $C = AB$ which is another $n \times n$ matrix with entries

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots a_{in}b_{nj} = \sum_k a_{ik}b_{kj}$$

The identity matrix

$$I = \begin{bmatrix} 1 & 0 & \dots \\ 0 & 1 & \dots \\ \dots \end{bmatrix}$$

has entries

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3.1. *Matrix multiplication is associative and I is the identity for it, i.e. $AI = IA = A$.*

Proof. Given matrices A, B, C, the ijth entries of $A(BC)$ and $(AB)C$ both work out to

$$\sum_k \sum_\ell a_{ik}b_{k\ell}c_{\ell j}$$

Also

$$a_{ij} = \sum_k a_{ik}\delta_{kj} = \sum_k \delta_{ik}a_{kj}$$

□

An $n \times n$ matrix A is *invertible* if there exists an $n \times n$ matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. It follows that:

Theorem 3.2. *The set of invertible $n \times n$ matrices with entries in $\mathbb{R}$ forms a group called the* general linear group *$GL_n(\mathbb{R})$.*

For 2×2 matrices there is a simple test for invertibility. We recall that the determinant

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

and

$$e \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea & eb \\ ec & ed \end{bmatrix}$$

Theorem 3.3. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be matrix over $\mathbb{R}$, then A is invertible if and only $\det(A) \neq 0$. In this case,*

$$A^{-1} = (\det(A))^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Proof. Let $\Delta = \det(A)$, and let $B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Then an easy calculation gives

$$AB = BA = \Delta I.$$

If $\Delta \neq 0$, then $\Delta^{-1}B$ will give the inverse of A by the above equation.

Suppose that $\Delta = 0$ and A^{-1} exists. Then multiply both sides of the above equation by A^{-1} to get $B = \Delta A^{-1} = 0$. This implies that $A = 0$, and therefore that $0 = AA^{-1} = I$. This is impossible. □

Let us study an important subgroup of this. A 2×2 rotation matrix is a matrix of the form

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This sends a column vector v in the plane $\mathbb{R}^2$ to the vector $R(\theta)v$ obtained by rotation through angle θ. We denote the set of these by $SO(2)$ (SO stands for special orthogonal).

Theorem 3.4. *$SO(2)$ forms a subgroup of $GL_2(\mathbb{R})$.*

Proof. It is easy to check that $\det R(\theta) = \cos^2\theta + \sin^2\theta = 1$ and of course, $R(0) = I \in SO(2)$. If we multiply two rotation matrices

$$\begin{aligned}
R(\theta)R(\phi) &= \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}\\
&= \begin{bmatrix}\cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi\\ \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi\end{bmatrix}\\
&= \begin{bmatrix}\cos(\theta+\phi) & -\sin(\theta+\phi)\\ \sin(\theta+\phi) & \cos(\theta+\phi)\end{bmatrix}\\
&= R(\theta+\phi)
\end{aligned}$$

Therefore $SO(2)$ is closed under multiplication. The last calculation also shows that $R(\theta)^{-1} = R(-\theta) \in SO(2)$ □

A matrix

$$A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

is called *orthogonal* if the columns are unit vectors $a^2 + c^2 = b^2 + d^2 = 1$ which are orthogonal in the sense that the dot product $ab + cd = 0$. Since the first column is on the unit circle, it can be written as $(\cos\theta, \sin\theta)^T$ (the symbol $(-)^T$, read *transpose*, turns a row into a column). The second column is on the intersection of the line perpendicular to the first column and the unit circle. This implies that the second column is $\pm(-\sin\theta, \cos\theta)^T$. So either $A = R(\theta)$ or

$$A = F(\theta) = \begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$$

In the exercises, you will find a pair of nonzero orthogonal vectors v_1, v_2, $F(\theta)v_1 = v_1$ and $F(\theta)v_2 = -v_2$. This means that $F(\theta)$ is a *reflection* about the line spanned by v_1. In the exercises, you will also prove that

Theorem 3.5. *The set of orthogonal matrices $O(2)$ forms a subgroup of $GL_2(\mathbb{R})$.*

Given a unit vector $v \in \mathbb{R}^2$ and $A \in O(2)$, Av is also a unit vector. So we can interpret $O(2)$ as the full symmetry group of the circle, including both rotations and reflections.

3.6 Exercises

1. Let $UT(2)$ be the set of upper triangular matrices

$$\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}$$

   Show this forms a subgroup of $GL_2(\mathbb{R})$.

2. Let $UT(3)$ be the set of upper triangular matrices

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

   Show this forms a subgroup of $GL_3(\mathbb{R})$.

3. Find a pair of nonzero orthogonal vectors v_1, v_2, $F(\theta)v_1 = v_1$ and $F(\theta)v_2 = -v_2$. (Hint: if $\theta = 0$ this is easy; when $\theta \neq 0$, try $v_1 = (\sin\theta, 1 - \cos\theta)^T$.)

4. Recall that the transpose of an $n \times n$ matrix A is the $n \times n$ matrix with entries a_{ji}. A matrix is called orthogonal if $A^TA = I = AA^T$ (the second equation is redundant but included for convenience).

   (a) Check that this definition of orthogonality agrees with the one we gave for 2×2 matrices.

   (b) Prove that the set of $n \times n$ orthogonal matrices $O(n)$ is a subgroup of $GL_n(\mathbb{R})$. You'll need to know that $(AB)^T = B^TA^T$.

5. Show that $SO(2)$ is abelian, but that $O(2)$ is not.

6. A 3×3 matrix is called a permutation matrix, if it can be obtained from the identity I by permuting the columns. Write $P(\sigma)$ for the permutation matrix corresponding to $\sigma \in S_3$. For example,

$$F = P\left((12)\right) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Check that $F^2 = I$. What can you conclude about the set $\{I, F\}$?

7. Prove that the set of permutations matrices in $GL_3(\mathbb{R})$ forms a subgroup. Prove the same thing for $GL_n(\mathbb{R})$, where permutations matrices are defined the same way. (The second part is not really harder than the first, depending how you approach it.)