Chapter 1

Some module theory

Basic Refs for this chapter.

1. Atiyah, Macdonald, Intro to commutative algebra
2. Harris, Algebraic geometry. A first course.
3. Rotman, Introduction to homological algebra.

1.1 Modules

We with work with not necessarily commutative rings, always with 1. There are many important examples which aren't commutative; matrix rings for example, and the following:

Example 1.2. *Let G be a group. The integral group ring $\mathbb{Z}G$ is the set of finite formal linear combinations $\sum_{g\in G} n_g g$, $n_g \in \mathbb{Z}$. The addition is obvious. The multiplication is*

$$(\sum n_g g)(\sum m_h h) = \sum_{g,h} n_g m_h gh$$

This is not commutative unless G is abelian. RG for any commutative ring R is defined the same way.

Let R be a ring. Since it may not be commutative, we have to be careful to distinguish left and right modules. Left R-module is an abelian group M with a multiplication $R \times M \to M$ satisfying

$$1 \cdot m = m$$

$$(r_1 r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$$

$$(r_1 + r_2) \cdot m = r_1 m + r_2 m$$

$$r \cdot (m_1 + m_2) = rm_1 + rm_2$$

A right module is an abelian group N with a multiplication $N \times R \to N$ satisfying a similar list of conditions. The opposite ring R^{op} is R as an additive group with multiplication reversed. We have $R \cong R^{op}$ if R is commutative, but not in general. A right module R is the same thing as a left R^{op}-module. Thus we may as well work with left modules, henceforth called modules, although there are few situations where it is convenient to work with right modules as well.

Example 1.3. *A left (right) $\mathbb{Z}G$ module is the same thing as an abelian group with a left (right) action by G.*

A homomorphism of R-modules $f : M \to N$ is an abelian group homomorphism satisfying $f(rm) = rf(m)$. The collection of R-modules and homomorphisms forms a category Mod_R (NB: Rotman denotes this by ${}_RMod$; if we want to consider right modules in these notes, we use $Mod_{R^{op}}$.) Let $Hom_R(M, N)$ be the set of homomorphisms. Since the sum of homomorphisms is a homomorphisms, this is an abelian group. However, in the noncommutative case it is not an R-module. It can be made into a module when M or N have additional structure. Given another ring S, an (R, S)-bimodule N is an abelian group with a left R-module structure and a right S-module structure such that $(rm)s = r(ms)$. Equivalently, N is left $R \times S^{op}$-module. In this case, $Hom_R(M, N)$ is a right S-module by $fs(m) = f(m)s$. Similarly when M is a (R, T)-bimodule, $Hom_R(M, N)$ is a left T-module.

Given a homomorphism $f : M \to N$, we get an induced homomorphisms

$$f_* : Hom_R(X, M) \to Hom_R(X, N)$$

$$f^* : Hom_R(N, Y) \to Hom_R(M, Y)$$

by $f_*(g) = f \circ g$ and $f^*(h) = h \circ f$. This maps $Hom(X.-)$ (resp. $Hom(-, Y)$) into a covariant (resp. contravariant) functor from $Mod_R \to Ab$ (cat. of abelian groups).

Recall that sequence of modules

$$\ldots L \xrightarrow{f} M \xrightarrow{g} N \ldots$$

is exact if $\ker g = \operatorname{im} g$ etc.

Theorem 1.4. *If*

$$0 \to L \to M \to N \to 0$$

is exact, then

$$0 \to Hom(X, L) \to Hom(X, M) \to Hom(X, N)$$

and

$$0 \to Hom(N, Y) \to Hom(M, Y) \to Hom(L, Y))$$

are exact.

