Homological Algebra

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April 1, 2020

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Chapter 1

Some module theory

Basic Refs for this chapter.

1. Atiyah, Macdonald, Intro to commutative algebra
2. Harris, Algebraic geometry. A first course.
3. Rotman, Introduction to homological algebra.

1.1 Modules

We with work with not necessarily commutative rings, always with 1. There are many important examples which aren't commutative; matrix rings for example, and the following:

Example 1.2. *Let G be a group. The integral group ring $\mathbb{Z}G$ is the set of finite formal linear combinations $\sum_{g\in G} n_g g$, $n_g \in \mathbb{Z}$. The addition is obvious. The multiplication is*

$$(\sum n_g g)(\sum m_h h) = \sum_{g,h} n_g m_h gh$$

This is not commutative unless G is abelian. RG for any commutative ring R is defined the same way.

Let R be a ring. Since it may not be commutative, we have to be careful to distinguish left and right modules. Left R-module is an abelian group M with a multiplication $R \times M \to M$ satisfying

$$1 \cdot m = m$$

$$(r_1 r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$$

$$(r_1 + r_2) \cdot m = r_1 m + r_2 m$$

$$r \cdot (m_1 + m_2) = rm_1 + rm_2$$

A right module is an abelian group N with a multiplication $N \times R \to N$ satisfying a similar list of conditions. The opposite ring R^{op} is R as an additive group with multiplication reversed. We have $R \cong R^{op}$ if R is commutative, but not in general. A right module R is the same thing as a left R^{op}-module. Thus we may as well work with left modules, henceforth called modules, although there are few situations where it is convenient to work with right modules as well.

Example 1.3. *A left (right) $\mathbb{Z}G$ module is the same thing as an abelian group with a left (right) action by G.*

A homomorphism of R-modules $f : M \to N$ is an abelian group homomorphism satisfying $f(rm) = rf(m)$. The collection of R-modules and homomorphisms forms a category Mod_R (NB: Rotman denotes this by ${}_RMod$; if we want to consider right modules in these notes, we use $Mod_{R^{op}}$.) Let $Hom_R(M,N)$ be the set of homomorphisms. Since the sum of homomorphisms is a homomorphisms, this is an abelian group. However, in the noncommutative case it is not an R-module. It can be made into a module when M or N have additional structure. Given another ring S, an (R,S)-bimodule N is an abelian group with a left R-module structure and a right S-module structure such that $(rm)s = r(ms)$. Equivalently, N is left $R \times S^{op}$-module. In this case, $Hom_R(M,N)$ is a right S-module by $fs(m) = f(m)s$. Similarly when M is a (R,T)-bimodule, $Hom_R(M,N)$ is a left T-module.

Given a homomorphism $f : M \to N$, we get an induced homomorphisms

$$f_* : Hom_R(X,M) \to Hom_R(X,N)$$

$$f^* : Hom_R(N,Y) \to Hom_R(M,Y)$$

by $f_*(g) = f \circ g$ and $f^*(h) = h \circ f$. This maps $Hom(X.-)$ (resp. $Hom(-,Y)$) into a covariant (resp. contravariant) functor from $Mod_R \to Ab$ (cat. of abelian groups).

Recall that sequence of modules

$$\ldots L \xrightarrow{f} M \xrightarrow{g} N \ldots$$

is exact if $\ker g = \operatorname{im} g$ etc.

Theorem 1.4. *If*

$$0 \to L \to M \to N \to 0$$

is exact, then

$$0 \to Hom(X,L) \to Hom(X,M) \to Hom(X,N)$$

and

$$0 \to Hom(N,Y) \to Hom(M,Y) \to Hom(L,Y))$$

are exact.

Proof. In class, or see Rotman, or better yet, check yourself. $\square$

The theorem says that $Hom(X, -)$ and $Hom(-, Y)$ are left exact functors. In general, these are not exact:

Example 1.5. *If $R = \mathbb{Z}$, consider*

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

Applying $Hom(\mathbb{Z}/2, -)$ yields

$$0 \to 0 \to 0 \to \mathbb{Z}/2$$

and $Hom(-, \mathbb{Z}/2)$ yields

$$0 \to \mathbb{Z}/2 \xrightarrow{=} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2$$

The final maps are not surjective in either case.

1.6 Projective modules

An R-module P is called projective if $Hom_R(P, -)$ is exact. More explicitly, this means given a diagram with solid arrows

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

We can find a not necessarily unique dotted arrow making this commute.

A module M is free if it isomorphic to a possibly infinite direct sum $\bigoplus_I R$. Equivalently M has a basis (which is a generating set with no relations). A map of a basis to any module extends, uniquely, to a homomorphism of the free module.

Lemma 1.7. *A free module is projective.*

Proof. Suppose that F is free with basis e_i. Given a diagram

$$\begin{tikzcd} & F \arrow[dl, dashed, "\tilde{f}"'] \arrow[d, "f"] & \\ M \arrow[r, "p"] & N \arrow[r] & 0 \end{tikzcd}$$

choose $m_i \in M$ such that $p(m_i) = f(e_i)$. Then $e_i \mapsto m_i$ extends to the dotted homomorphism. $\square$

Lemma 1.8. *If P is projective, then given any surjective homomorphism $f : M \to P$, there is a splitting i.e. a homomorphism $s : P \to M$ such that $f \circ s = id$.*

Proof. Use

$$\begin{tikzcd} & P \arrow[dl, dashed, "s"'] \arrow[d, "id"] & \\ M \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$

□

Theorem 1.9. *P is (finitely generated and) projective iff it is a direct summand of a (finitely generated) free module F, i.e. there exists K such that $F \cong P \oplus K$.*

Proof. Suppose that P is projective. We can choose a surjection $\pi : F \to P$, with F free. By the lemma, we have splitting $s : P \to F$. Note that s is injective, so $P \cong s(P)$. One checks that $F = \ker \pi \oplus s(P)$.

Suppose that $F = P \oplus F$ is free. Given

$$\begin{tikzcd} & P \arrow[dl, dashed, "\tilde{g}"'] \arrow[d, "g"] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

we can extend g to $f : F \to N$ by $f = g \oplus 0$. Since F is projective, we have a lift $\tilde{f} : F \to M$. So $\tilde{g} = \tilde{f}|_P$ will fill in the above diagrem.

If P is finitely generated then F can be chosen to be finitely generated, and visa versa.

□

Now let's assume basic constructions/facts about commutative rings, including localization and Nakayama's lemma, which can be found in Atiyah-Macdonald.

Theorem 1.10. *A finitely generated projective module over commutative noetherian local ring is free.*

Proof. Let (R, m) be a comm. noeth. local ring, and P a fin. gen. projective R-module. Let $k = R/m$ be the residue field, and let $n = \dim P \otimes_R k$. Choose a set of elements $p_1, \ldots, p_n \in P$ reducing to a basis of $P \otimes k$. By Nakayama's lemma p_i spans P. Therefore we have a surjection $f : R^n \to P$, sending $e_i \to p_i$. Let $K = \ker f$. Arguing as above, we see that

$$R^n = P \oplus K$$

We necessarily have $K \otimes k = 0$, so $K = 0$ by Nakayama.

□

Corollary 1.11. *If P is a fin. gen. projective module over commutative noetherian ring, then it is locally free, i.e. P_p is free for every $p \in \operatorname{Spec} R$.*

We will see the converse later.

1.12 Projective modules versus free modules

There exists projective modules which are not free. Here is a cheap class of examples.

Example 1.13. *Let R_1, R_2 be nontrivial rings, and let $n, m > 0$ be unequal integers. Set $R = R_1 \times R_2$. Then*

$$P_{nm} = R_1^n \times R_2^m$$

is an R module. It is projective because $P_{nm} \oplus P_{mn} = R^{n+m}$. However, it is not free. When R_i are commutative, we can argue as follows. If $p \in \operatorname{Spec} R_1 \subset \operatorname{Spec} R$, then $P_{nm,p} = R_p^n$, while the localization at $p \in \operatorname{Spec} R_2$ is R_p^m. A free module would have the same rank at each prime.

This sort of example is impossible if R is commutative with $\operatorname{Spec} R$ connected. Nevertheless other examples exist in such cases. Let us assume that R is commutative noetherian and that finitely generated projective modules are the same as locally free modules. If R is Dedekind domain with nontrivial class group (see Atiyah-Macdonald the definition), then we can find an ideal $I \subset R$ which is not principal. I would not be free, although it would be locally free because the localizations are PIDs.

Here we outline an important class of examples assuming a bit of algebraic geometry (see Harris). Let $f(x_1, \ldots, x_n)$ be a nonzero polynomial over an algebraically closed field k. Let $X = V(f) = \{a \in k^n \mid f(a) = 0\}$ be the hypersurface defined by f. The coordinate ring is $R = k[x_1, \ldots, x_n]/(f)$. X can be identified with the maximal ideal spectrum of R by the Nullstellensatz. Let us assume that X is smooth, which means that the gradient $(\frac{\partial f}{\partial x_i})$ is never zero on X. It follows that $U_i = X - V(\frac{\partial f}{\partial x_i}))$ is an open cover of X in the Zariski topology. The module of vector fields $T = \{(g_1, \ldots, g_n) \in R^n \mid \sum \frac{\partial f}{\partial x_i} g_i = 0\}$ The localization $T_{\partial f/\partial x_i}$ is free over $R_{\partial f/\partial x_i}$ with basis

$$(0, \ldots 0, \underbrace{-\frac{\partial f}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)^{-1}}_{i\text{th place}}, 0, \ldots, \underbrace{1}_{j\text{th place}}, \ldots 0), \quad j \neq i$$

It follows that T is locally free. For suitable f, one can show that T is not free by showing that the tangent bundle is nontrivial. It may be worth spelling out the dictionary

Algebra	Geometry
Projective module	Vector bundle
Free module	Trivial vector bundle

Topological vector bundles on affine space (over $\mathbb{C}$) are trivial because it is contractible. Using this analogy Serre conjectured that projective modules over polynomial rings were free. This was solved affirmatively by Quillen and Suslin

Theorem 1.14 (Quillen-Suslin)**.** *If k is a field, projective modules over $k[x_1, \ldots, x_n]$ are trivial.*

An account can be found in Rotman Section 4.8.

1.15 Injective modules

A module E is injective if $Hom(-, E)$ is exact. Equivalently given the solid diagram

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & M \arrow[u] \arrow[r] & N \arrow[ul, dashed] \end{tikzcd}$$

it can be filled in as indicated. Although the notion is dual to projectivity, it is harder to characterize. We only succeed in a special case.

Theorem 1.16 (Baer's criterion)**.** *E is injective if and only if the above property holds when $N = R$ and $M = I$ is a left ideal.*

Proof. Suppose that the extension property for an ideal. Given

$$\begin{tikzcd} E & \\ M \arrow[u, "g"] \arrow[r, "\subseteq"] & N \arrow[ul, dashed] \end{tikzcd}$$

we have to construct an extension as indicated. Let $M \subseteq M' \subseteq N$ with an extension $g' : M' \to E$ of g. We can assume this is maximal by Zorn's lemma. Suppose that $M' \neq N$. Choose $y \in N$, $y \notin M'$. Define

$$I = \{r \in R \mid ry \in M'\}$$

This is a left ideal. Let $h : I \to E$ be given by $h(r) = g'(ry)$. Then by assumption, we have an extension $\tilde{h} : R \to E$. Let $M'' = M + Ry$. The map $g'' : M'' \to E$ given by

$$g''(x + ry) = g'(x) + \tilde{h}(r), x \in M'$$

can be seen to be well defined (see Rotman pp 118-119). It extends g'. However, this contradicts the maximality of (M', g').

$\square$

Theorem 1.17. *If R is commutative integral domain, an injective module E is divisible, i.e. given $x \in E$, $r \in R$, $\exists y \in E, ry = x$. The converse holds if R is a PID.*

Proof. Since R is a domain $(r) \cong R$. Therefore $h : (r) \to E$ given by $h(r) = x$ is well defined. We have an extension $\tilde{h} : R \to E$. Then $y = \tilde{h}(1)$ satisfies $ry = x$.

Suppose that R is a PID and E is divisible. We have to check Baer's criterion. Any ideal $I = (r)$ for some $r \in R$. The above argument can be reversed to show that any $h : (r) \to E$ extends to $R \to E$. $\square$

Corollary 1.18. $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}, \mathbb{R}, \ldots$ *are injective* $\mathbb{Z}$*-modules.*

Given an abelian group A, the character group

$$A^* = Hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$$

This is divisible, and therefore injective as a $\mathbb{Z}$-module. We have a canonical map $A \to A^{**}$ given by sending a to

$$\hat{a}(f) = f(a)$$

Proposition 1.19. *If* $A \neq 0$*, then* $A^* \neq 0$*. The map* $A \to A^{**}$ *is injective.*

Proof. In general, if $a \in A$ is nonzero, let A_0 be the subgroup generated by a. Since $\mathbb{Q}/\mathbb{Z}$ has elements of arbitrary finite order, $A_0^* \neq 0$. Since $\mathbb{Q}/\mathbb{Z}$ is injective, the map

$$Hom(A, \mathbb{Q}/\mathbb{Z}) \to Hom(A_0, \mathbb{Q}/\mathbb{Z})$$

is surjective.

For the second statement, it is enough to observe that given $a \neq 0$, there exists $f \in A^*$ with $\hat{a}(f) = f(a) \neq 0$ by the previous argument. $\square$

Corollary 1.20. *Any abelian group embeds into an injective abelian group.*

As we'll see below, this holds more generally for similar reasons.

1.21 Tensor products

If you are familiar with tensor products over a commutative ring, then there are few necessary modifications to make things work in general.

- Tensor products only makes sense between right and left modules.
- The tensor product is only an abelian group in general.

Here is the precise statement.

Theorem/Def 1.22. *If* M *is a right* R*-module, and* N *a left* R*-module, there exists an abelian group and biadditive operation*

$$\otimes : M \times N \to M \otimes_R N$$

satisfying $mr \otimes n = m \otimes rn$*. Furthermore, this is the universal such object.*

See Rotman, Section 2.2, for the construction and precise explanation of the last part. The construction shows that elements of $M \otimes_R N$ are finite sums $\sum m_i \otimes n_i$. If M is an (T,R) bimodule, and N a (R,S) bimodule, then $M \otimes_R N$ is an (T,S)-bimodule satisfying $t(m \otimes n)s = tm \otimes ns$. The universal property of tensor products can be translated into the following adjointness statement.

Theorem 1.23. *Suppose that M is a right R-module, N an (R,S) bimodule, and Q a right S-module, then there is a natural isomorphism*

$$Hom_S(M \otimes_S N, Q) \cong Hom_R(M, Hom_S(N,Q))$$

where f on the left goes to the map

$$m \mapsto (n \mapsto f(m \otimes n))$$

on the right.

Proof. Rotman, Sect 2.2.1. □

Theorem 1.24. *If M (resp. N) is a right (resp. left) module, $M \otimes_R -$ (resp. $- \otimes_R N$) are right exact functors.*

Proof. Given an exact sequence

$$0 \to A \to B \to C \to 0$$

of left modules, we have to show that

$$M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

is exact. By prop 2.42 of Rotman, it suffices to prove the dual statement that

$$0 \to Hom_{\mathbb{Z}}(M \otimes A, X) \to Hom_{\mathbb{Z}}(M \otimes B, X) \to Hom_{\mathbb{Z}}(M \otimes C, X) \tag{1.1}$$

is exact for any abelian group X. From the initial sequence, we see that

$$0 \to Hom_{\mathbb{Z}}(C, X) \to Hom_{\mathbb{Z}}(B, X) \to Hom_{\mathbb{Z}}(C, X)$$

is exact. Therefore

$$0 \to Hom_R(M, Hom_{\mathbb{Z}}(C, X)) \to Hom_R(M, Hom_{\mathbb{Z}}(B, X)) \to Hom_R(M, Hom_{\mathbb{Z}}(C, X))$$

is exact. But this can be identified with (1.1) by the previous theorem.

The exactness statement for the left module N follows from working over R^{op} because we can identify

$$N \otimes_R A = A \otimes_{R^{op}} A$$

□

A right/left module is called flat if tensor product with respect to it is exact. For a left module X to be flat, it is enough to know that

$$M \otimes X \to N \otimes X$$

is injective whenever $M \to N$ is injective.

Theorem 1.25.

(a) X is flat if all of its finitely generated submodules are flat.

(b) Projective modules are flat.

(c) If R is a (commutative) PID, a module is flat if and only if it is torsion free.

Proof. Suppose that $M \to N$ is injective. If $M \otimes X \to N \otimes X$ is not injective, then some nonzero element $\sum m_i \otimes x_i$ lies in the kernel. This would lie in the kernel of $M \otimes X_0 \to N \otimes X_0$, where $X_0 \subset X$ is the submodule generated by the x_i. Therefore X_0 is not flat.

Suppose that X is projective. Then it is direct summand of R^I. Consider the commutative square

$$\begin{tikzcd} M^I \arrow[r, "d"] & N^I \\ M \otimes X \arrow[u, "c"] \arrow[r, "a"] & N \otimes X \arrow[u, "b"'] \end{tikzcd}$$

Then $M \otimes X$ is summand of M^I, so c is injective. Also d is injective because it is a sum of injective maps. Therefore a is injective by commutativity.

