

Chapter 16

Groups of Permutations

We can generalize the example of the triangle group. A permutation of a finite set X is a one to one onto map $f : X \rightarrow X$. Write S_X for the set of such permutations. We will be mainly interested in $X = \{1, 2, \dots, n\}$. In this case, we denote the set by S_n . For examples of a permutation in S_4 , let

$$f(1) = 2, f(2) = 3, f(3) = 1, f(4) = 4$$

$$g(1) = 1, g(2) = 1, g(3) = 4, g(4) = 3$$

It may be helpful to visualize this:

$$f = \left\{ \begin{array}{l} 1 \rightarrow 2 \\ 2 \rightarrow 3 \\ 3 \rightarrow 1 \\ 4 \rightarrow 4 \end{array} \right. \quad g = \left\{ \begin{array}{l} 1 \rightarrow 2 \\ 2 \rightarrow 1 \\ 3 \rightarrow 4 \\ 4 \rightarrow 3 \end{array} \right.$$

Instead of standard functional notation, it'll be more convenient to place functions *to the right* of the argument, as in

$$1 \cdot f = 2$$

(Think of the way you would compute $\sin(x)$ on calculator: you first enter x and then press the sin key.) We get new permutations by composition, in other words following one by another:

$$1 \cdot fg = 2 \cdot g = 1, \dots$$

and by forming the inverse:

$$2 \cdot f^{-1} = 1$$

We can visualize this by splicing the pictures, or reversing them:

$$fg = \left\{ \begin{array}{l} 1 \rightarrow 2 \rightarrow 1 \\ 2 \rightarrow 3 \rightarrow 4 \\ 3 \rightarrow 1 \rightarrow 2 \\ 4 \rightarrow 4 \rightarrow 3 \end{array} \right. = \left\{ \begin{array}{l} 1 \rightarrow 1 \\ 2 \rightarrow 4 \\ 3 \rightarrow 2 \\ 4 \rightarrow 3 \end{array} \right.$$

$$f^{-1} = \begin{cases} 2 & \rightarrow 1 \\ 3 & \rightarrow 2 \\ 1 & \rightarrow 3 \\ 4 & \rightarrow 4 \end{cases}$$

Define the identity function e by $i \cdot e = i$ for each i .

Lemma 16.1. S_n forms a group under composition with e as the identity. The order (i.e. number of elements) of this group is $n!$.

Proof. Let $f, g, h \in S_n$ and $i \in \{1, \dots, n\}$. Then

$$i \cdot f(gh) = (i \cdot f) \cdot gh = ((i \cdot f) \cdot g) \cdot h = (i \cdot fg) \cdot hi \cdot (fg)h$$

This proves the associative law $f(gh) = (fg)h$.

$$i \cdot fe = (i \cdot f) \cdot e = i \cdot f$$

$$i \cdot ef = (i \cdot e) \cdot f = i \cdot f$$

proves that $fe = ef = f$.

Let f^{-1} denote the inverse function. Then

$$i \cdot (ff^{-1}) = (i \cdot f) \cdot f^{-1} = i = i \cdot e$$

$$i \cdot (f^{-1}f) = (i \cdot f^{-1}) \cdot f = i = i \cdot e$$

implies that $ff^{-1}f^{-1}f = e$.

The last statement is a standard counting argument. Given a permutation $f \in S_n$, there are n choices for $1 \cdot f$, $n - 1$ choices for $2 \cdot f$... Leading to $n(n - 1) \dots 1$ choices for f . \square

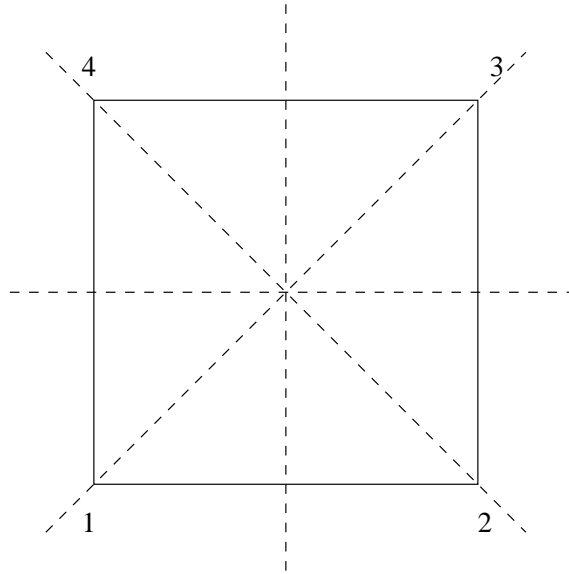
Since the triangle group can be identified with S_3 (exercise), this proves that it's a group. We get more examples of groups by generalizing the definition from chapter 7.