Proof. In class, or see Rotman, or better yet, check yourself. $\square$

The theorem says that $Hom(X,-)$ and $Hom(-,Y)$ are left exact functors. In general, these are not exact:

Example 1.5. *If $R = \mathbb{Z}$, consider*

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

Applying $Hom(\mathbb{Z}/2,-)$ yields

$$0 \to 0 \to 0 \to \mathbb{Z}/2$$

and $Hom(-,\mathbb{Z}/2)$ yields

$$0 \to \mathbb{Z}/2 \xrightarrow{=} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2$$

The final maps are not surjective in either case.

1.6 Projective modules

An R-module P is called projective if $Hom_R(P,-)$ is exact. More explicitly, this means given a diagram with solid arrows

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

We can find a not necessarily unique dotted arrow making this commute.

A module M is free if it isomorphic to a possibly infinite direct sum $\bigoplus_I R$. Equivalently M has a basis (which is a generating set with no relations). A map of a basis to any module extends, uniquely, to a homomorphism of the free module.

Lemma 1.7. *A free module is projective.*

Proof. Suppose that F is free with basis e_i. Given a diagram

$$\begin{tikzcd} & F \arrow[dl, dashed, "\tilde{f}"] \arrow[d, "f"] & \\ M \arrow[r, "p"] & N \arrow[r] & 0 \end{tikzcd}$$

choose $m_i \in M$ such that $p(m_i) = f(e_i)$. Then $e_i \mapsto m_i$ extends to the dotted homomorphism. $\square$

Lemma 1.8. *If P is projective, then given any surjective homomorphism $f : M \to P$, there is a splitting i.e. a homomorphism $s : P \to M$ such that $f \circ s = id$.*

Proof. Use

$$\begin{tikzcd} & P \arrow[dl, dashed, "s"'] \arrow[d, "id"] & \\ M \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$

□

Theorem 1.9. *P is (finitely generated and) projective iff it is a direct summand of a (finitely generated) free module F, i.e. there exists K such that $F \cong P \oplus K$.*

Proof. Suppose that P is projective. We can choose a surjection $\pi : F \to P$, with F free. By the lemma, we have splitting $s : P \to F$. Note that s is injective, so $P \cong s(P)$. One checks that $F = \ker \pi \oplus s(P)$.

Suppose that $F = P \oplus F$ is free. Given

$$\begin{tikzcd} & P \arrow[dl, dashed, "\tilde{g}"'] \arrow[d, "g"] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

we can extend g to $f : F \to N$ by $f = g \oplus 0$. Since F is projective, we have a lift $\tilde{f} : F \to M$. So $\tilde{g} = \tilde{f}|_P$ will fill in the above diagrem.

If P is finitely generated then F can be chosen to be finitely generated, and visa versa.

□

Now let's assume basic constructions/facts about commutative rings, including localization and Nakayama's lemma, which can be found in Atiyah-Macdonald.

Theorem 1.10. *A finitely generated projective module over commutative noetherian local ring is free.*

Proof. Let (R, m) be a comm. noeth. local ring, and P a fin. gen. projective R-module. Let $k = R/m$ be the residue field, and let $n = \dim P \otimes_R k$. Choose a set of elements $p_1, \ldots, p_n \in P$ reducing to a basis of $P \otimes k$. By Nakayama's lemma p_i spans P. Therefore we have a surjection $f : R^n \to P$, sending $e_i \to p_i$. Let $K = \ker f$. Arguing as above, we see that

$$R^n = P \oplus K$$

We necessarily have $K \otimes k = 0$, so $K = 0$ by Nakayama.

□

Corollary 1.11. *If P is a fin. gen. projective module over commutative noetherian ring, then it is locally free, i.e. P_p is free for every $p \in \operatorname{Spec} R$.*

We will see the converse later.

1.12 Projective modules versus free modules

There exists projective modules which are not free. Here is a cheap class of examples.