Suppose that R is a PID. If X is torsion free, all of its finitely generated submodules are free. Therefore X is flat. Suppose that X is not torsion free. Then $tx = 0$ for some nonzero $x \in X, t \in R$. Then $1 \otimes x$ would lie in the kernel of $t : R \to R$ tensored with X. So X would not be flat. $\square$

The converse to (b) is not true.

Example 1.26. *$\mathbb{Q}$ is a flat $\mathbb{Z}$-module by (c) above. However, it is not projective, because $\mathbb{Q}$ is divisible but a submodule of a free module cannot be.*

The converse does hold under appropriate finiteness conditions, see Rotman theorem 3.56.

Now suppose that M is a left R-module, then

$$M^* = Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$$

is naturally a right R-module. Applying this to R^{op}, we see that this operation also takes right R-modules to left modules.

Proposition 1.27. *If F is a free R-module, then F^* is injective.*

Proof. We have a natural isomorphism

$$Hom_R(-, F^*) \cong Hom_{\mathbb{Z}}(- \otimes_R F, \mathbb{Q}/\mathbb{Z})$$

Since F is flat, and $\mathbb{Q}/\mathbb{Z}$ is divisible, the functor on the right is exact. □

The following is of fundamental importance. It generalizes what we proved for abelian groups.

Theorem 1.28. *Every R-module embeds into an injective module.*

Proof. Let M be a module. Choose a surjection $F \to M^*$, with F a free right module. Then we have injections

$$M \to M^{**}, \quad M^{**} \to F^*$$

Composing these gives an injection of M into F^*, which is an injective module.
□

Chapter 2

Homology

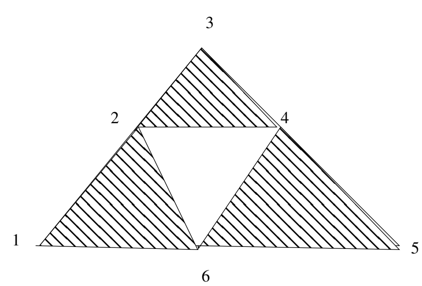
Basic Refs:

1. Hatcher, Algebraic topology
2. Rotman, Intro to homological algebra
3. Spanier, Algebraic topology
4. Weibel, An introduction to homological algebra

2.1 Simplicial complexes

Homology came out of algebraic topology. So we review the basic constructions for intuition and motivation. Recall that a (simple) graph consists of a set of vertices V, and a set of edges E between pairs of vertices. An edge can be regarded as a 2-element subset of V. A simplicial complex is a generalization, where one also allows triangles etc. More formally, it is a pair $S = (V, \Sigma)$ consisting of a set V and a collection of finite nonempty subsets Σ of V called simplices. We require that all singletons are in Σ, and any nonempty subset of $\sigma \in \Sigma$ is also in Σ. If $\sigma \in \Sigma$, has cardinality $i+1$, it is called an i-simplex.

Example 2.2. *In the example below*



$V = \{1, 2, \ldots 6\}$ *and* Σ *consists of the* 2*-simplices* $\{1, 2, 6\}, \{2, 3, 4\}, \{4, 5, 6\}$ *and all nonempty subsets of them.*

Simplices in the above sense, are combinatorial models for simplices in the geometric sense. The standard geometric n-simplex Δ^n is the convex hull of unit vectors $(1, 0, \ldots), (0, 1, 0 \ldots), \ldots \in \mathbb{R}^{n+1}$. Just as a graph gives rise to a topological space, where edges are replaced by arcs, a simplicial complex can also be turned in a topological space $|S|$, where n-simplices are replaced by spaces homeomorphic to geometric simplices. (see Spanier Chap 3 for details).

An orientation on a 2-simplex $\{v_1, v_2\}$ is a simply an ordering: either $[v_1, v_2]$ or $[v_2, v_1]$. In general, an orientation of $\sigma = \{v_1, v_2, \ldots v_n\}$ is an A_n-orbit of orderings (where $A_n \subset S_n$ is the alternating group). Thus every simplex has exactly two orientations. Given an oriented simplex $[v_0, \ldots, v_n]$, we identify $-[v_0, \ldots, v_n]$ with the same simplex with opposite orientation. Its boundary is the formal sum

$$\partial[v_0, \ldots, v_n] = [v_1, \ldots, v_n] - [v_0, v_2, \ldots, v_n] + \ldots = \sum_i (-1)^i [v_0, \ldots, \hat{v}_i, \ldots v_n]$$

We call finite formal linear combination of n-simplices, as above, an n-chain. These form a free abelian group $C_n(S)$. The above formula determines a homomorphism

$$\partial_n : C_n(S) \to C_{n-1}(S)$$

We usually drop the subscript, and write ∂. Here is the key fact.

Proposition 2.3. $\partial_{n-1}\partial_n = 0$*, or more succinctly* $\partial^2 = 0$*.*

Proof. We do this when $n = 2$

$$\partial^2[v_0v_1v_2] = \partial([v_1v_2] - [v_0v_2] + [v_0v_1]) = (v_1 - v_2) - (v_0 - v_2) + (v_0 - v_1) = 0$$

The general case is not essentially harder. Expand $\partial^2[v_0 \ldots v_n]$, then one can see that the term $[v_0, \ldots \hat{v}_i \ldots \hat{v}_j \ldots]$ occurs twice with opposite sign. □

Elements of the kernel ∂ are called cycles, and elements of the image of ∂ are called the boundaries.

Corollary 2.4. *Every boundary is a cycle.*

One can ask about the converse. In general, the answer is no. A measure of the failure is

Definition 2.5. *The nth homology group of* S *is*

$$H_n(S) = \frac{Z_n(S)}{B_n(S)}$$

where

$$Z_n(S) = \ker \partial_n$$

$$B_n(S) = \operatorname{im} \partial_{n+1}$$

Example 2.6. *Let S be the simplicial complex of example 2.2. Then $\gamma = [2,4] + [4,6] + [6,2]$ is a cycle which is not a boundary, so $H_1(S) \neq 0$. In fact, with enough patience, one can show that $H_1(S)$ is the infinite cyclic group generated by γ.*

There is a dual notion. The group of cochains

$$C^n(S) = Hom_{\mathbb{Z}}(C_n(S), \mathbb{Z})$$

This has a coboundary homomorphism

$$d : C^n(S) \to C^{n+1}(S)$$

defined by the dual to ∂.

Definition 2.7. *The nth cohomology group of S is*

$$H^n(S) = \frac{\ker d : C^n(S) \to C^{n+1}(S)}{\operatorname{im} d : C^{n-1}(S) \to C^n(S)}$$

Cohomology is roughly dual to homology (this is correct when homology is torsion free, but otherwise the precise relation is more subtle), so it may not be clear at first why it is useful. However, cohomology does carry extra structure, namely a product, called cup product

$$H^n(S) \times H^m(S) \to H^{n+m}(S)$$

which makes cohomology into a graded ring. Given n and m cochains f and g, their product is given by the formula

$$(f \cup g)[v_0, ..., v_{n+m}] = f[v_0, ..., v_n]g[v_n, ..., v_{n+m}]$$

A fact, which is at first glance, is surprising is that homology and cohomology on depends on the topological space $|S|$, and not on the triangulation. This can be done comparing to singular (co)homology, which doesn't depend on a triangulation. The group of singular chains $S_n(X)$, of a space X, is the free abelian groups generated by continuous maps from $\Delta^n \to X$. The boundary is essentially identical to the formula given previously. We refer to Hatcher or Spanier for a detailed treatment.

2.8 Complexes

We now abstract the ideas from the first section.

Definition 2.9. *A chain complex, or just complex, is a collection of abelian groups (or modules) $C_n, n \in \mathbb{Z}$ and homomorphisms (called differentials) $d : C_n \to C_{n-1}$ satisfying $d^2 = 0$. The nth homology is*

$$H_n(C_\bullet) = \frac{\ker d : C_n \to C_{n-1}}{\operatorname{im} d : C_{n+1} \to C_n}$$

It is technically convenient to allow the index to lie in $\mathbb{Z}$. Although in practice, we may only be given $C_n, n \geq 0$. In which case, we set $C_n = 0$ when $n < 0$ We will refer to such complexes as positive.

The following is obvious.

Lemma 2.10. *The sequence $C_\bullet$ is exact iff $H_n(C_\bullet) = 0$ for all n. In this case, $C_\bullet$ is also called acyclic.*

One can define a cochain complex $C^\bullet$ in similar fashion, except that differentials go the other way. Its cohomology

$$H^n(C^\bullet) = \frac{\ker d : C^n \to C^{n+1}}{\operatorname{im} d : C^{n-1} \to C^n}$$

We note that using a change of variable

$$C_n = C^{-n}$$

allows us to convert cochain complexes to complexes. Thus there is no real difference between these notions.

Definition 2.11. *A morphism of complexes, or chain map, $f : C_\bullet \to D_\bullet$ is a collection of homomorphisms $f : C_n \to D_n$, such that $df = fd$. With this notion, the collection of complexes of R-modules becomes a category $C(Mod_R)$.*

The following is straightforward.

Lemma 2.12. *A morphism of complexes $f : C_\bullet \to D_\bullet$ induces a homomorphism of homology groups $f_* : H_n(C_\bullet) \to H_n(D_\bullet)$. In fact, H_n gives a functor from $C(Mod_R) \to Mod_R$.*

A simplicial map of simplicial complexes $f : S = (V, \Sigma) \to S' = (V', \Sigma')$ is a map of sets $f : V \to V'$ such that the image of any simplex of S is a simplex of S'. It should be clear that a simplicial map f induces a morphism $C_\bullet(S) \to C_\bullet(S')$, and therefore homomorphisms $f_* : H_n(S) \to H_n(S')$. More generally, continuous maps for space induce chain maps on the singular chain complex, and therefore homomorphisms on homology.

We define an sequence of morphisms of complexes

$$C_\bullet \to C'_\bullet \to C''_\bullet$$

be exact if each sequence

$$C_n \to C'_n \to C''_n$$

is exact in the usual sense. The following result is fundamental. It will be used many times over.

Theorem 2.13. *If $0 \to C_\bullet \to C'_\bullet \to C''_\bullet \to 0$ is an exact sequence of complexes, then there is a long exact sequence*

$$\ldots H_n(C_\bullet) \to H_n(C'_\bullet) \to H_n(C''_\bullet) \xrightarrow{\partial} H_{n-1}(C_\bullet) \ldots$$

The unlabelled maps in the above sequence are the obvious ones, the map ∂, called a connecting map, is somewhat more mysterious, but it will be explained below. Rotman (prop 6.9) gives a proof of this theorem. We will give different argument. The starting point is the following standard fact, which is in fact a special case of the theorem.

Proposition 2.14 (Snake lemma)**.** *Given a commutative diagram*

$$\begin{tikzcd} & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' & \end{tikzcd}$$

with exact rows, there is an exact sequence

$$\ker f \to \ker g \to \ker h \xrightarrow{\partial} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h$$

The first (resp. last) map above is injective (resp. surjective) if $A \to B$ (resp. $B' \to C'$) is injective (resp. surjective).

Proof. We only explain the connecting map. The remain details are straightforward, and best checked in private. Given $c \in \ker h \subseteq C$, we can lift it to $b \in B$. Since $g(b)$ maps to 0, in C', it lies in A'. One can check that the image a of $g(b)$ in coker f does not depend on the choice of b. Set $\partial c = a$. □

Proof of theorem 2.13. Let us write

$$Z_n = \ker d : C_n \to C_{n-1}$$

$$B_n = \operatorname{im} d : C_{n-1} \to C_n$$

etc. Apply the snake lemma to

$$\begin{tikzcd} 0 \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "d"] & C'_{n+1} \arrow[r] \arrow[d, "d'"] & C''_{n+1} \arrow[r] \arrow[d, "d''"] & 0 \\ 0 \arrow[r] & C_n \arrow[r] & C'_n \arrow[r] & C''_n \arrow[r] & 0 \end{tikzcd}$$

to get an exact sequence of kernels

$$0 \to \ker d \to \ker d' \to \ker d''$$

and an exact sequence of cokernels

$$\operatorname{coker} d \to \operatorname{coker} d' \to \operatorname{coker} d'' \to 0$$

(We won't use the fact that these sequences fit together.) These can be rewritten as

$$0 \to Z_{n+1} \to Z_{n+1} \to Z''_{n+1}$$

and

$$C_n/B_n \to C'_n/B'_n \to C''_n/B''_n \to 0$$

Using these for $m = n, n+2$ yields a diagram

$$\begin{tikzcd}
 & C_m/B_m \arrow[r] \arrow[d, "d"] & C'_m/B'_m \arrow[r] \arrow[d, "d'"] & C''_m/B''_m \arrow[r] \arrow[d, "d''"] & 0 \\
0 \arrow[r] & Z_{m-1} \arrow[r] & Z'_{m-1} \arrow[r] & Z''_{m-1} &
\end{tikzcd}$$

Apply the snake lemma on more time to get a six term exact sequence

$$H_m(C_\bullet) \to H_m(C'_\bullet) \to H_m(C''_\bullet) \stackrel{\partial}{\to} H_{m-1}(C_\bullet) \ldots$$

These can be spliced together to obtain the infinite sequence. $\square$

2.15 Homotopy

We go back to topology to borrow another key idea. Let $I = [0, 1]$. Two continuous maps $f, g : X \to Y$ between topological spaces are homotopic if there is a continuous map $F : X \times I \to Y$, called a homotopy, such that $f = F|_{X\times\{0\}}$ and $g = F|_{X\times\{1\}}$. This means that f can be deformed to g. It's easy to check that it is an equivalence relation. The importance stems from the following fact

Theorem 2.16. *If $f, g : X \to Y$ are homotopic, then the induced maps $f_*, g_* : H_n(X) \to H_n(Y)$ are identical.*

Here is an extremely useful consequence.

Corollary 2.17. *Given a pair of continuous map $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identities, then f induces an isomorphism between the homology of X and Y.*

A space is contractible X if the identity is homotopic to a constant map. For example, $\mathbb{R}^n$ is contractible.

Corollary 2.18. *A contractible space has zero homology in positive degrees.*

The key idea for proving the theorem is to introduce and algebraic version of homotopy, which will be very important for us.

Definition 2.19. *If $f, g : C_\bullet \to D_\bullet$ are two morphisms between complexes, a chain homotopy between them is a collection of homorphisms $F : C_n \to D_{n+1}$ such that $dF + Fd = f - g$. f and g are called chain homotopic if F exists.*

To make sense of the last equation, we can draw the diagram

$$\begin{tikzcd}
 & C_n \arrow[r, "d"] \arrow[dl, "F"'] \arrow[d, "f-g"] & C_{n-1} \arrow[dl, "F"] \\
D_{n+1} \arrow[r, "d"] & D_n &
\end{tikzcd}$$

A couple of remarks: After the theorem is proved, we will drop "chain" and just say that f and g are homotopic, and we will refer to F as a homotopy. Some authors take $dF - Fd = f - g$. It's easy to go from one convention to the other by $F_n \mapsto (-1)^n F_n$, where F_n denotes the map in degree n.

Theorem 2.20. *If $f, g : C_\bullet \to D_\bullet$ are chain homotopic, then $f_* : H_n(C_\bullet) \to H_n(D_\bullet)$ and $g_* : H_n(C_\bullet) \to H_n(D_\bullet)$ coincide.*

Proof. Let F be chain homotopy. Given $a \in H_n(C_\bullet)$, we can represent it by $\alpha \in C_n$ such that $d\alpha = 0$. $f_*(a)$ is the coset of $f(\alpha)$. We have

$$f(\alpha) = dF(\alpha) + Fd(\alpha) + g(\alpha) = d(F(\alpha)) + g(\alpha)$$

which implies that $f(a) = g(a)$. $\square$

Let us indicate to proof of theorem 2.16, referring to pp 112-113 of Hatcher for precise details. The continuous maps f, g induce chain maps $\tilde{f}, \tilde{g} : S_\bullet(X) \to S_n(Y)$ on the singular chain complex. We have to construct a chain homotopy $\tilde{F}$ between these. Recall that elements of $S_n(X)$ are linear combinations of continuous maps $\Delta^n \to X$. Such a map induces a continuous map from the prism $\Delta^n \times I \to X \times I$. There is a natural, and purely combinatorial, way to subdivide the prism $\Delta^n \times I$ into a finite union of $n+1$ simplicies. When composed with F, these simplices give elements of $S_{n+1}(Y)$. Let $\tilde{F}(\Delta^n \to X)$ denote the sum of these elements with appropriate coefficients of the form ± 1 (chosen so that adjacent interior faces cancel). Then one checks that this gives the desired chain homotopy.

