Chapter 2

Homology

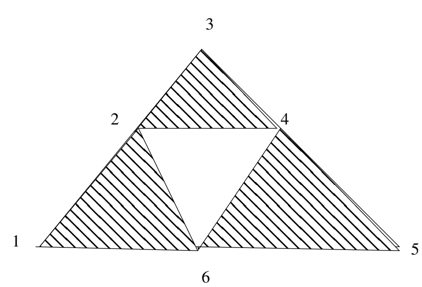
Basic Refs:

1. Hatcher, Algebraic topology
2. Rotman, Intro to homological algebra
3. Spanier, Algebraic topology
4. Weibel, An introduction to homological algebra

2.1 Simplicial complexes

Homology came out of algebraic topology. So we review the basic constructions for intuition and motivation. Recall that a (simple) graph consists of a set of vertices V, and a set of edges E between pairs of vertices. An edge can be regarded as a 2-element subset of V. A simplicial complex is a generalization, where one also allows triangles etc. More formally, it is a pair $S = (V, \Sigma)$ consisting of a set V and a collection of finite nonempty subsets Σ of V called simplices. We require that all singletons are in Σ, and any nonempty subset of $\sigma \in \Sigma$ is also in Σ. If $\sigma \in \Sigma$, has cardinality $i+1$, it is called an i-simplex.

Example 2.2. *In the example below*



$V = \{1, 2, \dots 6\}$ *and* Σ *consists of the* 2*-simplices* $\{1, 2, 6\}, \{2, 3, 4\}, \{4, 5, 6\}$ *and all nonempty subsets of them.*

Simplices in the above sense, are combinatorial models for simplices in the geometric sense. The standard geometric n-simplex Δ^n is the convex hull of unit vectors $(1, 0, \dots), (0, 1, 0 \dots), \dots \in \mathbb{R}^{n+1}$. Just as a graph gives rise to a topological space, where edges are replaced by arcs, a simplicial complex can also be turned in a topological space $|S|$, where n-simplices are replaced by spaces homeomorphic to geometric simplices. (see Spanier Chap 3 for details).

An orientation on a 2-simplex $\{v_1, v_2\}$ is a simply an ordering: either $[v_1, v_2]$ or $[v_2, v_1]$. In general, an orientation of $\sigma = \{v_1, v_2, \dots v_n\}$ is an A_n-orbit of orderings (where $A_n \subset S_n$ is the alternating group). Thus every simplex has exactly two orientations. Given an oriented simplex $[v_0, \dots, v_n]$, we identify $-[v_0, \dots, v_n]$ with the same simplex with opposite orientation. Its boundary is the formal sum

$$\partial[v_0, \dots, v_n] = [v_1, \dots, v_n] - [v_0, v_2, \dots, v_n] + \dots = \sum_i (-1)^i [v_0, \dots, \hat{v}_i, \dots v_n]$$

We call finite formal linear combination of n-simplices, as above, an n-chain. These form a free abelian group $C_n(S)$. The above formula determines a homomorphism

$$\partial_n : C_n(S) \to C_{n-1}(S)$$

We usually drop the subscript, and write ∂. Here is the key fact.

Proposition 2.3. $\partial_{n-1}\partial_n = 0$*, or more succinctly* $\partial^2 = 0$*.*

Proof. We do this when $n = 2$

$$\partial^2[v_0 v_1 v_2] = \partial([v_1 v_2] - [v_0 v_2] + [v_0 v_1]) = (v_1 - v_2) - (v_0 - v_2) + (v_0 - v_1) = 0$$

The general case is not essentially harder. Expand $\partial^2[v_0 \dots v_n]$, then one can see that the term $[v_0, \dots \hat{v}_i \dots \hat{v}_j \dots]$ occurs twice with opposite sign. □

Elements of the kernel ∂ are called cycles, and elements of the image of ∂ are called the boundaries.