Example 1.13. *Let R_1, R_2 be nontrivial rings, and let $n, m > 0$ be unequal integers. Set $R = R_1 \times R_2$. Then*

$$P_{nm} = R_1^n \times R_2^m$$

is an R module. It is projective because $P_{nm} \oplus P_{mn} = R^{n+m}$. However, it is not free. When R_i are commutative, we can argue as follows. If $p \in \operatorname{Spec} R_1 \subset \operatorname{Spec} R$, then $P_{nm,p} = R_p^n$, while the localization at $p \in \operatorname{Spec} R_2$ is R_p^m. A free module would have the same rank at each prime.

This sort of example is impossible if R is commutative with $\operatorname{Spec} R$ connected. Nevertheless other examples exist in such cases. Let us assume that R is commutative noetherian and that finitely generated projective modules are the same as locally free modules. If R is Dedekind domain with nontrivial class group (see Atiyah-Macdonald the definition), then we can find an ideal $I \subset R$ which is not principal. I would not be free, although it would be locally free because the localizations are PIDs.

Here we outline an important class of examples assuming a bit of algebraic geometry (see Harris). Let $f(x_1, \ldots, x_n)$ be a nonzero polynomial over an algebraically closed field k. Let $X = V(f) = \{a \in k^n \mid f(a) = 0\}$ be the hypersurface defined by f. The coordinate ring is $R = k[x_1, \ldots, x_n]/(f)$. X can be identified with the maximal ideal spectrum of R by the Nullstellensatz. Let us assume that X is smooth, which means that the gradient $(\frac{\partial f}{\partial x_i})$ is never zero on X. It follows that $U_i = X - V(\frac{\partial f}{\partial x_i}))$ is an open cover of X in the Zariski topology. The module of vector fields $T = \{(g_1, \ldots, g_n) \in R^n \mid \sum \frac{\partial f}{\partial x_i} g_i = 0\}$ The localization $T_{\partial f/\partial x_i}$ is free over $R_{\partial f/\partial x_i}$ with basis

$$(0, \ldots 0, \underbrace{-\frac{\partial f}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)^{-1}}_{i\text{th place}}, 0, \ldots, \underbrace{1}_{j\text{th place}}, \ldots 0), \quad j \neq i$$

It follows that T is locally free. For suitable f, one can show that T is not free by showing that the tangent bundle is nontrivial. It may be worth spelling out the dictionary

Algebra	Geometry
Projective module	Vector bundle
Free module	Trivial vector bundle

Topological vector bundles on affine space (over $\mathbb{C}$) are trivial because it is contractible. Using this analogy Serre conjectured that projective modules over polynomial rings were free. This was solved affirmatively by Quillen and Suslin

Theorem 1.14 (Quillen-Suslin). *If k is a field, projective modules over $k[x_1, \ldots, x_n]$ are trivial.*

An account can be found in Rotman Section 4.8.

1.15 Injective modules

A module E is injective if $Hom(-, E)$ is exact. Equivalently given the solid diagram

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & M \arrow[u] \arrow[r] & N \arrow[ul, dashed] \end{tikzcd}$$

it can be filled in as indicated. Although the notion is dual to projectivity, it is harder to characterize. We only succeed in a special case.

Theorem 1.16 (Baer's criterion). *E is injective if and only if the above property holds when $N = R$ and $M = I$ is a left ideal.*

Proof. Suppose that the extension property for an ideal. Given

$$\begin{tikzcd} E & \\ M \arrow[u, "g"] \arrow[r, "\subseteq"] & N \arrow[ul, dashed] \end{tikzcd}$$

we have to construct an extension as indicated. Let $M \subseteq M' \subseteq N$ with an extension $g' : M' \to E$ of g. We can assume this is maximal by Zorn's lemma. Suppose that $M' \neq N$. Choose $y \in N$, $y \notin M'$. Define

$$I = \{r \in R \mid ry \in M'\}$$

This is a left ideal. Let $h : I \to E$ be given by $h(r) = g'(ry)$. Then by assumption, we have an extension $\tilde{h} : R \to E$. Let $M'' = M + Ry$. The map $g'' : M'' \to E$ given by

$$g''(x + ry) = g'(x) + \tilde{h}(r), x \in M'$$

can be seen to be well defined (see Rotman pp 118-119). It extends g'. However, this contradicts the maximality of (M', g').