Definition 16.2. A subset H of a group $(G, *, e)$ is a subgroup if

1. $e \in B$.
2. B is closed under $*$.
3. B is closed under inversion.

A subgroup of a group is also group.

Example 16.3. The symmetry group D_4 of the square



is the subgroup of permutations of the vertices containing all rotations (e.g. $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$) and flips about the dotted lines. (e. g. the vertical flip is $1 \rightarrow 2, 2 \rightarrow 1, 3 \rightarrow 4, 4 \rightarrow 3$).

The notation we have been using for writing down permutations is very cumbersome. The most efficient notation is *cycle* notation which is based on the following:

Lemma 16.4. Let $p \in S_n$, then for $i \in \{1, \dots, n\}$, the sequence

$$i \cdot p, i \cdot p \cdot p, \dots$$

must contain i .

The pattern $i \rightarrow i \cdot p \rightarrow i \cdot p \cdot p \rightarrow \dots i$ is called a cycle, and it's denoted by $(i \ i \cdot p \dots)$. Start with 1, write down the cycle containing it. Pick the next number, say k , not in the previous cycle write down the corresponding cycle. Repeat until all numbers are accounted for. In cycle notation:

$$p = (1 \ i \cdot p \dots)(k \ k \cdot p \dots) \dots,$$

although normally one omits cycles of length one. In the previous examples,

$$f = (123), g = (12)(34), fg = (243)$$

It's possible to multiply permutations in Maple. First you have to tell it to load the group theory package:

```
> with(group):
```

Permutations are entered in Maple's version of cycle notation:

```
> f := [[1,2,3]]; g := [[1,2],[3,4]];
```

$$f := [[1, 2, 3]]$$

$$g := [[1, 2], [3, 4]]$$

> mulperms(f,g);

$$[[2, 4, 3]]$$

The identity would be denoted by $[]$ and the inverse is computed by the command *invperm*.

16.5 Exercises

1. Check that the triangle groups coincides with S_3 . $R_+ = (123)$ and $F_{12} = (12)$. Check that $F_{12}R_+F_{12} = R_+^2$.
2. Write down all the elements of D_4 . Is this an abelian group?
3. Write down all elements of T . Is this an abelian group?
4. Let $c = (a_1a_2 \dots a_k)$ be a cycle of length k in S_n . Prove that $c^k = e$.

Chapter 17

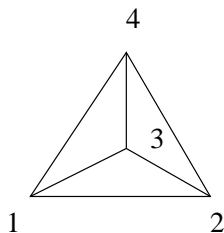
Symmetries of Platonic Solids

There five polyhedra with perfect symmetry. These are the Platonic solids. The last book of Euclid is devoted to their study.

Perfect symmetry means that it is possible to rotate any vertex to any other vertex. Group theory enters at this point. The symmetry group G of a polyhedron is the group of rotations which takes takes the polyhedron to itself. We will view it as a subgroup of the permutation group of vertices which will be labelled $1, 2, 3, \dots$. We can capture the notion of perfect symmetry by the following:

Definition 17.1. *A subgroup $G \subseteq S_n$ is called transitive if for each pair $i, j \in \{1, \dots, n\}$, there exists $f \in G$ such that $i \cdot f = j$.*

Let us analyse the symmetry of the tetrahedron



which is the simplest Platonic solid.