Corollary 2.4. *Every boundary is a cycle.*

One can ask about the converse. In general, the answer is no. A measure of the failure is

Definition 2.5. *The nth homology group of* S *is*

$$H_n(S) = \frac{Z_n(S)}{B_n(S)}$$

where

$$Z_n(S) = \ker \partial_n$$

$$B_n(S) = \operatorname{im} \partial_{n+1}$$

Example 2.6. *Let S be the simplicial complex of example 2.2. Then $\gamma = [2,4]+[4,6]+[6,2]$ is a cycle which is not a boundary, so $H_1(S) \neq 0$. In fact, with enough patience, one can show that $H_1(S)$ is the infinite cyclic group generated by γ.*

There is a dual notion. The group of cochains

$$C^n(S) = Hom_{\mathbb{Z}}(C_n(S), \mathbb{Z})$$

This has a coboundary homomorphism

$$d : C^n(S) \to C^{n+1}(S)$$

defined by the dual to ∂.

Definition 2.7. *The nth cohomology group of S is*

$$H^n(S) = \frac{\ker d : C^n(S) \to C^{n+1}(S)}{\operatorname{im} d : C^{n-1}(S) \to C^n(S)}$$

Cohomology is roughly dual to homology (this is correct when homology is torsion free, but otherwise the precise relation is more subtle), so it may not be clear at first why it is useful. However, cohomology does carry extra structure, namely a product, called cup product

$$H^n(S) \times H^m(S) \to H^{n+m}(S)$$

which makes cohomology into a graded ring. Given n and m cochains f and g, their product is given by the formula

$$(f \cup g)[v_0, ..., v_{n+m}] = f[v_0, ..., v_n]g[v_n, ..., v_{n+m}]$$

A fact, which is at first glance, is surprising is that homology and cohomology on depends on the topological space $|S|$, and not on the triangulation. This can be done comparing to singular (co)homology, which doesn't depend on a triangulation. The group of singular chains $S_n(X)$, of a space X, is the free abelian groups generated by continuous maps from $\Delta^n \to X$. The boundary is essentially identical to the formula given previously. We refer to Hatcher or Spanier for a detailed treatment.

2.8 Complexes

We now abstract the ideas from the first section.

Definition 2.9. *A chain complex, or just complex, is a collection of abelian groups (or modules) $C_n, n \in \mathbb{Z}$ and homomorphisms (called differentials) $d : C_n \to C_{n-1}$ satisfying $d^2 = 0$. The nth homology is*

$$H_n(C_\bullet) = \frac{\ker d : C_n \to C_{n-1}}{\operatorname{im} d : C_{n+1} \to C_n}$$

It is technically convenient to allow the index to lie in $\mathbb{Z}$. Although in practice, we may only be given $C_n, n \geq 0$. In which case, we set $C_n = 0$ when $n < 0$ We will refer to such complexes as positive.

The following is obvious.

Lemma 2.10. *The sequence $C_\bullet$ is exact iff $H_n(C_\bullet) = 0$ for all n. In this case, $C_\bullet$ is also called acyclic.*

One can define a cochain complex $C^\bullet$ in similar fashion, except that differentials go the other way. Its cohomology

$$H^n(C^\bullet) = \frac{\ker d : C^n \to C^{n+1}}{\operatorname{im} d : C^{n-1} \to C^n}$$

We note that using a change of variable

$$C_n = C^{-n}$$

allows us to convert cochain complexes to complexes. Thus there is no real difference between these notions.

Definition 2.11. *A morphism of complexes, or chain map, $f : C_\bullet \to D_\bullet$ is a collection of homomorphisms $f : C_n \to D_n$, such that $df = fd$. With this notion, the collection of complexes of R-modules becomes a category $C(Mod_R)$.*

The following is straightforward.

Lemma 2.12. *A morphism of complexes $f : C_\bullet \to D_\bullet$ induces a homomorphism of homology groups $f_* : H_n(C_\bullet) \to H_n(D_\bullet)$. In fact, H_n gives a functor from $C(Mod_R) \to Mod_R$.*

A simplicial map of simplicial complexes $f : S = (V, \Sigma) \to S' = (V', \Sigma')$ is a map of sets $f : V \to V'$ such that the image of any simplex of S is a simplex of S'. It should be clear that a simplicial map f induces a morphism $C_\bullet(S) \to C_\bullet(S')$, and therefore homomorphisms $f_* : H_n(S) \to H_n(S')$. More generally, continuous maps for space induce chain maps on the singular chain complex, and therefore homomorphisms on homology.