$\square$

Theorem 1.17. *If R is commutative integral domain, an injective module E is divisible, i.e. given $x \in E$, $r \in R$, $\exists y \in E, ry = x$. The converse holds if R is a PID.*

Proof. Since R is a domain $(r) \cong R$. Therefore $h : (r) \to E$ given by $h(r) = x$ is well defined. We have an extension $\tilde{h} : R \to E$. Then $y = \tilde{h}(1)$ satisfies $ry = x$.

Suppose that R is a PID and E is divisible. We have to check Baer's criterion. Any ideal $I = (r)$ for some $r \in R$. The above argument can be reversed to show that any $h : (r) \to E$ extends to $R \to E$. $\square$

Corollary 1.18. $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}, \mathbb{R}, \ldots$ *are injective* $\mathbb{Z}$*-modules.*

Given an abelian group A, the character group

$$A^* = Hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$$

This is divisible, and therefore injective as a $\mathbb{Z}$-module. We have a canonical map $A \to A^{**}$ given by sending a to

$$\hat{a}(f) = f(a)$$

Proposition 1.19. *If* $A \neq 0$*, then* $A^* \neq 0$*. The map* $A \to A^{**}$ *is injective.*

Proof. In general, if $a \in A$ is nonzero, let A_0 be the subgroup generated by a. Since $\mathbb{Q}/\mathbb{Z}$ has elements of arbitrary finite order, $A_0^* \neq 0$. Since $\mathbb{Q}/\mathbb{Z}$ is injective, the map

$$Hom(A, \mathbb{Q}/\mathbb{Z}) \to Hom(A_0, \mathbb{Q}/\mathbb{Z})$$

is surjective.

For the second statement, it is enough to observe that given $a \neq 0$, there exists $f \in A^*$ with $\hat{a}(f) = f(a) \neq 0$ by the previous statement. $\square$

Corollary 1.20. *Any abelian group embeds into an injective abelian group.*

As we'll see below, this holds more generally for similar reasons.

1.21 Tensor products

If you are familiar with tensor products over a commutative ring, then there are few necessary modifications to make things work in general.

- Tensor products only makes sense between right and left modules.
- The tensor product is only an abelian group in general.

Here is the precise statement.

Theorem/Def 1.22. *If* M *is a right* R*-module, and* N *a left* R*-module, there exists an abelian group and biadditive operation*

$$\otimes : M \times N \to M \otimes_R N$$

satisfying $mr \otimes n = m \otimes rn$*. Furthermore, this is the universal such object.*

See Rotman, Section 2.2, for the construction and precise explanation of the last part. The construction shows that elements of $M \otimes_R N$ are finite sums $\sum m_i \otimes n_i$. If M is an (T, R) bimodule, and N a (R, S) bimodule, then $M \otimes_R N$ is an (T, S)-bimodule satisfying $t(m \otimes n)s = tm \otimes ns$. The universal property of tensor products can be translated into the following adjointness statement.

Theorem 1.23. *Suppose that M is a right R-module, N an (R, S) bimodule, and Q a right S-module, then there is a natural isomorphism*

$$Hom_S(M \otimes_S N, Q) \cong Hom_R(M, Hom_S(N, Q))$$

where f on the left goes to the map

$$m \mapsto (n \mapsto f(m \otimes n))$$

on the right.