Let's try and list the elements. There is the identity I . We have two rotations which turning the base and keeping 4 fixed:

$$(123), (132)$$

There are six more rotations which keeps vertices 1, 2 and 3 fixed:

$$(234), (243), (134), (143), (124), (142)$$

But this isn't all. Since T is a group, we need to include products. For example,

$$(13)(24) = (123)(234)$$

We can do these by hand, but instead we get Maple 8 to produce all possible products of these elements (actually, it suffices to use $(123), (234)$.)

```
> P := permgroup(4, {[[1,2,3]], [[2,3,4]]}):
> elements(P);
```

The output is

```
{[], [[1, 3, 4]], [[1, 2, 3]], [[1, 2, 4]], [[1, 3, 2]], [[1, 4], [2, 3]], [[1, 2],
[3, 4]], [[1, 3], [2, 4]], [[2, 3, 4]], [[1, 4, 2]], [[1, 4, 3]], [[2, 4, 3]]}
```

Certainly, $P \subseteq T$. We want to show these are same. We need a method for computing the number of elements, or order, of T , in advance. First, we make a definitions.

Definition 17.2. *Given subgroup $G \subseteq S_n$ and $i \in \{1, \dots, n\}$, the stabilizer of i , is $\{f \in G \mid i \cdot f = i\}$*

The stablizer of 4 for T is the set

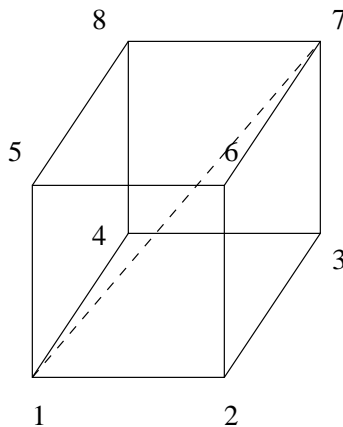
$$\{I, (123), (132)\}$$

with 3 elements.

Theorem 17.3. *Given a transitive subgroup $G \subseteq S_n$, let H be the stabilizer of some element i , then $|G| = n|H|$.*

The proof will be postponed until chapter 19. As a corollary, we get $|T| = (4)(3) = 12$.

Next, consider the cube



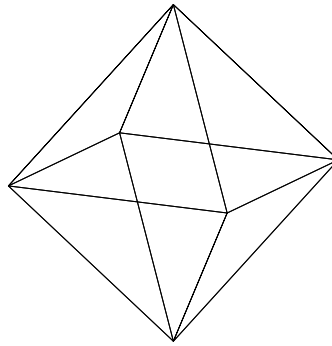
The symmetry group C is the group which takes the cube to itself. This can be viewed as a subgroup of S_8 . We need to calculate the stabilizer of 1. Aside from the identity, the only rotations which keep 1 fixed are those with the line joining 1 and 7 as its axis. Thus the stabilizer consists of

$$\{I, (254)(368), (245)(386)\}$$

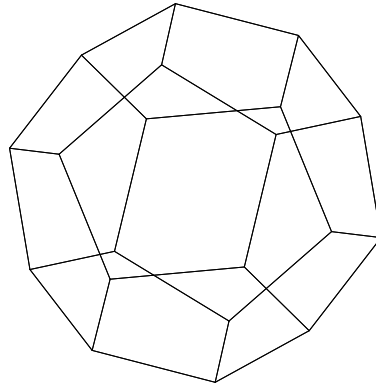
Therefore $|C| = (3)(8) = 24$

17.4 Exercises

1. Show that T is not abelian.
2. Calculate the order of the symmetry group for the octahedron



3. Calculate the order of the symmetry group for the dodecahedron



(There are 20 vertices, and 12 pentagonal faces.)

4. Let $G \subset S_n$ be a subgroup. Prove that the stabilizer H of an element i is a subgroup of G .

Chapter 18

Counting Problems involving Symmetry

Group theory can be applied to counting problems involving symmetry. Here are a few such problems.

Example 18.1. *How many dice can be constructed by labeling the face of a cube by the numbers $1, \dots, 6$?*

Example 18.2. *Suppose 3 identical decks of 52 cards are combined into a big deck. How many 3 card hands can be dealt out of the big deck?*

Example 18.3. *How many ways can a necklace be constructed with 2 black and 2 white beads?*

To analyze the first problem, let's first keep track of the order in which cards are dealt. Let's suppose that the kinds of cards are labeled by numbers $1, \dots, 52$. Since we have three decks we can be confident about not running out of kinds. Thus the set of ordered hands can be identified with

$$H = \{(a, b, c) \mid a, b, c = 1, 2 \dots 52\}$$

The cardinality of this set is 52^3 . Of course, we want to disregard order. As first guess, we might think that we should divide this by the number of ways of permuting the cards to get $52^3/6$. Unfortunately, this is not an integer so it doesn't make sense. To explain the correct answer, we will introduce some more group theory.