Definition 2.21. *A contracting homotopy of a complex $C_\bullet$ is a homotopy between identity and 0. A morphism $f : C_\bullet \to D_\bullet$ is a homotopy equivalence if there exists a morphism $g : D_\bullet \to C_\bullet$ such that $g \circ f$ and $f \circ g$ are homotopic to the identities of $C_\bullet$ and $D_\bullet$.*

As a corollary to theorem 2.20, we obtain

Proposition 2.22. *A complex is acyclic if it possesses a contracting homotopy. A homotopy equivalence induces an isomorphism on homology.*

2.23 Mapping cones

We can define the category $C(Mod_R)$ of complexes of R-modules, where the objects are complexes and morphisms were defined above. Given complexes $C_\bullet, D_\bullet$, $Hom(C_\bullet, D_\bullet)$ has the structure of an abelian group compatible with composition. Furthermore, standard constructions and notions such as direct sums, kernels, cokernels and exact sequences make sense within $C(Mod_R)$. This amounts to saying that this is an abelian category. See Rotman section 5.5 for the precise definition. Given complexes $C_\bullet, D_\bullet$, let $Null(C_\bullet, D_\bullet) \subset Hom(C_\bullet, D_\bullet)$ be the subset of morphisms homotopic to 0. This is easily seen to be a subgroup. Let $K(Mod_R)$ denote the category with the same objects as before, but morphisms from $C_\bullet$ to $D_\bullet$ are homotopy classes of morphisms in $C(Mod_R)$, or equivalently cosets $Hom(C_\bullet, D_\bullet)/Null(C_\bullet, D_\bullet)$. This is still an additive category since for example, the set of morphisms form and abelian group, but it is not abelian. Among other problems, the kernel and cokernel of a morphism need not exist in the homotopy category. If f and g are homotopic maps, then the complexes $\ker f$ (resp. $\operatorname{coker} f$) and $\ker g$ (resp. $\operatorname{coker} g$) need not be isomorphic in $K(Mod_R)$. Fortunately, there is a reasonable substitute. Given a morphism $f : A_\bullet \to B_\bullet$, we form a new complex $C(f)_\bullet$ called the mapping cone by

$$C(f)_n = B_n \oplus A_{n-1}$$

with differential

$$d(x, y) = (dx - f(y), dy)$$

This is also analogue of a topological notion, which is explained on pp 18-24 of Weibel's book.

Lemma 2.24. *If f and g are homotopic, then $C(f)_\bullet$ and $C(g)_\bullet$ are isomorphic.*

Proof. Let F be a homotopy from f to g. Then $(x, y) \mapsto (x - F(y), y)$ is morphism of $C(f) \to C(g)$ with inverse $(x, y) \mapsto (x + F(y), y)$. $\square$

The mapping cone can play the role of either the kernel or the cokernel under appropriate conditions. Let us explain the second. Suppose that

$$0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$$

is exact in $C(Mod_R)$, and that for each n there are splittings

$$s_n : C_n \to B_n$$

for g. Note that we do not require that the splittings are compatible with differentials.

Lemma 2.25. *The morphism $C(g)_\bullet \to C_\bullet$ given by $(x, y) \mapsto g(y)$ is an isomorphism in $K(Mod_R)$ with inverse*

$$z \mapsto (s_n(z), s_{n-1}d(z) - ds_n(z))$$

We omit the proof, which is a long calculation. Under these conditions, we see that the connecting map $H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ is induced by the projection $C(g)_\bullet \to A_{\bullet-1}$.

Chapter 3

Ext groups

Refs.

1. Atiyah-Macdonald, Commutative algebra
2. Rotman, Homological algebra

3.1 Extensions

Given two R-modules A and C, an extension of C by A is a short exact sequence

$$0 \to A \to B \to C \to 0$$

(NB: This terminology is opposite of what Rotman uses, but it is better aligned with the notation to be introduced.) Let us say that another extension

$$0 \to A \to B' \to C \to 0$$

is equivalent to the first if they can be put into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "id"] & B \arrow[r] \arrow[d, "\phi"] & C \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Lemma 3.2. *The map ϕ above is an isomorphism. Equivalence of extensions is an equivalence relation.*

Proof. The first statement, which is a special case of the 5-lemma, is an easy diagram chase. We will omit the proof. Since this implies that ϕ^{-1} exists, we see that this relation is symmetric. It is obviously reflexive, and transitive (use the composite of ϕ and the corresponding map in the third extension). $\square$

Let $ext(C, A)$ denote the set of equivalence classes of extensions. Our goal is to compute this. First observe that this set has a distinguished element

$$0 \to A \to A \oplus C \to C \to 0$$

which we call the trivial extension, and denote this by 0. We say that an extension

$$0 \to A \xrightarrow{j} B \xrightarrow{p} C \to 0$$

splits if there is a homomorphism $i : C \to B$ such that $p \circ i = id$.

Lemma 3.3. *An extension splits iff it is equivalent to the trivial extension.*

Proof. Given a split extension as above, define $\phi : A \oplus C \to B$ by $\phi(a, c) = j(a) + s(c)$. Conversely, if we have such a morphism the $s(c) = \phi(0, c)$ gives a splitting. □

We can now compute it in one case.

Proposition 3.4. *C is projective if and only if $ext(C, A) = \{0\}$ for every A.*

Proof. If C is projective, we proved early that any surjective morphism to C splits. Therefore $ext(C, A) = 0$.

Conversely, suppose $ext(C, A)$ for every A. Given

$$\begin{tikzcd} & & & P \arrow[d, "f"] & \\ 0 \arrow[r] & K \arrow[r] & M \arrow[r, "\pi"] & N \arrow[r] & 0 \end{tikzcd}$$

let $L = \{(m, p) \in (M, P) \mid f(m) = \pi(p)\}$ be the pullback. Then we have an extension

$$0 \to K \to L \to P \to 0$$

This has a splitting $s : P \to L$ by assumption. Composing this with the projection $L \to M$, yields a map $P \to M$ lifting f. □

In order to try to compute $ext(C, A)$ in general, we can try to reduce C to a projective module. We choose a surjection $\pi : P \to C$, with P projective. We could take P to be a free module on a set of generators for P, for example, Then form the sequence

$$0 \to K \xrightarrow{i} P \xrightarrow{\pi} C \to 0$$

Define

$$_{\pi}Ext(C, A) = \mathrm{coker}(Hom(P, A) \to Hom(K, A))$$

We will prove the following later in more form.

Proposition 3.5. *The isomorphism class of $_{\pi}Ext(C, A)$ is independent of f.*

Henceforth, we write $Ext(C, A)$ for ${}_{\pi}Ext(C, A)$.

Theorem 3.6. *There is a bijection $ext(C, A) \cong Ext(C, A)$ preserving* 0.

Proof. Given $f : K \to A$, let

$$Q_f = P \oplus A/\{(i(k), f(k)) \mid k \in K\}$$

be the pushout. The fits into an extension

$$0 \to A \to Q_f \to C \to 0$$

If $f = F|_K$, with $F \in Hom(P, A)$, then $\phi(p, a) = (F(p), a)$ is an equivalence to the trivial extension. Similarly, one can check that if $g : K \to A$ is another map such that $g - f$ lies in the image of $Hom(P, A)$, then

$$0 \to A \to Q_g \to C \to 0$$

is equivalent to the previous extension. Therefore we have constructed a map from $Ext(C, A) \to ext(C, A)$ preserving 0.

Given an extension of C by A,

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, dashed, "f"] & P \arrow[r] \arrow[d, dashed, "g"] & C \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

we can find g and therefore f using the projectivity of P. This can be checked to give the inverse $ext(C, A) \to Ext(C, A)$.

□

Corollary 3.7. *$ext(C, A)$ has the structure of an abelian group.*

See Rotman section 7.2.1 for an explicit description of the group structure in terms of extensions.

Example 3.8. *Let $R = \mathbb{Z}$. Consider the exact sequence*

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

where $n \neq 0$. Then "Hom-ing" into A yields

$$A \xrightarrow{n} A \to Ext(\mathbb{Z}/n\mathbb{Z}, A) \to 0$$

Therefore $Ext(\mathbb{Z}/n\mathbb{Z}, A) \cong A/nA$. The calculation can be upgraded to calculate $Ext(B, A)$ for any finitely generated abelian group, Writing $B = \bigoplus \mathbb{Z}/n_{\mathbb{Z}} \oplus \mathbb{Z}^N$, $Ext(\mathbb{Z}/n\mathbb{Z}, A) \cong \bigoplus A/n_i A$.

So far we have been borrowing ideas from topology. Now we are in a position to repay the debt. We defined the cohomology of a simplicial complex earlier, and said that it is roughly dual to homology. Here is a the precise statement.

Theorem 3.9 (Universal coefficient theorem)**.** *Given a simplicial complex S, there is an isomorphism*

$$H^n(S) \cong Hom(H_n(S), \mathbb{Z}) \oplus Ext(H_{n-1}(S), \mathbb{Z})$$

The argument is slightly simpler for finite simplicial complexes. So let us assume this. Then the result will be a consequence of the following result from pure homological algebra.

Theorem 3.10. *If $F_\bullet$ is a complex of finitely generated free abelian groups, there is an isomorphism*

$$H^n(Hom(F_\bullet, \mathbb{Z})) \cong Hom(H_n(F_\bullet), \mathbb{Z}) \oplus Ext(H_{n-1}(F_\bullet), \mathbb{Z})$$

Proof. Let $B_n \subseteq Z_n \subseteq F_n$ be the subgroups of boundaries and cycles. These are free abelian by basic algebra. Therefore the exact sequences

$$0 \to Z_n \to F_n \to B_{n-1} \to 0$$

is split. It follows that

$$0 \to Hom(B_{n-1}, \mathbb{Z}) \to Hom(F_n, \mathbb{Z}) \to Hom(Z_n, \mathbb{Z}) \to 0$$

is also split exact. This can be viewed as an exact sequence of cochain complexes where the complexes on the left and right have zero differential. Having zero differential implies that $Hom(B_{n-1}, \mathbb{Z})$ and $Hom(Z_n, \mathbb{Z})$ are the cohomology groups. The long exact sequence for cohomology is

$$Hom(Z_{n-1}, \mathbb{Z}) \to Hom(B_{n-1}, \mathbb{Z}) \to H^n(Hom(F_n, \mathbb{Z})) \to Hom(Z_n, \mathbb{Z}) \to Hom(B_n, \mathbb{Z})$$

Using the exact sequences

$$0 \to Z_n \to B_n \to H_n(F_\bullet) \to 0$$

we can write the previous sequence as

$$0 \to Ext(H_{n-1}, \mathbb{Z}) \to H^n(Hom(F_n, \mathbb{Z})) \to Hom(H_n, \mathbb{Z}) \to 0$$

Finally, note that Ext is a torsion group and Hom is torsion free, so this must split canonically. $\square$

3.11 Projective resolutions

Let M be an R-module. Choose a projective module P_0 and a surjection $P_0 \to M$. Let K_0 be the kernel. Choose a surjection from another projective module $P_1 \to K_0$. Let K_1 be the kernel of this, and repeat. Composing $P_i \to K_{i-1}$ with $K_{i-1} \to P_{i-1}$ yields an exact sequence

$$\ldots P_2 \to P_1 \to P_0 \to M \to 0$$

where each P_i is projective. This is called a projective resolution of M. We have proved that such things exist.

Lemma 3.12. *Every module possesses a projective resolution.*

Such resolutions are not unique, because choices are involved. However, they are unique in a weaker sense that any two projective resolutions are homotopy equivalent.

Theorem 3.13. *If $Q_\bullet \to M \to 0$ is an exact sequence, so perhaps another projective resolution. Then there exists a morphism $f : P_\bullet \to Q_\bullet$ such that*

$$\begin{tikzcd} P_0 \arrow[d, "f"] \arrow[dr] & \\ Q_0 \arrow[r] & M \end{tikzcd}$$

commutes. This is unique up to homotopy, i.e. any other morphism is homotopic to f.

Proof. A morphism f is a collection of homomorphisms $f_n : P_n \to Q_n$, which can be inductilvely The first map f_0 exists by projectivity of P_0

$$\begin{tikzcd} & P_0 \arrow[d] \arrow[dl, dashed, "f_0"'] \\ Q_0 \arrow[r] & M \end{tikzcd}$$

Suppose $f_n, f_{n-1} \dots$ have been constructed. Let us write $d_\bullet$ and $d'_\bullet$ for the differentials of $P_\bullet$ and $Q_\bullet$. Then we have that $f_{n-1}d_n = d'_n f_n$. So that $d'_n f_n d_{n+1} = f_{n-1} d_n d_{n+1} = 0$. Therefore $f_n d_{n+1} \subseteq \ker d'_n = \operatorname{im} d'_{n+1}$. So we have a diagram

$$\begin{tikzcd} & P_{n+1} \arrow[d] \arrow[dl, dashed, "f_{n+1}"'] \\ Q_{n+1} \arrow[r] & \operatorname{im} d'_{n+1} \end{tikzcd}$$

Projectivity of P_{n+1} shows the existence of f_{n+1} making this commute.

Given a second morphism $g : P_\bullet \to Q_\bullet$, we have to construct a homotopy h between, that is sequence of maps $h_n : P_n \to Q_{n+1}$ satisfying

$$f_n - g_n = d_{n+1}h_n + h_{n-1}d_n$$

This is again constructed by induction, using projectivity of each P_n. See p342 of Rotman for details. □

Remark 3.14. *The same proof actually something stronger, namely that if $P_\bullet \to M$ is a complex, with each P_n projective, then $f : P_\bullet \to Q_\bullet$ exists and is unique up to homotopy.*

Corollary 3.15. *If $Q_\bullet \to M$ is another projective resolution, there exists a homotopy equivalence $f : P_\bullet \to Q_\bullet$. (Recall that this means that there is $g : Q_\bullet \to P_\bullet$ such that $f \circ g$ and $g \circ f$ are homotopic to the identities.)*

3.16 Higher Ext groups

Given a pair of modules M and N fix a projective resolution $P_\bullet \to M$. Let $\partial : P_n \to P_{n-1}$ denote the maps. Since $P\bullet$ is exact, it forms a complex i.e. $\partial^2 = 0$. Then

$$C^n = Hom(P_n, N)$$

carries maps

$$d : C^n \to C^{n+1}$$

dual to ∂. We necessarily have $d^2 = 0$, so $C^\bullet$ forms a cochain complex.

Theorem/Def 3.17. *The isomorphism classes of the cohomology groups*

$$Ext^n_R(M, N) = H^n(Hom_R(P_\bullet, N))$$

depend only on M and not on the choice of resolution $P_\bullet$.

Proof. If $Q_\bullet$ is another projective resolution, we have morphisms $f : P_\bullet \to Q_\bullet$ and $g : Q_\bullet \to P_\bullet$ such that $f \circ g$ and $g \circ f$ are homotopic to the identities. These induces morphisms between $Hom(P^\bullet, N)$ and $Hom(Q_\bullet, N)$ whose compositions are again homotopic to the identities. This implies that they have isomorphic cohomology by proposition 2.22.

□

Corollary 3.18. *There are isomorphisms*

$$Ext^0_R(M, N) \cong Hom_R(M, N)$$

and

$$Ext^1_R(M, N) \cong Ext(M, N)$$

where the last group is the one constructed in a previous section.

Proof. Given a projective resolution $P_\bullet \to M$, we can form an exact sequence

$$0 \to K \to P_0 \to M \to 0$$

where $K = \operatorname{im} P_1 \to P_0$. Then

$$0 \to Hom(M, N) \to Hom(P_0, N) \to Hom(K, N)$$

and

$$0 \to Hom(K, N) \to Hom(P_1, N)$$

are exact. This implies that

$$Hom(M, N) = H^0(Hom(P_\bullet, N))$$

The proof of the second isomorphism is similar. □

The previous theorem is not that useful as stated. In fact, we will show that $Ext^n(-,-)$ is a functor in both variables, and that it fits into natural exact sequences. It are these properties that make it a powerful tool.

Theorem 3.19. *If $g : N \to N'$ is a morphism there is an induced morphism $g_* : Ext^n_R(M,N) \to Ext^n_R(M,N')$. This makes $Ext^n_R(M,-)$ a covariant functor from $Mod_R \to Ab$. If*

$$0 \to N \to N' \to N'' \to 0$$

is exact, then there is a long exact sequence

$$\ldots Ext^n_R(M,N) \to Ext^n_R(M,N') \to Ext^n_R(M,N'') \to Ext^{n+1}_R(M,N) \ldots$$

Proof. Fix a projective resolution $P_\bullet \to M$. Then we get a morphism of complexes

$$Hom(P_\bullet, N) \to Hom(P_\bullet, N')$$

The induced map on cohomology yields

$$Ext^n_R(M,N) \to Ext^n_R(M',N)$$

Suppose that

$$0 \to N \to N' \to N'' \to 0$$

is exact. Since P_i is projective, $Hom(P_i,-)$ is an exact functor. Therefore we get a short exact sequence of complexes

$$0 \to Hom(P_\bullet, N) \to Hom(P_\bullet, N') \to Hom(P_\bullet, N'') \to 0$$

This yields a long exact sequence

$$\ldots Ext^n_R(M,N) \to Ext^n_R(M,N') \to Ext^n_R(M,N'') \to Ext^{n+1}_R(M,N) \ldots$$

□

Theorem 3.20. *If $h : M \to M'$ is a morphism, there is an induced morphism $h^* : Ext^n_R(M',N) \to Ext^n_R(M,N)$. This makes $Ext^n_R(-,N)$ into a contravariant functor from $Mod_R \to Ab$. If*

$$0 \to M \to M' \to M'' \to 0$$

is exact, then there is a long exact sequence

$$\ldots Ext^n_R(M'',N) \to Ext^n_R(M',N) \to Ext^n_R(M,N) \to Ext^{n+1}_R(M'',N) \ldots$$

Proof. If $P'_\bullet \to M'$ is a projective resolution, the above remark 3.14 allows us to construct a morphism $\tilde{h} : P_\bullet \to P'_\bullet$ unique up to homotopy. This induces a morphism

$$Hom(P'_\bullet, N) \to Hom(P_\bullet, N)$$

which induces h^*. If $\ell : M' \to M''$ is another morphism. Choose a projective resolution $P''_\bullet \to M''$ and construct the corresponding morphism $\tilde{\ell} : P'_\bullet \to P''_\bullet$.