We define an sequence of morphisms of complexes

$$C_\bullet \to C'_\bullet \to C''_\bullet$$

be exact if each sequence

$$C_n \to C'_n \to C''_n$$

is exact in the usual sense. The following result is fundamental. It will be used many times over.

Theorem 2.13. *If $0 \to C_\bullet \to C'_\bullet \to C''_\bullet \to 0$ is an exact sequence of complexes, then there is a long exact sequence*

$$\ldots H_n(C_\bullet) \to H_n(C'_\bullet) \to H_n(C''_\bullet) \xrightarrow{\partial} H_{n-1}(C_\bullet) \ldots$$

The unlabelled maps in the above sequence are the obvious ones, the map ∂, called a connecting map, is somewhat more mysterious, but it will be explained below. Rotman (prop 6.9) gives a proof of this theorem. We will give different argument. The starting point is the following standard fact, which is in fact a special case of the theorem.

Proposition 2.14 (Snake lemma)**.** *Given a commutative diagram*

$$\begin{array}{ccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \end{array}$$

with exact rows, there is an exact sequence

$$\ker f \to \ker g \to \ker h \xrightarrow{\partial} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h$$

The first (resp. last) map above is injective (resp. surjective) if $A \to B$ (resp. $B' \to C'$) is injective (resp. surjective).

Proof. We only explain the connecting map. The remain details are straightforward, and best checked in private. Given $c \in \ker h \subseteq C$, we can lift it to $b \in B$. Since $g(b)$ maps to 0, in C', it lies in A'. One can check that the image a of $g(b)$ in $\operatorname{coker} f$ does not depend on the choice of b. Set $\partial c = a$. $\square$

Proof of theorem 2.13. Let us write

$$Z_n = \ker d : C_n \to C_{n-1}$$

$$B_n = \operatorname{im} d : C_{n-1} \to C_n$$

etc. Apply the snake lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_{n+1} & \longrightarrow & C'_{n+1} & \longrightarrow & C''_{n+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d''} & & \\ 0 & \longrightarrow & C_n & \longrightarrow & C'_n & \longrightarrow & C''_n & \longrightarrow & 0 \end{array}$$

to get an exact sequence of kernels

$$0 \to \ker d \to \ker d' \to \ker d''$$

and an exact sequence of cokernels

$$\operatorname{coker} d \to \operatorname{coker} d' \to \operatorname{coker} d'' \to 0$$

(We won't use the fact that these sequences fit together.) These can be rewritten as

$$0 \to Z_{n+1} \to Z_{n+1} \to Z''_{n+1}$$

and

$$C_n/B_n \to C'_n/B'_n \to C''_n/B''_n \to 0$$

Using these for $m = n, n+2$ yields a diagram

$$\begin{array}{ccccccccc}
 & & C_m/B_m & \longrightarrow & C'_m/B'_m & \longrightarrow & C''_m/B''_m & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d''} & & \\
0 & \longrightarrow & Z_{m-1} & \longrightarrow & Z'_{m-1} & \longrightarrow & Z''_{m-1} & &
\end{array}$$

Apply the snake lemma on more time to get a six term exact sequence

$$H_m(C_\bullet) \to H_m(C'_\bullet) \to H_m(C''_\bullet) \stackrel{\partial}{\to} H_{m-1}(C_\bullet) \dots$$

These can be spliced together to obtain the infinite sequence. □

2.15 Homotopy

We go back to topology to borrow another key idea. Let $I = [0, 1]$. Two continuous maps $f, g : X \to Y$ between topological spaces are homotopic if there is a continuous map $F : X \times I \to Y$, called a homotopy, such that $f = F|_{X\times\{0\}}$ and $g = F|_{X\times\{1\}}$. This means that f can be deformed to g. It's easy to check that it is an equivalence relation. The importance stems from the following fact

Theorem 2.16. *If $f, g : X \to Y$ are homotopic, then the induced maps $f_*, g_* : H_n(X) \to H_n(Y)$ are identical.*

Here is an extremely useful consequence.

