Chapter 15

Nonabelian groups

Let's start with definition.

Definition 15.1. *A group consists of a set A with an associative operation $*$ and an element $e \in A$ satisfying*

$$a * e = e * a = a,$$

and such that for every element $a \in A$, there exists an element $a' \in A$ satisfying

$$a * a' = a' * a = e$$

A better title for this chapter would have been *not necessarily Abelian groups*, since Abelian groups are in fact groups.

Lemma 15.2. *An Abelian group is a group.*

Proof. The extra conditions $e * a = a * e$ and $a' * a = a * a'$ follow from the commutative law. □

Before giving more examples, let's generalize some facts about Abelian groups.

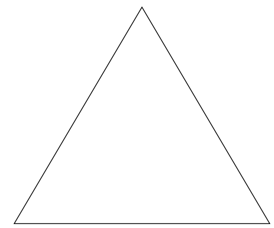
Lemma 15.3. *If $a * b = a * c$ then $b = c$. If $b * a = c * a$ then $b = c$.*

Corollary 15.4. *Given a, there is a unique element a', called the inverse such that $a * a' = a' * a = e$.*

We want give some examples of genuinely nonabelian groups. The next example should already be familiar from linear algebra class (where F is usually taken to be $\mathbb{R}$ or maybe $\mathbb{C}$). We'll say more about this example later on.

Example 15.5. *The set of $n \times n$ invertible (also known as nonsingular) matrices over a field F forms a group denoted by $GL_n(F)$ and called the $n \times n$ general linear group. The operation is matrix multiplication, and the identity element is the identity matrix When $n = 1$, this is just F^* which is Abelian. However, this group is not Abelian when $n > 1$.*

We want to consider a more elementary example next. Consider the equilateral triangle.



We want to consider various motions which takes the triangle to itself (changing vertices). We can do nothing I. We can rotate once counterclockwise.

$$R_+ : 1 \to 2 \to 3 \to 1.$$

We can rotate once clockwise

$$R_- : 1 \to 3 \to 2 \to 1.$$

We can also flip it in various ways

$$F_{12} : 1 \to 2, 2 \to 1, 3 \text{ fixed}$$

$$F_{13} : 1 \to 3, 3 \to 1, 2 \text{ fixed}$$

$$F_{23} : 2 \to 3, 3 \to 2, 1 \text{ fixed}$$

To multiply means to follow one motion by another. For example doing two R rotations takes 1 to 2 and then to 3 etc. So

$$R_+ R_+ = R_+^2 = R_-$$

Let's do two flips, F_{12} followed by F_{13} takes $1 \to 2 \to 2$, $2 \to 1 \to 3$, $3 \to 3 \to 1$, so

$$F_{12} F_{13} = R_+$$

Doing this the other way gives

$$F_{13} F_{12} = R_-$$

Therefore this multiplication is not commutative. The following will be proved in the next section.

Lemma 15.6. $\{I, R_+, R_-, F_{12}, F_{13}, F_{23}\}$ *is a group with* I *as the identity. It is called the triangle group.*

The full multiplication table can be worked out.

.	I	F_{12}	F_{13}	F_{23}	R_+	R_-
I	I	F_{12}	F_{13}	F_{23}	R_+	R_-
F_{12}	F_{12}	I	R_+	R_-	F_{13}	F_{23}
F_{13}	F_{13}	R_-	I	R_+	F_{23}	F_{12}
F_{23}	F_{23}	R_+	R_-	I	F_{12}	F_{13}
R_+	R_+	F_{23}	F_{12}	F_{13}	R_-	I
R_-	R_-	F_{13}	F_{23}	F_{12}	I	R_+

where each entry represents the product in the following order

.	a	b	...
a	$a \cdot a$	$a \cdot b$	...

This is latin square (chapter 3), but it isn't symmetric because the commutative law fails. Do all latin squares arise this way? The answer is no. For convenience, let's represent a latin square by an $n \times n$ matrix M with entries $1, \dots n$. Let's say that there is a group $G = \{g_1, \dots g_n\}$ with $g = e$ such that M has entry ijth entry k if $g_i * g_j = g_k$. Since $g_1 = e$, we would want the first row and column to be $1, 2, \dots n$. Such a latin square is called normalized. However, this is still not enough since the associative law needs to hold. I don't know any way to visualize this. Here's a Maple procedure for checking this.

```
assoc := proc(n::posint, M::matrix) i,j,k, associative;
associative := true;
for i from 1 to n do
  for j from 1 to n do
    for k from 1 to n do
      if (M[i,M[j,k]] <> M[M[i,j],k]) then
       associative := false;
       printf("g%d*(g%d*g%d)=g%d\n",i,j,k,M[i,M[j,k]]);
       printf("(g%d*g%d)*g%d=g%d\n\n",i,j,k,M[M[i,j],k]);
      fi;
    od;
  od;
od;
if associative then print("It is associative") fi;
end;
```

Typing $assoc(n, M)$ either tells you if M is associative, or else reports violations to the associative property.

15.7 Exercises

1. 1. Determine the inverse for every element of the triangle group.

2. Prove lemma 15.3.
3. Let $(G, *, e)$ be a group. Prove that the inverse of the product $(x * y)' = y' * x'$.
4. The commutator of x and y is the expression $x * y * x' * y'$. Prove that $x * y = y * x$ if and only if the commutator $x * y * x' * y' = e$.
5. Prove that the multiplication table for a group is always a latin square (see the proof of lemma 3.10 for hints).
6. Test to see if the normalized latin square corresponds to a group:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \\ 3 & 4 & 2 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 3 & 4 & 1 & 2 \end{bmatrix}$$