The uniqueness shows that $\widetilde{\ell \circ h}$ and $\tilde{\ell} \circ \tilde{h}$ are homotopy equivalent. This implies $(\ell \circ h)^* = h^* \circ \ell^*$. Therefore we have a functor.

For the last statement, we claim that we can construct projective resolutions fitting into a diagram

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] \arrow[dl, dashed] & \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d, "i"] & P'_0 \arrow[r] \arrow[d, "f"] & P''_0 \arrow[r] \arrow[d] \arrow[dl, dashed, "g"] & 0 \\
0 \arrow[r] & M \arrow[r] & M' \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

with exact rows. To prove this, choose resolutions $P_\bullet$ and $P''_\bullet$, and set $P'_\bullet = P_\bullet \oplus P''_\bullet$ as a graded module. Since P''_0 is projective, we can construct g above. Set $f : P_0 \oplus P''_0 \to M'$ to $i + g$. The differentials of $P'_\bullet$ are built similarly.

From the claim, we have an exact sequence of complexes

$$0 \to P_\bullet \to P'_\bullet \to P''_\bullet \to 0$$

which, by construction, is split as a sequence of graded modules. It follows that

$$0 \to Hom(P''_\bullet, N) \to Hom(P'_\bullet, N) \to Hom(P_\bullet, N) \to 0$$

is an exact sequence of complexes. Applying theorem 2.13 to this, gives a long exact sequence of Ext groups.

$\square$

Example 3.21. *If $R = \mathbb{Z}$, using the projective resolution,*

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

we find that

$$Ext^1(\mathbb{Z}/n\mathbb{Z}, A) = A/nA$$

and

$$Ext^i(\mathbb{Z}/n\mathbb{Z}, A) = 0$$

for $i > 1$.

3.22 Characterization of projectives and injectives

Theorem 3.23. *Let P be an R-module. The following are equivalent.*

(a) P is projective.

(b) $Ext^n_R(P, M) = 0$ for all $n > 0$ and for all modules M.

(c) $Ext^1_R(P, M) = 0$ *for all modules* M.

Proof. If P is projective, then $P = P$ is a projective resolution. Therefore (b) follows. Clearly (b) implies (c). If (c) holds, then for any exact sequence

$$0 \to K \to M \to N \to 0$$

we have

$$Hom_R(P, M) \to Hom_R(P, N) \to Ext^1_R(P, K) = 0$$

This implies that P is projective. $\square$

We prove an analogous characterization for injectives. However, due to the asymmetry of the definition, the proof will be completely different.

Theorem 3.24. *Let* E *be an* R*-module. The following are equivalent.*

(a) E *is injective.*

(b) $Ext^1_R(M, E) = 0$ *for all modules* M.

(c) $Ext^n_R(M, E) = 0$ *for all* $n > 0$ *and for all modules* M.

Proof. Suppose that E is injective. Injectivity will imply that given an exact sequence

$$0 \to E \xrightarrow{i} N \to M \to 0$$

we can find a homomorphism $r : N \to E$ such that $r \circ i = id$. This means that the sequence splits. By an earlier characterization, $Ext^1_R(M, E)$ is the equivalence class of extensions as above. Therefore it must be zero. Conversely, if (b) holds then any extension must split. So E can be seen to be injective.

Clearly (c) implies (b). We just have to prove the converse. We use induction on n and a trick called "dimension shifting". Following Grothendieck, algebraic geometers also refer this type of argument more broadly as "devissage", which translates roughly as "untwisting". Suppose that (c) holds for a fixed $n > 0$ for all M. Given M we can find an exact sequence

$$0 \to K \to P \to M \to 0$$

with P projective. Then we have an exact sequence

$$Ext^n_R(K, E) \to Ext^{n+1}_R(M, E) \to Ext^{n+1}_R(P, E)$$

The group on the left is zero by induction, while the group on the right is zero by projectivity of P.

$\square$

For the remainder of this section, let us assume that R is commutative. Then $Hom_R(M, N)$ is naturally an R-module via $(rf)(m) = rf(m) = f(rm)$. Therefore

$$Ext^n_R(M, N) = H^n(Hom_R(P_\bullet, N))$$

is also an R-module. Moreover, the previous arguments can be modified to show that this structure is independent of the resolution.

Recall that if $S \subset R$ is a multiplicatively closed set, we can form a new ring $S^{-1}R$ by inverting elements of S. This operation extends to an exact functor $S^{-1} : Mod_R \to Mod_{S^{-1}R}$. See Atiyah-Macdonald for details.

Lemma 3.25. *If P is projective, then $S^{-1}P$ is projective.*

Proof. P is projective if and only if it is a summand of a free module. The last condition is stable under localization. $\square$

Suppose now in addition that R is noetherian. If M is finitely generated over R, then we can find a surjection

$$R^{n_0} \to M \to 0$$

for some n_0. Since the kernel is finitely generated (by noetherianness), we can prolong this to an exact sequence

$$R^{n_1} \to R^{n_0} \to M \to 0$$

and so on to obtain

Lemma 3.26. *If M is finitely generated, then it has a free resolution by finitely generated free modules.*

Lemma 3.27. *If M is finitely generated, then for any multiplicative set*

$$S^{-1}Hom_R(M, N) \cong Hom_{S^{-1}R}(S^{-1}M, S^{-1}N)$$

Proof. If $M = R^n$, then this amounts to the isomorphism

$$S^{-1}(M^n) = (S^{-1}M)^n$$

We can form a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S^{-1}Hom(M,N) \arrow[r] \arrow[d, "f"] & S^{-1}Hom(R^{n_0},N) \arrow[r] \arrow[d, "\cong"] & S^{-1}Hom(R^{n_1},N) \arrow[d, "\cong"] \\
0 \arrow[r] & Hom(S^{-1}M, S^{-1}N) \arrow[r] & Hom(S^{-1}R^{n_0}, S^{-1}N) \arrow[r] & Hom(S^{-1}R^{n_1}, S^{-1}N)
\end{tikzcd}$$

The last two maps are isomorphisms by what we said above. Therefore f is an isomorphism by a diagram chase.

$\square$

Combining the last two lemmas, we find that

Theorem 3.28. *If M is finitely generated, then*

$$S^{-1}Ext^n_R(M, N) \cong Ext^n_{S^{-1}R}(M, N)$$

Corollary 3.29. *A finitely generated R-module P is projective if and only if it is locally free.*

Proof. Suppose that P is finitely generated and locally free. We have to show that $E = Ext^1_R(P, N) = 0$ for any N. It suffices to prove that localizations of $E_p = 0$ at primes $p \in \operatorname{Spec} R$. By the theorem

$$E_p = Ext^1_{R_p}(P_p, N_p) = 0$$

for any $p \in \operatorname{Spec} R$. $\square$

Chapter 4

Cohomology of groups

Refs.

1. Brown, Cohomology of groups
2. Rotman, Intro to homological algebra
3. Weibel, An intro to homological algebra

4.1 Group cohomology

Given a group G, a left $\mathbb{Z}G$-module will simply be called a G-module. It is the same thing as an abelian group with an action by G. Let $\mathbb{Z}$ stand for the group of integers with trivial G-action. Fix a G-module A. We define the 0th cohomology by

$$H^0(G, A) = Hom_{\mathbb{Z}G}(\mathbb{Z}, A)$$

Lemma 4.2. *$H^0(G, A)$ is isomorphic to the subgroup of invariant elements $A^G = \{a \in A \mid \forall g \in G, ga = a\}$.*

Proof. The image of $1 \in \mathbb{Z}$ under an element of $H^0(G, A) = Hom_{\mathbb{Z}G}(\mathbb{Z}, A)$ lies in A^G, and conversely. $\square$

Corollary 4.3. *The functor $A \mapsto A^G$ is left exact, i.e. given a sequence of G-modules*

$$0 \to A \to B \to C \to 0$$

we obtain an exact sequence

$$0 \to A^G \to B^G \to C^G$$

The answer to the question of what comes next is higher cohomology

$$H^n(G, A) = Ext^n_{\mathbb{Z}G}(\mathbb{Z}, A)$$

Using the properties of Ext established earlier

Theorem 4.4. *Given a short exact sequence of G-modules, as above, we have a long exact sequence*

$$\dots H^n(G,A) \to H^n(G,B) \to H^n(G,C) \to H^{n+1}(G,A)\dots$$

extending the previous sequence.

Example 4.5. *If $G = \{1\}$ is trivial, then $H^n(G,A) = 0$ for any A and $n > 0$. This because $\mathbb{Z}G = \mathbb{Z}$ and $\mathbb{Z}$ is projective over it.*

Nontrivial examples will have to wait.

4.6 Bar resolution

Group cohomology can be computed using an explicit projective resolution, that we now define. The map

$$\varepsilon : \mathbb{Z}G \to \mathbb{Z}$$

defined by

$$\sum n_g g = \sum n_g$$

is a surjective G-module homomorphism. We will extend this to a free resolution of $\mathbb{Z}$. Let B_n is the free abelian group generated by the $(n+1)$ fold product $G^{n+1} = G \times G \times \dots G$. This is a G-module, where $g \in G$ acts by $g(g_0, \dots, g_n) = (gg_0, \dots, gg_n)$. We define a maps $\varepsilon : B_0 \to \mathbb{Z}$ as above, and

$$d : B_n \to B_{n-1}$$

by

$$d(g_0, \dots, g_n) = \sum (-1)^i (g_0, \dots, \hat{g}_i, \dots g_n)$$

Lemma 4.7. $d^2 = 0$.

Proof. The calculation is similar to what we did for the simplicial chain complex. □

Proposition 4.8. *The complex*

$$\dots B_1 \to B_0 \to \mathbb{Z}_0$$

is exact.

Proof. Set $B_{-1} = \mathbb{Z}$ and $d_0 = \epsilon$. We have to show that the extended complex $B_\bullet$ is acyclic. Let $h : B_n \to B_{n+1}$ be defined by

$$h(g_0, \dots, g_n) = (1, g_0, \dots, g_n), \quad n \geq 0$$

$$h(1) = 1, \quad n = -1$$

One sees that

$$(dh + hd)(g_0, \dots, g_n) = (g_0, \dots, g_n)$$

so that h is a contracting homotopy. □

Since B_n is a free $\mathbb{Z}G$-module with basis $(1, g_1, \ldots, g_n)$, we obtain

Corollary 4.9. *$B_\bullet$ is a free resolution of $\mathbb{Z}$*

$B_\bullet$ is called the bar or standard resolution. The first name comes from the bar notation

$$[g_1|\ldots,|g_n] = (1, g_1, g_1g_2, g_1g_2g_3, \ldots)$$

With this notation, the differential is given by

$$d[g_1,\ldots,|g_n] = g_1[g_2|\ldots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\ldots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\ldots] + (-1)^n[g_1|\ldots|g_{n-1}]$$

It is often convenient to work with a smaller complex called the normalized bar resolution

$$\bar{B}_\bullet = \frac{B_\bullet}{\{[g_1|g_2|\ldots] \mid \ldots] \mid \exists i, g_i = 1\}}$$

Proposition 4.10. *$\bar{B}_\bullet$ is also a free resolution of $\mathbb{Z}$.*

Proof. Rotman, theorem 9.38. □

4.11 Low degree cohomology

Recall

$$H^n(G, A) = Ext^n_{\mathbb{Z}G}(\mathbb{Z}, A)$$

We have already seen what this means when $n = 0$. Let us look at the next few cases. A derivation or crossed homomorphism is a map $f : G \to A$ satisfying $f(gh) = f(g) + gf(h), g, h \in G$. If $a \in A$, then $f(g) = ga - a$ is an example of a derivation, called an inner derivation.

Lemma 4.12. *$H^1(G, A)$ is isomorphic to the quotient of the group of derivations from G to A by the subgroup of inner derivations.*

Proof. We can compute

$$H^1(G, A) = H^1(Hom(B_\bullet, A)) = \frac{\ker Hom(B_1, A) \to Hom(B_2, A)}{\operatorname{im} Hom(B_0, A) \to Hom(B_1, A)}$$

Elements of the numerator, called 1-cocycles, are G-homomorphisms $f : B_1 \to A$ such that

$$f(d[g|h]) = f(g[h] - [gh] + [g]) = 0$$

This means that 1-cocycles are derivations. We have to divide this space of these by the space of 1-coboundaries, which are elements in $\operatorname{im} Hom(B_0, A)$. These can be seen to be inner derivations. □

Corollary 4.13. *If G acts trivially on A, then $H^1(G, A) = Hom_{Groups}(G, A)$.*

Given groups G and N, an extension of G by N is an exact sequence of groups

$$1 \to N \to E \to G \to 1$$

This means that N is a normal subgroup of E, such that $E/N = G$. We will focus on the case where $N = A$ is abelian. The action of E on itself by conjugation stabilizes A, because it is normal. Therefore we have a homomorphism $E \to Aut(A)$. The image of A is trivial because A is abelian, this homomorphism factors through G. Therefore A is naturally a G-module. Conversely, given a G-module, we can ask whether it arises from an extension. The answer is yes.

Proposition 4.14. *The semidirect product $G \ltimes A$ is $G \times A$ as a set. When equipped with a product*

$$(g, a)(h, b) = (gh, a + gb)$$

it becomes a group, fitting into an extension

$$0 \to A \to G \ltimes A \to G \to 1$$

such that the G-module structure on A is the given structure.

Proof. See Rotman theorem 9.5. □

We can now ask what are all possible extensions of G by A, with given G-module structure? Of course, to get a reasonable answer we work up to the equivalence, where two extensions are equivalent if they fit into the following diagram

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, "="] & E \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & A \arrow[r] & E' \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

Theorem 4.15 (Schreier)**.** *There is a bijection between the set of equivalence classes of extensions of G by A with given G-module structure, and $H^2(G, A)$. The equivalence class of $G \ltimes A$ corresponds to 0.*

Proof. First we compute $H^2(G, A)$ using the normalized bar resolution. It is the group of 2-cocycles modulo the subgroup of 2-coboundaries. A 2-cocycle is a map $f : \bar{B}_2 \to A$ such that $f \circ d = 0$, and a 2-coboundary is a map of the form $k \circ d$ for some $k : \bar{B}_1 \to A$. More explicity a 2-cocycle is given by a map $f : G \times G \to A$ satisfying

$$\begin{aligned} gf(h, \ell) - f(gh, \ell) + f(g, h\ell) - f(g, h) = 0 \\ f(g, 1) = f(1, g) = 0 \end{aligned} \tag{4.1}$$

Classically a 2-cocycle is also called a "factor set". A cocycle f is a 2-coboundary if there exists a function $k : G \to A$, called a 1-cochain, such that $k(1) = 0$ and

$$f(g, h) = gk(h) - k(gh) + k(g) := dk(g, h)$$

We now summarize the basic construction, and refer to Rotman section 9.1.2 for the remaining details. Given a 2-cocycle f, define $E(f) = G \times A$ as a set with product

$$(g,a)(h,b) = (gh, a + gb + f(g,h))$$

Note that $E(0) = G \ltimes A$. Using the identities (4.1), we can see that multiplication is associative

$$[(g,a)(h,b)](\ell,c) = (gh\ell, a + gb + ghc + f(g,h) + f(gh,\ell))$$

$$= (gh\ell, a + gb + ghc + gf(h,\ell) + f(g,h\ell)) = (g,a)[(h,b)(\ell,c)],$$

$(1,0)$ is the identity, and

$$(g,a)^{-1} = (g^{-1}, -g^{-1}a - g^{-1}f(g,g^{-1}))$$

So $E(f)$ is a group. Furthermore, it fits into an extension

$$0 \to A \to E(f) \xrightarrow{\pi} G \to 1$$

where π is projection on the first factor. If we modify f by adding a coboundary associated to a cochain $k : G \to A$, then $(g,a) \mapsto (g, a + k(g))$ defines an equivalence $E(f) \cong E(f + dk)$. So the equivalence class of the above extension depends only on the cohomology class associated to f. This gives the map from H^2 to the set of equivalence classes of extensions.

Conversely, given an extension

$$0 \to A \to E \xrightarrow{\pi} G \to 1$$

choose a set theoretic map $s : G \to E$ such that $\pi \circ s = id$. Define $f(g,h) = s(x)s(y)s(xy)^{-1}$. Since $\pi(f(g,h)) = 1$, we must have $f(g,h) \in A$. This can be seen to define a 2-cocycle such that $E = E(f)$. A different choice of s will define another cocycle differing from f by a coboundary. This gives the inverse from equivalence classes of extensions to $H^2(G,A)$.

□

4.16 Applications to finite groups

Given a subgroup $H \subset G$ of a group, a G-module M is naturally also an H-module. $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module. Therefore the bar resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ can be viewed as a resolution by free $\mathbb{Z}H$-modules. This yields a natural map, called restriction

$$Res : H^n(G,M) \to H^n(H,M)$$

for any G-module.