Definition 18.4. *We say that a group $(G, *, e)$ acts on a set X if there is an operation $\cdot : X \times G \rightarrow X$ satisfying:*

1. $x \cdot e = x$.
2. $x \cdot (g * h) = (x \cdot g) \cdot h$

(We're are really defining a right action here, there is also a notion of left action, but we won't need that.)

Definition 18.5. *Given a group G acting on a set X , the orbit of $x \in X$ is the set $x \cdot G = \{x \cdot g \mid g \in G\}$. The set of orbits is denoted by X/G .*

Definition 18.6. *An element x is called a fixed point of g is $x \cdot g = x$.*

Returning to example 18.2. S_3 acts on H by moving the positions of the cards. For example,

$$f = \begin{cases} 1 & \rightarrow 2 \\ 2 & \rightarrow 3 \\ 3 & \rightarrow 1 \end{cases}$$

then $(a_1, a_2, a_3) \cdot f = (a_2, a_3, a_1)$. We want to treat two hands as the same if you can permute one to get the other, i.e. if they lie in the same orbit. Therefore our problem is to count the set of orbits H/S_3 . If the first two cards are repeated, so $a_1 = a_2$, then (a_1, a_2, a_3) is a fixed point for (12).

Theorem 18.7 (Burnside). *If G is a finite group acting on a finite set X , then*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} (\# \text{ of fixed points of } g)$$

This will be proved in the next chapter.

Corollary 18.8. *Suppose that every element other than the identity has no fixed points, then $|X/G| = |X|/|G|$.*

Now, we can solve the problems mentioned earlier. For problem 18.1, we first choose an initial labelling, or marking, of our blank cube. This allows us to talk about the first face, second face and so on. Let X be the set of labellings of the faces of this marked cube. There are 6 choices for the first face, 5 for the second..., therefore there are $6! = 720$ elements of X . $G = C$ is the symmetry group for the cube which has order 24. G acts by rotating the cube. Clearly there are no fixed points for any rotation (other than I). Therefore the number of labellings for a *blank* cube is

$$|X/G| = \frac{720}{24} = 30$$

Next consider problem 18.2. Let H_{ij} be the set of ordered triples (a_1, a_2, a_3) with $a_i = a_j$. This is the set of fix points of (ij) . Since there only two free choices, we have

$$|H_{ij}| = 52^2$$

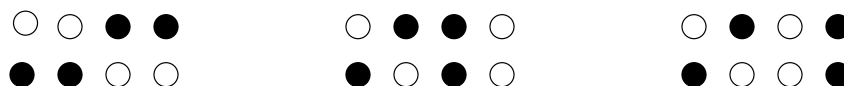
Let H_{123} be the set of triples where all the cards are the same. This is the set of fixed points for (123) and (132). Clearly

$$|H_{123}| = 52$$

Therefore

$$\begin{aligned}
 H/S_3 &= \frac{1}{6}[|H| + |H_{12}| + |H_{13}| + |H_{23}| + |H_{123}| + |H_{123}|] \\
 &= \frac{1}{6}[52^3 + 3(52^2) + 2(52)] \\
 &= 24804
 \end{aligned}$$

Finally, consider problem 18.3. Let us first arrange the beads in sequence. There are 6 possibilities



and we let X be the set of these. The group in this problem is the symmetry group of the square $D_4 \subset S_4$ which permutes the positions of the beads. All of these are fixed by the identity. The rotations (1234) and (1432) have no fixed points. The element $(12)(34)$ has two fixed points, namely the leftmost sequences in the above diagram. The remaining 4 elements also have two fixed points a piece (exercise). Therefore

$$|X/D_4| = \frac{1}{8}[6 + 5(2)] = 2$$

18.9 Exercises

1. Complete the analysis of problem 18.3.
2. How many necklaces can be constructed using 4 different colored beads?
3. In how many ways can the sides of a tetrahedron be labelled by the numbers 1, 2, 3, 4?
4. Suppose 2 identical decks of 52 cards are combined into a big deck. How many 3 card hands can be dealt out of the big deck?