Proof. Rotman, Sect 2.2.1. □

Theorem 1.24. *If M (resp. N) is a right (resp. left) module, $M \otimes_R -$ (resp. $- \otimes_R N$) are right exact functors.*

Proof. Given an exact sequence

$$0 \to A \to B \to C \to 0$$

of left modules, we have to show that

$$M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

is exact. By prop 2.42 of Rotman, it suffices to prove the dual statement that

$$0 \to Hom_{\mathbb{Z}}(M \otimes A, X) \to Hom_{\mathbb{Z}}(M \otimes B, X) \to Hom_{\mathbb{Z}}(M \otimes C, X) \tag{1.1}$$

is exact for any abelian group X. From the initial sequence, we see that

$$0 \to Hom_{\mathbb{Z}}(C, X) \to Hom_{\mathbb{Z}}(B, X) \to Hom_{\mathbb{Z}}(C, X)$$

is exact. Therefore

$$0 \to Hom_R(M, Hom_{\mathbb{Z}}(C, X)) \to Hom_R(M, Hom_{\mathbb{Z}}(B, X)) \to Hom_R(M, Hom_{\mathbb{Z}}(C, X))$$

is exact. But this can be identified with (1.1) by the previous theorem.

The exactness statement for the left module N follows from working over R^{op} because we can identify

$$N \otimes_R A = A \otimes_{R^{op}} A$$

□

A right/left module is called flat if tensor product with respect to it is exact. For a left module X to be flat, it is enough to know that

$$M \otimes X \to N \otimes N$$

is injective whenever $M \to N$ is injective.

Theorem 1.25.

(a) X is flat if all of its finitely generated submodules are flat.

(b) Projective modules are flat.

(c) If R is a (commutative) PID, a module is flat if and only if it is torsion free.

Proof. Suppose that $M \to N$ is injective. If $M \otimes X \to N \otimes X$ is not injective, then some nonzero element $\sum m_i \otimes x_i$ lies in the kernel. This would lie in the kernel of $M \otimes X_0 \to N \otimes X_0$, where $X_0 \subset X$ is the submodule generated by the x_i. Therefore X_0 is not flat.

Suppose that X is projective. Then it is direct summand of R^I. Consider the commutative square

$$\begin{tikzcd}
M^I \arrow[r, "d"] & N^I \\
M \otimes X \arrow[u, "c"] \arrow[r, "a"] & N \otimes X \arrow[u, "b"']
\end{tikzcd}$$

Then $M \otimes X$ is summand of M^I, so c is injective. Also d is injective because it is a sum of injective maps. Therefore a is injective by commutativity.

Suppose that R is a PID. If X is torsion free, all of its finitely generated submodules are free. Therefore X is flat. Suppose that X is not torsion free. Then $tx = 0$ for some nonzero $x \in X, t \in R$. Then $1 \otimes x$ would lie in the kernel of $t : R \to R$ tensored with X. So X would not be flat. $\square$

The converse to (b) is not true.

Example 1.26. *$\mathbb{Q}$ is a flat $\mathbb{Z}$-module by (c) above. However, it is not projective, because $\mathbb{Q}$ is divisible but a submodule of a free module cannot be.*

The converse does hold under appropriate finiteness conditions, see Rotman theorem 3.56.

Now suppose that M is a left R-module, then

$$M^* = Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$$

is naturally a right R-module. Applying this to R^{op}, we see that this operation also takes right R-modules to left modules.

Proposition 1.27. *If F is a free R-module, then F^* is injective.*

Proof. We have a natural isomorphism

$$Hom_R(-, F^*) \cong Hom_{\mathbb{Z}}(- \otimes_R F, \mathbb{Q}/\mathbb{Z})$$

Since F is flat, and $\mathbb{Q}/\mathbb{Z}$ is divisible, the functor on the right is exact. $\square$

The following is of fundamental importance. It generalizes what we proved for abelian groups.

Theorem 1.28. *Every R-module embeds into an injective module.*

Proof. Let M be a module. Choose a surjection $F \to M^*$, with F a free right module. Then we have injections

$$M \to M^{**}, \quad M^{**} \to F^*$$

Composing these gives an injection of M into F^*, which is an injective module. $\square$