If H has finite index, then we can define map in the opposite direction called corestriction or transfer

$$Cor : H^n(H,M) \to H^n(H,M)$$

The idea is as follows. Choose a set of representatives $g_1, \dots, g_N$ of G/H. For $n = 0$, we define

$$Cor : M^H = H^0(H, M) \to H^0(G, M) = M^G$$

by

$$Cor(m) = \sum_i g_i m$$

This is independent of the choice of representatives. For higher n, we can use a dimension shifting technique. Embed M into an injective $\mathbb{Z}G$-module E (which we proved to exist) Then

$$H^1(G, M) = H^0(G, E/M)/\operatorname{im} H^0(H, E)$$
$$H^n(G, M) = H^{n-1}(G, E/M), \quad n > 1$$

So the definition of Cor can be reduced to $n = 0$. A detailed construction can be found in Rotman section 9.6, or also in Brown's book.

Lemma 4.17. *If $H \subset G$ is a subgroup of index $N < \infty$, the composition of restriction and corestriction*

$$H^n(G, M) \to H^n(H, M) \to H^n(G, M)$$

is multiplication by N.

Sketch. This can be reduced to the case $n = 0$ by construction. If $m \in M^G$, then the composition of the above maps sends

$$m \mapsto \sum_{i=1}^{N} g_i m = \sum_{i=1}^{N} m = Nm$$

□

Theorem 4.18. *If G is a finite group of order N, then for any G-module M and $n > 0$, we have $NH^n(G, M) = 0$.*

Proof. Let $H = \{1\}$. Then multiplication by N is the same as the composite

$$H^n(G, M) \to H^n(1, M) \to H^n(G, M)$$

But $H^n(1, M) = 0$ when $n > 0$.

□

Theorem 4.19 (Schur-Zassenhaus)**.** *Let G be a finite group of order mn, where $(m, n) = 1$. If K is a normal subgroup of order n, then G is a semidirect product of K by G/K.*

Proof. We prove this when K is abelian. The general case can be reduced to this with additional work. If K is abelian, it suffices to prove that $H^2(G/K, K) = 0$. Let $\mu : K \to K$ be multiplication by m. This is an isomorphism of G-modules, because m is coprime to $|K|$. Therefore this induces an isomorphism $\mu_* : H^2(G/K, K) \to H^2(G/K, K)$. On the other hand μ_* is the same as multiplication by m. But this is zero by the previous theorem.

□

4.20 Topological interpretation

Suppose that X is a topological space. Let G be a group such that each $g \in G$, defines a homeomorphism $g : X \to X$. Also suppose that $g_1(g_2(x)) = (g_1g_2)(x)$ for $g_1, g_2 \in G$. Then we say that G acts on X. We give the set of orbits X/G the quotient topology. In general, this can be quite wild, even when X is nice. However, it has reasonable properties if the action is fixed point free and proper, which means that every point $x \in X$ has an open neighbourhood U such that $gU \cap U = \emptyset$ when $g \neq 1$. For example, the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ by translation satisfies these conditions, and $\mathbb{R}^n/\mathbb{Z}^n$ is the n-torus.

Theorem 4.21. *If X is contractible, and G has a fixed point free and proper action, then $H^*(X) \cong H^*(G, \mathbb{Z})$.*

Proofs can be found in Brown's or Weibel's books. Since $\mathbb{R}^n$ is contractible, we obtain:

Corollary 4.22. *Group cohomology of $\mathbb{Z}^n$ is given by $H^i(\mathbb{Z}^n, \mathbb{Z}) = H^i(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{Z})$*

Standard methods from algebraic topology allows us to compute the cohomology of the torus.

$$H^i(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{Z}) = \mathbb{Z}^{\binom{n}{i}} \tag{4.2}$$

More canonically, the answer can be expressed as an exterior power.

When $n = 1$, we can prove this algebraically as follows. Let us write $G = \mathbb{Z}$. We can identify $\mathbb{Z}G \cong \mathbb{Z}[t, t^{-1}]$ where $N \in G$ corresponds to t^N. Then

$$0 \to \mathbb{Z}[t, t^{-1}] \stackrel{t-1}{\to} \mathbb{Z}[t, t^{-1}] \to \mathbb{Z} \to 0$$

gives a free resolution. Therefore for any $\mathbb{Z}[t, t^{-1}]$-module M, we can Hom this resolution into it to obtain the complex

$$M \stackrel{t-1}{\to} M$$

So that

$$H^i(G, M) = \begin{cases} \ker(t-1) : M \to M & i = 0 \\ \mathrm{coker}(t-1) : M \to M & i = 1 \\ 0 & i > 1 \end{cases}$$

When $M = \mathbb{Z}$, we obtain $\mathbb{Z}$ in degrees 0 and 1, which is consistent with (4.2). We will return to deal with the the case $n > 1$ later on.

Chapter 5

Derived Functors and Tor

Refs.

1. Cartan, Eilenberg, Homological algebra
2. Grothendieck, Sur quelques points d'algèbre homologique, Tohoku 1957
3. Mitchell, Theory of categories
4. Rotman, Intro to homological algebra.
5. Weibel, An introduction to homological algebra

5.1 Abelian categories

We start with some category theory. A category A is called abelian if it behaves like the category Mod_R. Rotman section 5.5 treats abelian categories in some detail. Most other books on homological algebra do as well. Let's write down a long list list of conditions on category A, which hold when $A = Mod_R$.

A1. $Hom_A(M, N)$ is an abelian group for every pair of objects M, N.

A2. Composition satisfies $f \circ (g + h) = f \circ g + f \circ h$ whenever both sides are defined. Similary, $(g + h) \circ f = g \circ f + h \circ f$ when this makes sense.

A3. There is a zero object satisfying $Hom_A(0, M) = Hom_A(M, 0) = 0$ for all M.

A4. For any pair of objects M, N we can form a direct sum, characterized up to isomorphism by $Hom(M \oplus N, T) = Hom(M, T) \oplus Hom(N, T)$ and $Hom(T, M \oplus N) = Hom(T, M) \oplus Hom(T, N)$

A5. Given a morphism $f : M \to N$, we can form an object $\ker f$ with a morphism $\ker f \to M$ characterized by $Hom(T, \ker f) = \ker Hom(T, M) \to Hom(T, N)$.

A6. Given $f : M \to N$, we can form an object $\operatorname{coker} f$ with a morphism $M \to \operatorname{coker} f$, characterized by $Hom(\operatorname{coker} f, T) = \ker Hom(N,T) \to Hom(M,T)$.

A7. Given $f : M \to N$, there exists an object $\operatorname{im} f$ with morphisms $M \to \operatorname{im} f$ and $\operatorname{im} f \to N$ such that their composition is f. We also require that $\operatorname{im} f$ is both $\operatorname{coker}(\ker f \to M)$ and $\ker(N \to \operatorname{coker} f)$. (A bit more precisely, these are canonically isomorphic.)

A category is called additive if A1-A4 hold, and it is called abelian if they all hold. The last axiom is the hardest to fathom. It is trying to capture the idea that in Mod_R, f can be factored through a surjective homomorphism $M \to \operatorname{im} f$ followed by an injective homomorphism $\operatorname{im} f \to N$. Since injectivity and surjectivity are not categorical notions, we replace them by saying that they are kernels or cokernels. To appreciate further subtleties, see example 5.5.

Example 5.2. *Mod_R is an abelian category.*

Example 5.3. *The category of finitely generated modules over a left noetherian ring is abelian. In particular, this applies to finitely generated abelian groups.*

Example 5.4. *The category of free abelian groups is additive but not abelian, because cokernels need not exist.*

Example 5.5. *The category of Hausdorff topological abelian groups and continuous homomorphisms satisfies A1-A6. The operations are the usual ones except for the cokernel. The cokernel of $f : M \to N$ in this category is the quotient $N/\overline{f(M)}$. However, if $f(M)$ is not closed, the map from $\operatorname{coker}(\ker f \to M) = M/\ker f$ to $\ker(N \to \operatorname{coker} f) = \overline{f(M)}$ is not an isomorphism. So A7 fails.*

Here is a simple yet powerful observation.

Proposition 5.6. *If A is abelian (resp. additive), then so is the opposite category A^{op}. This has the same objects as A but arrows are reversed, so that $Hom_{A^{op}}(N,M) = Hom_A(M,N)$.*

Proof. The axioms are self dual. $\square$

Therefore

Example 5.7. *Mod_R^{op} is an abelian category. (NB: This should not be confused with $Mod_{R^{op}}$.)*

Given an abelian category, we can do most of what we have done so far in class. In particular, we can talk about exact sequences, injectives, projectives, complexes, and homology. We also note the following remarkable fact:

Theorem 5.8 (Freyd-Mitchell). *Any small*[1] *abelian category can be embedded into a category modules over a ring in such a way that Hom's are the same, and exact sequences are the same.*

A proof can be found in Mitchell's book. Since we will mostly be working with explicit examples, we won't really need it. But it is reassuring to know that one can pretend that an abstract abelian category is a category of modules, without loosing too much. Also this means that various standard results such as the 5-lemma, snake lemma, etc. can be extended to an arbitrary abelian category.

In order to do more homological algebra, we need the following.

Definition 5.9. *An abelian category has enough injectives if for every object M, there exists an injective object I and a morphism $f : M \to I$ such that* $\ker f = 0$.

Example 5.10. Mod_R *has enough injectives.*

Example 5.11. Mod_R^{op} *has enough injectives. This is because an injective in* Mod_R^{op} *is a projective module, and every module is the quotient of a projective module.*

Example 5.12. *The category of finitely generated abelian groups does not have enough injectives.*

5.13 Derived functors

Definition 5.14. *A functor $F : A \to B$ between additive categories is called additive if $F(f + g) = F(f) + F(g)$, for every $f, g \in Hom(M, N)$.*

Lemma 5.15. *If F is additive, then $F(M \oplus N) \cong F(M) \oplus F(N)$.*

Proof. There are morphisms $i : M \to M \oplus N$, $j : N \to M \oplus N$, $p : M \oplus N \to M$, and $q : M \oplus N \to N$ such that $pi = id_M$, $qj = id_N$, $pj = 0$, $qi = 0$, and $ip + jq = id_{M\oplus N}$. The existence of such a collection of morphisms satisfying these relations characterizes the direct sum. The collection $F(i), \dots$ would satisfy the same relations, therefore $F(M \oplus N)$ must be isomorphic to $F(M) \oplus F(N)$. $\square$

Definition 5.16. *An additive (covariant) functor $F : A \to B$ from one abelian category from one category to another is left (right) exact if whenever*

$$0 \to M \to N \to P \to 0$$

[1] This is a set theoretic condition. In Gödel-Bernays, or similar set theory, one distinguishes between sets and classes. Classes are allowed to be very big, but sets are not. For example, one can form the class of all sets, but it wouldn't be a set. One is not allowed to form the class of all classes, thus avoiding the standard paradox of Cantor's set theory. A category is called small if the collection of the objects and morphisms form a set as opposed to a proper class.

is exact,

$$0 \to F(M) \to F(N) \to F(P)$$

(resp.

$$F(M) \to F(N) \to F(P) \to 0$$

is exact.) A functor which both right and left exact is called exact.

We can also handle contravariant functors $F : A \to B$ by treating them as covariant functors from $F : A^{op} \to B$. These are left or right exact if the second form is.

Let us fix a left exact functor $F : A \to B$, and let assume that A has enough injectives. An injective resolution of an object M is an exact sequence

$$0 \to M \to I^0 \to I^1 \dots$$

with I^i injective. By an argument dual to what we did for projective resolutions, we can see

Lemma 5.17. *Every M possesses an injective resolution.*

By arguments similar to what we did for Ext, we have

Theorem/Def 5.18. *We define the right derived functors*

$$R^i FM = H^i(F(I^\bullet))$$

The isomorphism class of these objects do not depend on the resolution.

Theorem 5.19. *R^iF extend to additive functors from $A \to B$ with $R^0F = F$. Given a short exact sequence*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

there is a long exact sequence

$$\dots R^iFM_1 \to R^iFM_2 \to R^iFM_3 \to R^{i+1}FM_1 \dots$$

Derived functors were introduced by Cartan and Eilenberg in their book in the mid 1950's in order to unify several disparate theories. Grothendieck carried the story further in his landmark paper shortly thereafter.

Example 5.20. *Fix a module N, and consider the left exact functor $Hom_R(-, N) : Mod_R^{op} \to Ab$. The right derived functors*

$$R^i Hom_R(-, N) = Ext^i_R(-, N)$$

by definition.

However, if we fix M, and consider $Hom_R(M, -) : Mod_R \to Ab$ we can also take derived functors. A much less obvious fact is

Theorem 5.21.
$$R^i Hom_R(M,-) \cong Ext^i_R(M,-)$$

Since Rotman does not appear to do this, we indicate the proof. A δ-functor is a sequence of functors $F^i : A \to B$ such that for any exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

there is a long exact sequence

$$\dots F^i M_1 \to F^i M_2 \to F^i M_3 \to F^{i+1} M_1 \dots$$

such that the connecting maps are natural in the appropriate sense. For example, the sequence of right derived functors $F^i = R^i F$ forms a delta functor. A functor F is called effacable if for any M, there exists an exact sequence $0 \to M \to I$ such that $F(I) = 0$.

Theorem 5.22. *Suppose that if F^i is a δ-functor such that for any $i > 0$ F^i is effacable. Then $F^i = R^i F^0$.*

Proof. This follows from the results of chap II sections 2.2-2.3 of Grothendieck. □

Proof of theorem 5.21. By results proved earlier $Ext^i(M,-)$ is a δ-functor. Furthermore if $i > 0$ and I is injective, $Ext^i(M, I) = 0$. Therefore $Ext^i(M,-)$ is effacable. So the result follows from the previous theorem. □

A right exact functor $F : A \to B$ is the same thing as a left exact functor $F' : A^{op} \to B^{op}$. So that we can take right derived of F'. When the story is translated back to F, we arrive at the notion of a left derived functor. To be explicit, given M, choose a projective resolution

$$\dots P_1 \to P_0 \to M \to 0$$

We need to assume that A has enough projectives to guarantee this exists. Set

$$L_i F = H_i(F(P_\bullet))$$

The key properties are:

- These are independent of the choice of resolution.
- These are additive functors from $A \to B$ such that $L_0 F = F$.
- A short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

gives rise to a long exact sequence

$$\dots L_i M_1 \to L_i M_2 \to L_i M_3 \to L_{i-1} M_1 \dots$$

5.23 Tor functors

Given a right R-module M, consider the functor $T : Mod_R \to Ab$ defined by

$$T(N) = M \otimes_R N$$

This is a right exact functor. We define

$$Tor_i^R(M,N) = L_i T$$

Then given

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

we get an exact sequence

$$\ldots Tor_1^R(M,N_1) \to M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3 \to 0$$

This will allow us to compute this in principle, but we still need a few more tricks.

Proposition 5.24. *If N is flat, then $Tor_i(M,N) = 0$ for $i > 0$.*

Proof. This follows from the construction

$$Tor_i(M,N) = L_i T(N) = H_i(P_\bullet \otimes N)$$

where $P_\bullet \to M \to 0$ is a projective resolution. Since N is flat,

$$\ldots P_1 \otimes N \to P_0 \otimes N \to M \otimes N \to 0$$

is exact. This means that $P_\bullet \otimes N$ has no homology in positive degrees.

□

Theorem 5.25. *Suppose that R is a (commutative) integral domain with field of fractions K. If $f \in R$ is nonzero,*

$$Tor_i(M, R/fR) = \begin{cases} \{m \in M \mid fm = 0\} & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$$

$$Tor_i(M, K/R) = \begin{cases} \{m \in M \mid \exists f \in R, fm = 0\} & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$$

Remark 5.26. *"Tor" is short for "torsion". The theorem partly explains why this name makes sense.*

Proof. We have an exact sequence

$$0 \to R \xrightarrow{f} R \to R/fR \to 0$$

Since R is flat, we obtain

$$0 = Tor_i(M, R) \to Tor_i(M, R/fR) \to Tor_{i-1}(M, R) = 0$$

for $i > 1$. We can identify $M \otimes R = M$ and the map $1 \otimes f$ with f. Therefore we also have

$$0 \to Tor_1(M, R/fR) \to M \xrightarrow{f} M$$

The proof of the second isomorphism is similar. We use the sequence

$$0 \to R \to K \to K/R \to 0$$

and the fact that K is flat. □

We note the following useful symmetry property.

Theorem 5.27. *Under the identification of left (right) R-modules with right (left) R^{op}-modules,*

$$Tor_i^R(M, N) \cong Tor_i^{R^{op}}(N, M)$$

In particular, if R is commutative,

$$Tor_i^R(M, N) \cong Tor_i^R(N, M)$$

Comments about the proof. The result is stated in theorem 7.1 in Rotman, but the proof given there is incomplete. What's missing is the fact that one can compute Tor using a projective resolution of the second variable. See theorem 2.7.2 of Weibel for this. (We may do this later, if there is time.) □

5.28 Homology of a group

Fix a group G and G-module i.e. $\mathbb{Z}G$-module M. We regard $\mathbb{Z}$ as a left (and also right) G-module with trivial G-action. Earlier we defined

$$H^i(G, M) = Ext^i_{\mathbb{Z}G}(\mathbb{Z}, M)$$

In the current language, we could also define it as the right derived functors of the left exact functor

$$M \mapsto M^G$$

Recall that $M^G \subset M$ is the submodule of element invariant under G. It is the largest submodule on which G acts trivially. Let M_G be the largest quotient module on which G acts trivially. More explicitly

$$M_G = M/\{gm - m \mid m \in M, g \in G\}$$

Lemma 5.29. *Treating $\mathbb{Z}$ as a right $\mathbb{Z}G$-module with trivial G action,*

$$M_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M$$

Proof. We define a surjective ring homomorphism $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ by

$$\epsilon(\sum_i n_i g_i) = \sum_i n_i$$

Let $I = \ker \epsilon$. This is the two sided ideal generated by $g - 1$ with $g \in G$. Consider the exact sequence

$$0 \to I \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

Tensoring with M gives a sequence

$$I \otimes M \to \mathbb{Z}G \otimes M \to \mathbb{Z} \otimes M \to 0$$

We can identify the middle module with M, and the image of the first map with $\{(g-1)m \mid g \in G, m \in M\}$. So the lemma is now proved. $\square$

Corollary 5.30. *$M \mapsto M_G$ is right exact.*

We define group homology by

$$H_i(G, M) = Tor_i^{\mathbb{Z}G}(\mathbb{Z}, M)$$

The lemma shows that

$$H_0(G, M) = M_G$$

Before describing the next result, we recall that the commutator (or derived) subgroup $[G, G] \subseteq G$ is the normal subgroup generated by all commutators $ghg^{-1}h^{-1}$. The quotient $G/[G, G]$ can be characterized as the largest abelian quotient of G.

Theorem 5.31. $H_1(G, \mathbb{Z}) \cong G/[G, G]$

Proof. With the above notation, we have an exact sequence

$$Tor_1^{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z}) \to Tor_1^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) \to I \otimes_{\mathbb{Z}G} \mathbb{Z} \to \mathbb{Z}G \otimes_{\mathbb{Z}G} \mathbb{Z} \xrightarrow{r} \mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z} \to 0$$

By theorem 6.2

$$Tor_1^{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z}) \cong Tor_1^{\mathbb{Z}G^{op}}(\mathbb{Z}, \mathbb{Z}G)$$

In fact $g \mapsto g^{-1}$ induces an isomorphism between $\mathbb{Z}G$ and $\mathbb{Z}G^{op}$. Therefore

$$Tor_1^{\mathbb{Z}G^{op}}(\mathbb{Z}, \mathbb{Z}G) \cong Tor_1^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G) = 0$$

because $\mathbb{Z}G$ is flat. The map marked r above can be identified with the identity $\mathbb{Z} \to \mathbb{Z}$. By definition $H_1(G, \mathbb{Z}) = Tor_1(\mathbb{Z}, \mathbb{Z})$. Therefore, we can conclude

$$H_1(G, \mathbb{Z}) \cong I \otimes_{\mathbb{Z}G} \mathbb{Z} = I \otimes_{\mathbb{Z}G} \mathbb{Z}G/I = I/I^2$$

Let $f : G \to I/I^2$ be given by $f(g) = g - 1 \, mod \, I^2$. Since

$$(gh - 1) - (g - 1) - (h - 1) = (g - 1)(h - 1) \in I^2$$

f is a homomorphism. Since I/I^2 is abelian, it factors through a homomorphism $\bar{f} : G/[G, G] \to I/I^2$. An explicit inverse is constructed on page 540 of Rotman, So $\bar{f}$ is an isomorphism. $\square$

Chapter 6

Further techniques

Refs.

1. Matsumura, Commutative algebra
2. Rotman, Intro. to homological algebra
3. Weibel, An intro to homological algebra

6.1 Double complexes

Recall that we defined $Tor_i^R(N, -)$ as the right derived functor of $N \otimes_R -$. Given a left R-module M, consider the right exact functor

$$- \otimes_R M : Mod_{R^{op}} \to Ab$$

Let ${}^*Tor_i^R(-, M)$ denote the left derived functors. We claim

Theorem 6.2. *There is a natural isomomorphism* ${}^*Tor_i^R(N, M) \cong Tor_i^R(N, M)$

We prove this by introducing a technique that is important on its own. A double complex, or bicomplex, of modules (or more generally objects in an abelian category) is a collection of modules (objects) $C_{\bullet\bullet}$ indexed by $\mathbb{Z} \times \mathbb{Z}$, and morphisms $d_h : C_{i,j} \to C_{i-1,j}$ and $d_v : C_{i,j} \to C_{i,j-1}$ satisfying

$$d_h^2 = d_v^2 = d_h d_v + d_v d_h = 0 \tag{6.1}$$

It is helpful to visualize this in the ij-plane. Then d_h maps horizontally, and d_v vertically. If $C_{ij} = 0$ for negative i or j, then we say this is a first quadrant double complex. Here is a basic example.

Example 6.3. *Suppose that $P_\bullet$ and $Q_\bullet$ are complexes of right and left modules over a ring R. Set $C_{ij} = P_i \otimes_R Q_j$ with differentials*

$$d_h(p \otimes q) = dp \otimes q$$

$$d_v(p \otimes q) = (-1)^i p \otimes dq$$

The first two relations of (6.1) say that the horizontal lines $C_{\bullet,j}$ and vertical lines $C_{i,\bullet}$ are complexes. The total complex

$$Tot(C)_p = \bigoplus_{i+j=p} C_{ij}$$

Note that this is a finite direct sum, assuming our double complex is first quadrant. In general, this construction is only available for abelian categories with infinite direct sums, such as Mod_R. We define the differential $d : Tot(C)_p \to Tot(C)_{p-1}$ by $d = d_h + d_v$. The relations (6.1) imply that $d^2 = 0$.

Example 6.4. *Continuing the previous example, $Tot(P_\bullet \otimes Q_\bullet)$ is called the tensor product of the complexes $P_\bullet$ and $Q_\bullet$.*

The last relation of (6.1) also implies that after adjusting signs $\pm d_v : C_{\bullet,j} \to C_{\bullet,j-1}$ is a map of complexes. This implies that d_v induces a map

$$\bar{d}_v : H_i(C_{\bullet,j}, d_h) \to H_i(C_{\bullet,j-1}, d_h)$$

It is easy to see that $\bar{d}_v^2 = 0$ so that

$$\ldots H_i(C_{\bullet,j+1}, d_h) \to H_i(C_{\bullet,j}, d_h) \to H_i(C_{\bullet,j-1}, d_h) \ldots$$

forms a complex. We now come to the key problem: *Suppose we are given $H_i(C_{\bullet,j}, d_h)$ and the maps $\bar{d}_v$ for all i, j, can we compute the homology of the total complex?* The answer is almost. The precise statement is very complicated in that it involves the notion of a *spectral sequence*. Rather than getting into that now, we consider a special case which is sufficient to prove theorem 6.2.

Theorem 6.5. *Suppose that $C_{\bullet\bullet}$ is a first quadrant double complex, and that $H_i(C_{\bullet,j}) = 0$ for $i > 0$ and all j. Then $H_i(Tot(C)_\bullet)$ is isomorphic to the ith homology of*

$$\ldots H_0(C_{\bullet,i+1}) \to H_0(C_{\bullet,i}) \to H_0(C_{\bullet,i-1}) \ldots$$

Sketch. We define a new double complex $\tilde{C}_{\bullet\bullet}$ by adding the vertical line over $i = -1$ to $C_{\bullet\bullet}$ as indicated below

$$\begin{tikzcd}
H_0(C_{\bullet,j}) \arrow[d, "\bar{d}_v"] & C_{0,j} \arrow[l, "\pm\pi"'] \arrow[d, "d_v"] & C_{1,j} \arrow[l, "d_h"'] \arrow[d, "d_v"'] \\
H_0(C_{\bullet,j-1}) & C_{0,j-1} \arrow[l, "\pm\pi"'] & C_{1,j-1} \arrow[l, "d_h"']
\end{tikzcd}$$

where π denotes the natural projections. We have an exact sequence of complexes

$$0 \to Tot(C) \to Tot(\tilde{C}) \to (H_0(C), \bar{d}_v)[-1] \to 0$$

where $[-1]$ indicates a shift by -1. Applying lemma 2.73 of Weibel on pp 59-60 shows that the complex in the middle is acyclic. The long exact sequence for homology does the rest. $\square$

There is an obvious symmetry here. If we interchange the roles of d_h and d_v, we get an analogous statement.

Corollary 6.6. *Suppose that $H_j(C_{i,\bullet}) = 0$ for $j > 0$ and all i. Then $H_i(Tot(C)_\bullet)$ is isomorphic to the ith homology of*

$$\ldots H_0(C_{i+1,\bullet}) \to H_0(C_{i,\bullet}) \to H_0(C_{i-1,\bullet}) \ldots$$

Proof of theorem 6.2. Let $P_\bullet \to N$ and $Q_\bullet \to M$ be projective resolutions. Let $C_{ij} = P_\bullet \otimes_R Q_\bullet$ be the double complex given by tensor product as in 6.3. Since Q_j is flat,

$$\ldots P_1 \otimes Q_j \to P_0 \otimes Q_j \to N \otimes Q_j \to 0$$

is exact. Therefore $H_i(P_\bullet \otimes Q_j) = 0$ for $i > 0$, and

$$H_0(P_\bullet \otimes Q_j) \cong N \otimes Q_j$$

Thus, from theorem 6.5, we conclude that $H_i(Tot(C)_\bullet)$ is isomorphic to

$$H_i(M \otimes Q_\bullet) \cong Tor_i(N, M)$$

For similar reasons, we have isomorphisms

$$H_i(Tot(C)_\bullet) \cong H_i(P_\bullet \otimes M) \cong {}^*Tor_i(N, M)$$

Putting these isomorphisms together proves the theorem.

□

6.7 Koszul complexes

Let R be a commutative ring for the rest of this section. Suppose that $x \in R$ is a nonzero divisor. Then

$$0 \to R \xrightarrow{x} R \to R/(x) \to 0$$

is a free resolution of $R/(x)$, which can be used to compute various Exts and Tors. Let us denote this projective resolution by $K(x)$, where K stands for Koszul. To be clear $K(x)_0 = K(x)_1 = R$ and all other terms are zero. This is the building block for more general Koszul complexes. Given two elements $x_1, x_2 \in R$, we define the Koszul complex as the tensor product $K(x_1, x_2) = K(x_1) \otimes K(x_2)$. Recall that this the total complex of the tensor product as a double complex, which after identifying $R \otimes R = R$ is

$$\begin{tikzcd} R \arrow[d, "x_2"] & R \arrow[l, "-x_1"'] \arrow[d, "x_2"] \\ R & R \arrow[l, "x_1"] \end{tikzcd}$$

Therefore

$$K(x_1, x_2) = R \to R^2 \to R$$

with differentials

$$a \mapsto (-x_1 a, x_2 a)$$

$$(a, b) \mapsto x_2 a + x_1 b$$

We say that the pair $x_1, x_2 \in R$ is regular if x_1 is a nonzero divisor in R, x_2 is (or more precisely reduces to) a nonzero divisor in $R/(x_1)$. Note that the definition depends on the order.

Proposition 6.8. *If x_1, x_2 is regular, then $K(x_1, x_2)$ is a free resolution of $R/(x_1, x_2)$.*

Proof. Since x_1 is a nonzero divisor in R, $K(x_1)$ is non homology in postive degrees. Therefore theorem 6.5 shows that $K(x_1, x_2)$ has the same homology as

$$R/(x_1) \xrightarrow{x_2} R/(x_1)$$

Since x_2 is a nonzero divisor in $R/(x_1)$. The last complex resolves $R/(x_1)/(x_2) = R/(x_1, x_2)$.

$\square$

Given a finite sequence $x_1, \ldots, x_n \in R$, define

$$K(x_1, \ldots, x_n) = K(x_1, \ldots, x_{n-1}) \otimes K(x_n)$$

inductively.

Lemma 6.9. *$K(x_1, \ldots, x_n)$ is a complex of free modules. It has rank $\binom{n}{i}$ in degree i.*

Proof. Since

$$K(x_1, \ldots, x_n)_i = K(x_1, \ldots, x_{n-1})_i \otimes R \oplus K(x_1, \ldots, x_{n-1})_{i-1} \otimes R$$

this follows by induction and Pascal's triangle identity. $\square$

A better proof of the previous lemma comes by identifying $K(x_1, \ldots, x_n)_i$ with the exterior power $\wedge^i R^n$. See Matsumura p 134.

A finite sequence $x_1, \ldots, x_n \in R$ is called regular if x_i is a nonzero divisor in $R/(x_1, \ldots, x_{i-1})$ for each i. Proposition 6.8 extends to this the more general case, with essentially the same proof.

Proposition 6.10. *If $x_1, \ldots, x_n$ is regular, then $K(x_1, \ldots, x_n)$ is a free resolution of $R/(x_1, \ldots x_n)$.*

Earlier, we gave a computation of the group cohomology of $G = \mathbb{Z}^n$ using topology. We are now in a position to do it algebraically.

Theorem 6.11. $H^i(G, \mathbb{Z}) = \mathbb{Z}^{\binom{n}{i}}$.

Proof. We can identify the group ring $\mathbb{Z}G = \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$. The sequence $x_1, \ldots, x_n$ is easily seen to be regular. Therefore $K = K(x_1, \ldots, x_n)$ gives a free resolution of $\mathbb{Z}$. So we can compute $H^i(G, \mathbb{Z})$ as $H^i(Hom(K, \mathbb{Z}))$. The complex $Hom(K, \mathbb{Z})$ is just

$$\ldots \mathbb{Z}^{\binom{n}{i+1}} \xrightarrow{0} \mathbb{Z}^{\binom{n}{i}} \xrightarrow{0} \mathbb{Z}^{\binom{n}{i-1}} \ldots$$

So the theorem follows.

$\square$

Chapter 7

Applications to commutative algebra

Refs.

1. Eisenbud, Commutative algebra
2. Rotman, Intro. to homological algebra
3. Weibel, Intro. to homological algebra

7.1 Global dimensions

For this section, R is a not necessarily commutative ring. Given an R-module M, we say that the projective dimension $pd(M) \leq n \in \mathbb{N} \cup \{\infty\}$ if there exists a projective resolution

$$\dots P_1 \to P_0 \to M \to 0,$$

such that $P_i = 0$ for $i > n$. We say $pd(M) = n$, if $pd(M) \leq n$ and not for anything smaller. Clearly $pd(M) = 0$ if and only if M is projective. We have to following test

Proposition 7.2. *The following are equivalent*

1. $pd(M) \leq n$
2. *For all* N, $Ext_R^{n+1}(M, N) = 0$
3. *For all* $m > n$ *and for all* N, $Ext_R^m(M, N) = 0$.

Proof. This was a (slightly misstated) homework problem; or see Rotman prop 8.6. □

We define the (left) global dimension of R to be

$$\operatorname{gldim}(R) = \sup pd(M)$$

From the previous proposition, we deduce

Corollary 7.3. $\operatorname{gldim}(R) \leq n$ *if and only if* $Ext^{n+1}(M, N) = 0$ *for all* M *and* N.

Example 7.4. *If R is a field, or more generally a division ring, all modules are free by standard arguments in linear algebra. Therefore* $\operatorname{gldim} R = 0$.

Example 7.5. *Let $R = \mathbb{Z}[t]/(t^2 - 1)$. A previous homework problem showed that group cohomology of $H^i(Z/2, Z/2) = Ext^i(\mathbb{Z}, \mathbb{Z}/2) \neq 0$ for all i. Therefore* $\operatorname{gldim} R = \infty$.

We can define the injective dimension $id(M)$ of a module as the length of the shortest injective resolution

$$0 \to M \to I^0 \to I^1 \dots I^n \to 0$$

This is the same the projective dimension in Mod_R^{op}. Therefore, we obtain a result dual to the previous proposition.

Proposition 7.6. *The following are equivalent*

1. $id(M) \leq n$
2. *For all N,* $Ext_R^{n+1}(N, M) = 0$
3. *For all $m > n$ and for all N,* $Ext_R^m(N, M) = 0$.

By combining this with the previous corollary, we obtain

Corollary 7.7. $\operatorname{gldim} R = \sup id(M)$.

Proposition 7.8. $\operatorname{gldim} R = \sup pd(M)$ *as M varies over finitely generated left modules.*

Proof. Let $n = \sup pd(M)$ over finitely generated modules. Given a module N, it suffices to show that $id(N) \leq n$. Let

$$0 \to N \to E^0 \to E^1 \dots E^{n-1} \to C \to 0 \tag{7.1}$$

be a resolution with E^i injective. If we can show that C is injective, then we are done. We can use Baer's criterion, which says that C is injective if any homomorphism from a left ideal $I \to C$ extends to a homomorphism from $R \to C$. From the sequence

$$Hom(R, C) \to Hom(I, C) \to Ext^1(R/I, C)$$

we see that it suffices to prove that $Ext^1(R/I, C) = 0$. By breaking up (7.1) into short exact sequences

$$0 \to N \to E^0 \to E^0/N \to 0$$

etc., we find that

$$Ext^n(R/I, E^0) \to Ext^n(R/I, E^0/N) \to Ext^{n+1}(R/I, N) \to Ext^{n+1}(R/I, E^0) = 0$$

etc. Therefore we get isomorphisms

$$Ext^{n+1}(R/I, N) \cong Ext^n(R/I, E^0/N) \ldots \cong Ext^1(R/I, C)$$

Since R/I is finitely generated, $Ext^{n+1}(R/I, N) = 0$. □

7.9 Global dimension of commutative rings

From now on, let us assume that R is commutative and noetherian.

Theorem 7.10. *The global dimension*

$$\text{gldim } R = \sup_{p \in \text{Spec } R} \text{gldim } R_p$$

Proof. Suppose that M is finitely generated. Earlier we proved that

$$Ext^i_R(M, N)_p = Ext^i_{R_p}(M_p, N_p) \tag{7.2}$$

Therefore $Ext^i_R(M, N) = 0$ for $i > \sup \operatorname{gldim} R_p$. This proves $\operatorname{gldim} R \leq \sup \operatorname{gldim} R_p$. So it remains to prove the opposite inequality. For this we need

Lemma 7.11. *Any (finitely generated) R_p-module is isomorphic to M_p for some (finitely generated) R-module M.*

Proof. If $\mathcal{M}$ is a finitely generated R_p-module, we can find a presentation

$$R_p^n \xrightarrow{A} R_p^m \to \mathcal{M} \to 0$$

We can find a matrix B over R such that $A = \frac{1}{f}B$ with $f \notin p$. It follows that $\mathcal{M} = M_p$ where $M = R^m/BR^n$.

If $\mathcal{M}$ is not finitely generated then we can take $M = \mathcal{M}$ regarded as an R-module. □

From the lemma along with (7.2), we obtain $\operatorname{gldim} R_p \leq \operatorname{gldim} R$.

□

To understand global dimensions of commutative noetherian rings, we can, by the previous theorem, focus on the case where R commutative noetherian local ring. We now assume this. Let m be the unique maximal ideal, and $k = R/m$ the residue field.

Proposition 7.12. *If M is a finitely generated R-module, then*

$$pd(M) \leq n \Leftrightarrow Tor^R_{n+1}(M, k) = 0$$

Proof. If $pd(M) \leq n$, then M has a projective resolution $P_\bullet$ of length $\leq n$, therefore $Tor_{n+1}(M, k) = H_i(P_\bullet \otimes k) = 0$.

For the converse, by dimension shifting, it is enough to consider $n = 0$. That is assuming $Tor_1(M, k) = 0$, we have to show that M is projective. Choose set of generators $m_i \in M$ which reduce to a basis of $M \otimes k$. Consider the map $R^n \to M$ sending the ith basis vector of R^n to m_i. We have an exact sequence

$$0 \to K \otimes R^n \to M \to 0$$

After tensoring with k, we obtain

$$0 = Tor_1(M, k) \to K \otimes k \to k^n \to M \otimes k \to 0$$

The last map is bijective by construction, therefore $K \otimes k = 0$. This implies $K = 0$ by Nakayama's lemma. $\square$

The ring R is called regular if m can be generated by d elements, where $d = \dim R$ is the Krull dimension (the maximal length of a chain of prime ideals $p_0 \subsetneq p_1 \ldots \subsetneq p_d$). The importance of regularity comes from algebraic geometry. A point $x \in X$ in algebraic variety is nonsingular if and only if the corresponding local ring is regular.

Example 7.13. *If k is a field, then the localization $= k[x_1, \ldots, x_d]_{(x_1,\ldots,x_d)}$ is regular, since the maximal is generated by x_i and $d = \dim R$. In geometric terms, this says that the origin of affine space $\mathbb{A}^d_k$ is nonsingular.*

The following fundamental theorem gives an elegant characterization.

Theorem 7.14 (Auslander-Buschsbaum-Serre)**.** *A noetherian local ring R is regular if and only if* $\operatorname{gldim} R < \infty$*. The global dimension of R coincides with the Krull dimension.*

Proof. We only prove one direction. Suppose that R is regular, then we can find $d = \dim R$ element $x_1, \ldots, x_d$ generating m. Then $x_1, \ldots, x_d$ is a regular sequence by Eisenbud corollary 10.15. Therefore the Koszul complex $K_\bullet = K(x_1, \ldots, x_d)$ gives a projective resolution of k of length d. It follows that $Tor_i(M, k) = H_i(M \otimes K_\bullet) = 0$ for $i > d$. If M is finitely generated, this implies that $pd(M) \leq d$ be the previous proposition. This implies $\operatorname{gldim} R \leq d$. Since $Tor_d(k, k) = k \neq 0$, we must have equality. $\square$

Corollary 7.15. *The localization of a regular local ring is regular local.*

We define a commutative ring to be regular if all of its local rings are regular. The last corollary says that regular local rings are regular in this sense. Given a field k, a prime ideal $I \subseteq k[x_1, \ldots, x_n]$ defines an affine algebraic variety

$$X = V(I) = \{a \in \mathbb{A}^n_k \mid \forall f \in I, f(a) = 0\}$$

The ring $k[x_1, \ldots, x_n]/I$ is called the coordinate ring. X is called nonsingular if the ring is regular.

Corollary 7.16. *A regular ring of finite Krull dimension has finite global dimension. In particular, this is the case for the coordinate ring of a nonsingular affine algebraic variety.*

7.17 Regular local rings are UFDs

We will continue to assume rings are commutative. Recall that an integral domain is a unique factorization domain, or a factorial ring, if every nonzero element is a product of a unit times a product of irreducible elements in an essentially unique way. We give some useful criteria for this property. First recall that a prime ideal p is a minimal prime of an ideal I if it contains I and there are no primes between I and p. The height of a prime ideal p is the length of the longest chain $p_0 \subsetneq p_1 \ldots \subsetneq p_n = p$. In algebro-geometric terms minimal primes correspond to irreducible components of $V(I)$, and the height of p is the codimension of the subvariety defined by p.

Proposition 7.18. *Let R be a noetherian domain. The following are equivalent*

1. *R is a UFD .*
2. *Every minimal prime of a principal ideal is itself principal.*
3. *Every height one prime is principal.*

If $x \in R$ generates a prime ideal, and if $R[x^{-1}]$ is a UFD, then so is R.

Proof. This follows from proposition 3.11, corollary 10.6, and lemma 19.20 of Eisenbud. □

We will use the next lemma below.

Lemma 7.19. *If P is a rank one projective module which admits a finite resolution by finitely generated free modules, then P is free.*

Proof. Combine proposition 19.16 and lemma 19.18 from Eisenbud. □

Theorem 7.20 (Auslander-Buchsbaum)**.** *A regular local ring is a UFD.*

Proof. If $x \in R$ generates a prime, then it is enough to show that $R[x^{-1}]$ is a UFD by the previous proposition. Suppose that $q \in \operatorname{Spec} R[x^{-1}]$ is height one, then it is enough to prove that it is principal by the same proposition. If $p \in \operatorname{Spec} R[x^{-1}]$, then $R[x^{-1}]_p$ is a localization of R. By the corollary 7.15, it is regular. Furthermore $\dim R[x^{-1}]_p < \dim R$. So by induction, we can assume that it is a UFD. This shows that each Q_p is principal, and therefore that Q is projective. To show that is principal, it is enough to show that it free. We can find an R-module Q' such that $Q = Q'[x^{-1}]$. Since R has finite global dimension, we can find a finite resolution of $F_\bullet \to Q'$ by finitely generated projective R-modules. Earlier we proved that any finitely generated projective module over a noetherian local ring is free. Therefore $F_\bullet[x^{-1}] \to Q$ is a free resolution. Lemma 7.19 implies that Q is free. $\square$

In algebraic geometry, it is important to study algebraic varieties in terms of their codimension one subvarieties, called Weil divisors. The best behaved among these are the Cartier divisors which are locally defined by a single equation $f = 0$. The above theorem guarantees than on a nonsingular variety (or more generally regular scheme), all Weil divisors are Cartier.

Chapter 8

Spectral sequences

Refs.

1. Brown, Cohomology of groups
2. Godement, Topologie algébrique et théorie de faisceux
3. Hochschild, Serre, Cohomology of group extensions, Trans AMS 1953
4. Rotman, Intro to homological algebra
5. Weibel, An intro to homological algebra

8.1 Filtrations

A decreasing filtration on an abelian group (or module) M is a collection of subgroups (submodules) $F^pM \subseteq M$, indexed by $p \in \mathbb{N}$, such that $F^{p+1}M \subseteq F^pM$. Let us say that a filtration is an n-step filtration if $F^{n+1}M = 0$, and finite this holds for some n. There are a couple of variants which arise in practice, although not in these notes. The index p might take values in $\mathbb{Z}$, or the filtration might be increasing.

Example 8.2. *Let R be a commutative ring, and I an ideal. The I-adic filtration on a module M is $F^pM = I^pM$. This is finite, if for example, I is nilpotent.*

We define associated graded modules by

$$Gr_F^pM = F^pM/F^{p+1}M$$

$$Gr_F(M) = \bigoplus_p Gr_F^pM$$

A 1-step filtration is equivalent to the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & F^1M \arrow[r] \arrow[d, dash, "="'] & M \arrow[r] & M/F^1 \arrow[r] \arrow[d, dash, "="] & 0 \\ & Gr^1M & & Gr^0M & \end{tikzcd}$$

So a filtration is a sort of generalization of this. We make another remark: to know $Gr(M)$ is not the same as knowing M, because there is an extension problem to worry about. However, $Gr(M)$ does give a lot of information.

We define a morphism of filtered modules $f : (M, F) \to (N, F)$ to be homomorphism $f : M \to N$ such that $f(F^pM) \subseteq F^pN$ for all p. It is easy to see that the category of filtered modules is an additive category. For instance the direct sum is the usual one with

$$F^p(M \oplus N) = F^pM \oplus F^pN$$

Moreover it has kernels. The kernel of $f : (M, F) \to (N, F)$ in this category is the usual kernel with filtration

$$F^p \ker f = F^pN \cap \ker f$$

It also has cokernels:

$$F^p \operatorname{coker} f = \operatorname{im} F^pM \to \operatorname{coker} f$$

However, it is *not* abelian:

Example 8.3. *Le $M = \mathbb{Z}$ with two filtrations* 1*-step filtrations $F^0M = M, F^1M = 0$ and $G^0M = M, G^1M = M$. Let $f : (M, F) \to (M, G)$ be the identity. Even though both* $\ker f$ *and* $\operatorname{coker} f$ *are zero, f is not an isomorphism of filtered abelian groups. Such an example cannot occur in an abelian category.*

While this may appear to be a mere technicality, the nonabelianness is the source of a lot of the complications in homological algebra.

8.4 Filtered complexes and double complexes

A complex in the category of filtered modules is called a filtered complex. To simplify the discussion, we focus exclusively on the case of cochain complexes. The case of chain complexes is similar. Then be more explicit, a filtered complex consists of a complex $(C^\bullet, d)$ such that each C^n carries a filtration F^pC^n satisfying $d(F^pC^n) \subset F^pC^{n+1}$.

Example 8.5. *Given any complex $C^\bullet$, define*

$$F^pC^j = \begin{cases} C^j & \text{if } j \geq p \\ 0 & \text{otherwise} \end{cases}$$

This defines a filtration called the stupid filtration (or bête *filtration if you want to sound sophisticated). $(C^\bullet, F)$ is a filtered complex.*

Before giving some more examples, we note that double cochain complex is as defined as earlier with a few modifications. We need a collection of modules $C^{ij}, i, j \in \mathbb{Z}$ and homomorphisms going "up" i.e. $d_h : C^{i,j} \to C^{i+1,j}$ and $d_v : C^{i,j} \to C^{i,j+1}$. These satisfy the same relations as before

$$d_h^2 = d_v^2 = d_h d_v + d_v d_h = 0$$

These imply that

$$Tot(C)^n = \bigoplus_{i+j=n} C^{ij}$$

becomes a complex under $d = d_h + d_v$.

Example 8.6. *$Tot(C)$ becomes a filtrated complex with respect to either the horizontal or vertical filtrations*

$$F_h^p Tot(C)^n = \bigoplus_{i+j=n,\, i \geq p} C^{ij}$$

$$F_v^p Tot(C)^n = \bigoplus_{i+j=n,\, j \geq p} C^{ij}$$

Given a filtered complex $(C^\bullet, F)$, $F^p C^\bullet \subset C^\bullet$ is a subcomplex, and $Gr_F^p C^\bullet$ is a quotient of that. So it is a complex in its own right, and it is often simpler than $C^\bullet$

Example 8.7. *Given the stupid filtration $Gr_F^p C^\bullet = C^p[-p]$, where the notation indicates C^p shifted p-places to the right, or in other words the complex*

$$\ldots 0 \to \underbrace{C^p}_{degree\ p} \to 0 \ldots$$

Example 8.8. *If $(T, F) = (Tot(C), F_h)$ as in example 8.6, then*

$$Gr_F^p T = C^{p,\bullet}[-p]$$

A similar statement holds for F_v.

A morphism of filtered complexes is a morphism of complexes preserving the given filtrations. This induces a morphism of the associated graded complexes. Recall that we call a complex $C^\bullet$ positive if $C^n = 0$ for $n < 0$.

Theorem 8.9. *Suppose that $C^\bullet$ and $D^\bullet$ are positive complexes with filtrations F, which are finite on each C^i and D^i. Suppose also that $f : (C^\bullet, F) \to (D^\bullet, F)$ is a morphism induces and isomorphism $H^i(Gr_F^p C^\bullet) \to H^i(Gr_F^p D^\bullet)$ for all i, p. Then f induces an isomorphism*

$$H^i(C^\bullet) \to H^i(D^\bullet)$$

for all i.

Proof. We give a proof under the stronger assumption that $F^{n+1}C^i = 0$ for n independently of i. Then we prove the theorem by induction on n. For $n = 0$, there is nothing to prove. Suppose that the theorem holds for $n-1$. Then consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & F^{n-1}C^\bullet \arrow[r] \arrow[d, "f'"] & C^\bullet \arrow[r] \arrow[d, "f"] & Gr_F^n C^\bullet \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & F^{n-1}D^\bullet \arrow[r] & D^\bullet \arrow[r] & Gr_F^n D^\bullet \arrow[r] & 0
\end{tikzcd}$$

where f', f'' are induced by f. We get an induced diagram

$$\begin{tikzcd}
\ldots \arrow[r] & H^i(F^{n-1}C^\bullet) \arrow[r] \arrow[d, "f'"] & H^i(C^\bullet) \arrow[r] \arrow[d, "f"] & H^i(Gr_F^n C^\bullet) \arrow[r] \arrow[d, "f''"] & \ldots \\
\ldots \arrow[r] & H^i(F^{n-1}D^\bullet) \arrow[r] & H^i(D^\bullet) \arrow[r] & H^i(Gr_F^n D^\bullet) \arrow[r] & \ldots
\end{tikzcd}$$

The arrows labelled by f' and f'' are isomorphisms by induction or the assumptions of the theorem. Therefore the arrow labelled by f is an isomorphism by the five lemma. $\square$

As a corollary, we obtained a cohomological version of theorem 6.5. The original theorem can be proved in a similar fashion.

Corollary 8.10. *Suppose that $C^{\bullet\bullet}$ is a first quadrant double complex, and that $H^i(C^{\bullet,j}) = 0$ for $i > 0$ and all j. Then $H^i(Tot(C)^\bullet)$ is isomorphic to the ith cohomology of*

$$\ldots H^0(C^{\bullet,i-1}) \to H^0(C^{\bullet,i}) \to H^0(C^{\bullet,i+1}) \ldots$$

Proof. Let

$$K^i = H^0(C^{\bullet,i}) = \ker C^{0,i} \to C^{1,i}$$

This defines a subcomplex of $Tot(C)$. The inclusion defines a morphism of filtered complexes $(K, F) \to (Tot(C)^\bullet, F_v)$, where F is the stupid filtration. The above assumptions imply that $H^*(Gr_F^p K^\bullet) \to H^*(Gr_{F_v} Tot(C)^\bullet)$ is an isomorphism. So the corollary follows from the theorem. $\square$

8.11 Spectral sequences

Now let us assume that $(C^\bullet, F)$ is a positive filtered complex, and that FC^i is finite for each i. The question we want to consider is: *Suppose that we know $H^*(Gr_F^p C^\bullet)$ for all p, can we determine $H^*(C^\bullet)$?* The answer is no in general. However, we can determine the associated graded $Gr_F^p H^i(C^\bullet)$ with respect to the filtration

$$\mathrm{im}[H^*(F^pC^\bullet) \to H^*(C^\bullet)]$$

and this is good enough for many applications. The method for doing this involves a so called spectral sequence. One can think of this as mathematical machine which starts with $E_1 = H^i(Gr_F^p C^\bullet)$ at stage $r = 1$, and computes groups $E_2, E_3 \dots$ at successive stages and arrive at the answer $E_\infty = Gr_F^p H^i(C^\bullet)$ at stage $r \gg 0$. For the bookkeeping, we need to keep track of three indices, the filtration index p, the cohomological degree i, and the stage or page number r. It is traditional to use the complementary degree $q = i - p$ instead of i (although not everyone does this). With this in mind, we can write down formulas. Define the cycle group

$$Z_r^{pq} = \{x \in F^p C^{p+q} \mid dx \in F^{p+r}\}$$

When $r = 1$, $x \mapsto [x]$ gives a surjection

$$Z_1^{pq} = \{x \in F^p C^{p+q} \mid dx \in F^{p+1}\} \to H^{p+q}(Gr^p C^\bullet) \tag{8.1}$$

When $r \gg 0$, $x \mapsto [x]$ gives a surjection

$$Z_r^{pq} = \{x \in F^p C^{p+q} \mid dx = 0\} \to Gr_F^p H^{p+q}(C^\bullet) \tag{8.2}$$

We need to divide Z_r^{pq} by subgroups of boundaries. Set

$$B_r^{pq} = dZ_{r-1}^{p-r+1,q+r-2} + Z_{r-1}^{p+1,q-1}$$

One can check that this is a subgroup of Z_r^{pq}, so we may take the quotient

$$E_r^{pq} = \frac{Z_r^{pq}}{B_r^{pq}} \tag{8.3}$$

These formulas are not used in practice. What is important is the structure:

Theorem 8.12.

(a)

$$E_1^{pq} = H^{p+q}(Gr_F^p C^\bullet)$$

(b) There is an inclusion $dZ_r^{pq} \subset Z_r^{p+r,q-r+1}$ preserving B_r^{**}. Therefore d induces a map

$$d_r : E_r^{pq} \to E_r^{p+r,q-r+1}$$

satisfying $d_r^2 = 0$. There is an isomorphism

$$E_{r+1}^{pa} \cong \frac{\ker d_r : E_r^{pq} \to E_r^{p+r,q-r+1}}{\operatorname{im} d_r : E_r^{p-r,q+r-1} \to E_r^{pq}}$$

(c) The sequence of groups $E_1^{pq}, E_2^{pq}, \dots$ become isomorphic at some point. The common value is denoted by E_∞^{pq}. One has

$$E_\infty^{pq} \cong Gr_F^p H^{p+q}(C^\bullet)$$

A collection (E_r^{pq}, d_r) satisfying (b) is called a spectral sequence. If (c) holds for some filtration on $C^\bullet$, we say that the spectral sequence abuts or converges to $H^*(C^\bullet)$. In symbols

$$E_1^{pq} \Rightarrow H^{p+q}(C^\bullet)$$

We now make a few remarks about the proof. It is clear from the above definition (8.3) that E_r^{pq} stabilizes as $r \to \infty$. It is also relatively straight forward to check that the kernel of (8.1) (resp. (8.2)) is B_1^{pq} (resp. B_∞^{pq}). The proof of (b) can be found on pp 77-78 of Godement, which is the source that we what followed (but be aware the notation there is slightly different). Rotman and Weibel also discuss spectral sequences.

Once part (b) of theorem is established, there is another way to see why the groups E_r^{pq} stabilize. Let us suppose for simplicity that $F^\bullet C^n$ is an n-step filtration for each n. Then plotting E_r^{pq} in the pq-plane, we see that the nonzero terms are concentrated in the first quadrant – consequently one calls this a first quadrant spectral sequence.

$$\begin{tikzcd}
E_r^{p-r,q+r-1} \arrow[dr, "d_r"] & & \\
& E_r^{pq} \arrow[dr, "d_r"] & \\
& & E_r^{p+r,q-r+1}
\end{tikzcd}$$

It is clear that for r large enough, the two groups on the end fall outside the first quadrant, and are therefore zero. This means that

$$E_{r+1}^{pq} = \frac{\ker E_r^{pq} \to 0}{0} = E_r^{pq}$$

We give two examples.

Example 8.13. *If $C^\bullet$ is a postive complex with the stupid filtration. Then*

$$E_1^{pq} = H^{p+q}(C^p[-p]) = H^q(C^p) = \begin{cases} C^p & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

Under the first isomorphism d_1 is the same as $d : C^p \to C^{p+1}$. Therefore the E_1 page consists of a copy of $C^\bullet$ on the p-axis, and zeros elsewhere. It follows that

$$E_2^{pq} = E_\infty^{pq} = \begin{cases} H^p(C^\bullet) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

Example 8.14. *Given a double complex $C^{\bullet\bullet}$, let $(T, F) = (Tot(C), F_h)$. Then the spectral sequence is*

$$E_1^{pq} = H^q(C^{p\bullet}, d_v) \Rightarrow H^{p+q}(T^\bullet)$$

One often just writes the E_2 page

$$E_2^{pq} = H^p(H^q(C^{\bullet\bullet}, d_v), d_h)$$

There is a similar spectral sequence with d_v, d_h reversed.

The first example is fairly trivial, but the second is important. Among other things, it can be used to give a complete proof of theorem 8.9.

8.15 The Hochschild-Serre spectral sequence

So far our the discussion has been somewhat abstract. It is helpful to look at an explicit example. Let G be a group with a normal subgroup K. Let $H = G/K$. Given a G module M, we can regard it as a K-module. Suppose that $m \in M^K$, $g \in G$ and $k \in K$. Then $g^{-1}kg \in K$ so $g^{-1}kgm = m$. This implies $kgm = gm$, so that $gm \in M^K$. Therefore M^K is a G-module. Since K acts trivially on it, $H^0(K, M) = M^K$ is an H-module. By dimension shifting, we get an natural H-module structure on higher cohomology $H^i(K, M)$ as well.

Theorem 8.16 (Lyndon-Hochschild-Serre)**.** *There is a spectral sequence*

$$E_2^{pq} = H^p(H, H^q(K, M)) \Rightarrow H^{p+q}(G, M)$$

Hochschild and Serre gave several constructions in their original paper. These include giving explicit filtrations on the normalized bar complex, and applying theorem 8.12. This can also be deduced from the more general Grothendieck spectral sequence explained in the next chapter. Rather than going into the details now, we want to explain some applications.

Corollary 8.17. *If the orders of K and H are finite and coprime, for $n > 0$ there is a split exact sequence*

$$0 \to H^n(H, M^K) \to H^n(G, M) \to H^n(K, M)^H \to 0$$

Proof. We proved earlier that when Γ is finite and $i > 0$, for any M, $H^i(\Gamma, M)$ is $|\Gamma|$-torsion. In particular, when $q > 0$, $H^q(K, M)$ are $|K|$-torsion, or equivalently a $\mathbb{Z}/|K|$-module. Since $|H|$ is invertible in $\mathbb{Z}/|K|$, it follows that multiplication by $|H|$ is an automorphism on $H^q(K, M)$. On the other hand if $p > 0$, then $H^p(H, H^q(K, M))$ is $|H|$-torsion. Therefore we conclude that E_2^{pq} is concentrated on the p and q axes. This must also hold for $E_\infty^{pq} = Gr^p H^{p+q}(G, M)$. This tells us that the filtration on $H^n(G, M)$ is a 1-step filtration, and so it gives rise to an exact sequence

$$0 \to E_\infty^{n0} \to H^n(G, M) \to E_\infty^{0n} \to 0$$

Since $d_2 : E_2^{10} \to E_s^{02}$ goes between an $|H|$-torsion and a $|K|$-torsion group, it must be zero. For similar reasons, we can conclude that all the differentials are zero. It follows that $E_2^{pq} = E_\infty^{pq}$. This proves that there is an exact sequence as claimed in the corollary. Let $V \subseteq H^n(G, M)$ be the maximal $|H|$-torsion submodule. Then $V \cap H^n(H, M^K) = 0$, so V maps isomorphically to $H^n(K, M)^H$. Therefore the sequence splits. ∎

The cohomological dimension of G

$$cd(G) = \sup\{i \mid \exists M, H^i(G,M) \neq 0\}$$

measures the size of G from the point of view of group cohomology. For a topologiclal interpretation, see Brown, chapter VIII. The following should be clear from the definition and earlier results

Proposition 8.18. *We have $cd(G) = pd(\mathbb{Z})$, where the second quantity is the projective dimension of $\mathbb{Z}$ as a G-module.*

Corollary 8.19. *$cd(\mathbb{Z}^n) = n$.*

Proof. The Koszul complex has length n, so $pd(\mathbb{Z}) \leq n$. We must have equality, since $H^n(\mathbb{Z}^n, \mathbb{Z}) \neq 0$, again using the Koszul complex. $\square$

Theorem 8.20. *Given an extension*

$$1 \to K \to G \to H \to 1$$

we have $cd(G) \leq cd(H) + cd(K)$.

Proof. If $i = p + q > cd(H) + cd(K)$, then $p > cd(H)$ or $q > cd(K)$. In either case

$$E_2^{pq} = H^p(H, H^q(K,M)) = 0$$

This implies

$$E_3^{pq} = 0 \Rightarrow \ldots \Rightarrow E_\infty^{pq} = Gr^p H^i(G,M) = 0$$

Since this holds for all p, $H^i(G,M) = 0$. $\square$

A group is called polycyclic if there exists a composition series

$$1 = G_0 \lhd G_1 \lhd G_2 \ldots \lhd G_n = G$$

such that each quotient G_i/G_{i-1} is cyclic. The condition is the same as solvable when G is finite, but it is much stronger in general. When G is also torsion free, we have that each $G_i/G_{i-1} \cong \mathbb{Z}$.

Corollary 8.21. *If G is torsion free polycyclic, then $cd(G) \leq n$, where n is the length of the composition series written above.*

Proof. This follows by the theorem, induction, and the fact that $cd(\mathbb{Z}) = 1$. $\square$

Chapter 9

Epilogue: Derived categories

Refs.

1. Gelfand, Manin, Methods of homological algebra
2. Iverson, Cohomology of sheaves
3. Weibel, An introduction to homological algebra.

9.1 Derived categories

Let A be an abelian category. Let $C(A)$ denote the category complexes in A viewed as cochain complexes. Given morphisms $f, g : C^\bullet \to D^\bullet$, recall that they are homotopic if there is a collection of morphisms $h : C^i \to D^{i-1}$ such that $dh + hd = f - g$. This defines an equivalence relation. Let $K(A)$ denote the the homotopy category, where objects are complexes, and morphisms are homotopy classes of morphisms in $C(A)$. The category $C(A)$ is abelian, but $K(A)$ is not. The second category does not have kernels or cokernels in general. There is, however, a substitute for exact sequences called distinguished triangles. These are diagrams of the form

$$C^\bullet \xrightarrow{f} D^\bullet \to Cone(f) \to C^\bullet[1]$$

The collection of these makes $K^+(A)$ into a so called triangulated category. We won't go into all that here, but instead refer to the books listed above for details. A morphism $f : C^\bullet \to D^\bullet$ is called a quasi-isomorphism if the induced map $H^i(C^\bullet) \to H^i(D^\bullet)$ is an isomorphism for each i. Examples of quasi-isomorphisms obviously include isomorphisms $C(A)$, and more generally in $K(A)$ (homotopy equivalences). A complex $C^\bullet$ is called bounded below if C^i

for $i \ll 0$. Let $C^+(A) \subset C(A)$ and $K^+(A) \subset K(A)$ denote the full subcategories of bounded below complexes. We can generalize the existence of injective resolutions in the following way.

Theorem 9.2. *Assume that A has enough injectives. Then every bounded below complex is quasi-isomorphic to a bounded below complex of injective objects.*

Proof. Since we can always shift a bounded below to a positive complex, there is no loss in proving the statement for positive complexes.Let $C^0 \to C^1 \to \dots$ be such a complex. Let

$$B^n(C^\bullet) = \operatorname{im} C^{n-1} \to C^n$$

$$Z^n(C^\bullet) = \ker C^n \to C^{n+1}$$

Of course,

$$H^n(C^\bullet) = Z^n(C^\bullet)/B^n(C^\bullet)$$

By induction, we will construct a complex of injectives $I^0 \to I^1 \dots I^n$ with a morphism f from $C^0 \to C^1 \dots C^n$ to the previous complex, such that f induces

$$H^i(C^\bullet) \cong H^i(I^\bullet), \quad i < n$$

$$0 \to C^n/B^n(C^\bullet) \xrightarrow{\bar{f}} I^n/B^n(I^\bullet)$$

For the initial step, we just choose an injective object I^0 with an embedding

$$0 \to C^0 \to I^0$$

For the induction step, form the pushout

$$\begin{tikzcd}
C^n/B^n(C^\bullet) \arrow[r, "g"] \arrow[d, "\bar{f}"] & Z^{n+1}(C^\bullet) \arrow[d] \\
I^n/B^n(I^\bullet) \arrow[r, "h"] & P
\end{tikzcd}$$

Note that $P = \operatorname{coker}(g, -\bar{f})$, so it certainly exists in A. Choose an injective object with an embedding

$$0 \to P \to I^{n+1}$$

Let

$$h' : I^n/B^n(I^\bullet) \to I^{n+1}$$

denote the composition of h with above embedding. Then

$$H^n(I^\bullet) \cong \ker h' \cong \ker h$$

A diagram chase (which is permitted by Freyd-Mitchell) show that

$$\ker h \cong \ker g \cong H^n(C^\bullet)$$

Since I^{n+1} is injective, we can choose a morphism $C^{n+1} \to I^{n+1}$ extending $Z^{n+1}(C^\bullet) \to I^{n+1}$. A further diagram chase shows that

$$0 \to C^{n+1}/B^{n+1}(C^\bullet) \to I^{n+1}/B^{n+1}(I^\bullet)$$

is exact $\square$

The complex $I^\bullet$ constructed in the proof will be called an injective resolution of $C^\bullet$. When $C^\bullet$ consists of a single object placed in degree 0, this is an injective resolution in the previous sense.

Theorem 9.3. *If $I^\bullet$ is a bounded below complex of injectives, and $C^\bullet \to D^\bullet$ is a quasi-isomorphism in $C^+(A)$, then*

$$Hom_{K^+(A)}(C^\bullet, I^\bullet) \cong Hom_{K^+(A)}(D^\bullet, I^\bullet)$$

Proof. See theorem I 6.1 of Iversen. $\square$

Corollary 9.4. *Any two injective resolutions are homotopy equivalent.*

Let A have enough injectives. Then we define the derived category $D^+(A)$ be the full subcategory of $K^+(A)$ consisting of injective complexes. For each $C^\bullet \in C^+(A)$, let $Q(C^\bullet)$ denote a fixed injective resolution. The previous theorem implies that for each morphism $f : C^\bullet \to D^\bullet$, there is a unique dotted arrow making the diagram

$$\begin{tikzcd} C^\bullet \arrow[r] \arrow[d, "f"] & Q(C^\bullet) \arrow[d, dashed, "Q(f)"] \\ D^\bullet \arrow[r] & Q(D^\bullet) \end{tikzcd}$$

commute. This shows that Q defines a functor from $K^+(A) \to D^+(A)$. One can show that Q takes quasi-isomorphisms to isomorphisms. In fact, this characterizes the derived category

9.5 Composition of derived functors

Let A and B be abelian categories with enough injectives. If $F : A \to B$ is a left exact functor, we can apply F term by term to get a functor $F : K^+(A) \to K^+(B)$. We define a new functor, called the right derived functor by $\mathbb{R}F : D^+(A) \to D^+(B)$ by $\mathbb{R}F(C^\bullet) = Q(F(Q(C^\bullet)))$. In other words, apply F term by term to an injective resolution of $C^\bullet$, and then take an injective resolution of the result. This fits into a commutative diagram

$$\begin{tikzcd} K^+(A) \arrow[r, "Q"] \arrow[d, "F"] & D^+(A) \arrow[d, "\mathbb{R}F"] \\ K^+(B) \arrow[r, "Q"] & D^+(B) \end{tikzcd}$$

This is related to the previous notion of derived functor as follows.

Proposition 9.6. *If M is an object, $H^i(\mathbb{R}F(M)) = R^iF(M)$.*

Proof. $H^i(\mathbb{R}F(M)) = H^i(Q(F(Q(M)))) = H^i(F(Q(M))) = R^iF(M)$. □

Let $G : B \to C$ be another left exact functor to an abelian category with enough injectives. Then we can compose them to get a third left exact functor $G \circ F : A \to C$. We can ask about the relationship between the derived functors. We give two answers. The cleanest uses derived categories. There is also an older formulation using spectral sequences. Let us say that an object M of B is G-acyclic if $R^iG(M) = 0$ for all $i > 0$. For example, injective objects are G-acyclic.

Theorem 9.7. *Suppose that F takes injective objects to G-acyclic objects. Then*

(a) (Verdier) $\mathbb{R}(G \circ F) \cong \mathbb{R}G \circ \mathbb{R}F$

(b) (Grothendieck) There exists a first quadrant spectral sequence

$$E_2^{pq} = R^pFR^qG(M) \Rightarrow R^{p+q}(F \circ G)M$$

Proof. See chapter III section 7 of Gelfand-Manin. □

Example 9.8. *Let $K \subset G$ be a normal subgroup of a group with quotient $H = G/K$. We can factor*

$$(-)^G : G\text{-}mod \to Ab$$

as

$$G\text{-}mod \stackrel{(-)^K}{\to} H\text{-}mod \stackrel{(-)^H}{\to} Ab$$

The first functor takes injective G-modules to injective H-modules. Therefore we have Grothendieck spectral sequence, which reduces to the Hochschild-Serre spectral sequence.