Corollary 2.17. *Given a pair of continuous map $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identities, then f induces an isomorphism between the homology of X and Y.*

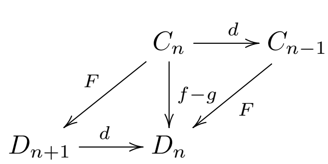
A space is contractible X if the identity is homotopic to a constant map. For example, $\mathbb{R}^n$ is contractible.

Corollary 2.18. *A contractible space has zero homology in positive degrees.*

The key idea for proving the theorem is to introduce and algebraic version of homotopy, which will be very important for us.

Definition 2.19. *If $f, g : C_\bullet \to D_\bullet$ are two morphisms between complexes, a chain homotopy between them is a collection of homorphisms $F : C_n \to D_{n+1}$ such that $dF + Fd = f - g$. f and g are called chain homotopic if F exists.*

To make sense of the last equation, we can draw the diagram



A couple of remarks: After the theorem is proved, we will drop "chain" and just say that f and g are homotopic, and we will refer to F as a homotopy. Some authors take $dF - Fd = f - g$. It's easy to go from one convention to the other by $F_n \mapsto (-1)^n F_n$, where F_n denotes the map in degree n.

Theorem 2.20. *If $f, g : C_\bullet \to D_\bullet$ are chain homotopic, then $f_* : H_n(C_\bullet) \to H_n(D_\bullet)$ and $g_* : H_n(C_\bullet) \to H_n(D_\bullet)$ coincide.*

Proof. Let F be chain homotopy. Given $a \in H_n(C_\bullet)$, we can represent it by $\alpha \in C_n$ such that $d\alpha = 0$. $f_*(a)$ is the coset of $f(\alpha)$. We have

$$f(\alpha) = dF(\alpha) + Fd(\alpha) + g(\alpha) = d(F(\alpha)) + g(\alpha)$$

which implies that $f(a) = g(a)$. □

Let us indicate to proof of theorem 2.16, referring to pp 112-113 of Hatcher for precise details. The continuous maps f, g induce chain maps $\tilde{f}, \tilde{g} : S_\bullet(X) \to S_n(Y)$ on the singular chain complex. We have to construct a chain homotopy $\tilde{F}$ between these. Recall that elements of $S_n(X)$ are linear combinations of continuous maps $\Delta^n \to X$. Such a map induces a continuous map from the prism $\Delta^n \times I \to X \times I$. There is a natural, and purely combinatorial, way to subdivide the prism $\Delta^n \times I$ into a finite union of $n + 1$ simplicies. When composed with F, these simplices give elements of $S_{n+1}(Y)$. Let $\tilde{F}(\Delta^n \to X)$ denote the sum of these elements with appropriate coefficients of the form ± 1 (chosen so that adjacent interior faces cancel). Then one checks that this gives the desired chain homotopy.

Definition 2.21. *A contracting homotopy of a complex $C_\bullet$ is a homotopy between identity and 0. A morphism $f : C_\bullet \to D_\bullet$ is a homotopy equivalence if there exists a morphism $g : D_\bullet \to C_\bullet$ such that $g \circ f$ and $f \circ g$ are homotopic to the identities of $C_\bullet$ and $D_\bullet$.*

As a corollary to theorem 2.20, we obtain

Proposition 2.22. *A complex is acyclic if it possesses a contracting homotopy. A homotopy equivalence induces an isomorphism on homology.*

2.23 Mapping cones

We can define the category $C(Mod_R)$ of complexes of R-modules, where the objects are complexes and morphisms were defined above. Given complexes $C_\bullet, D_\bullet$, $Hom(C_\bullet, D_\bullet)$ has the structure of an abelian group compatible with composition. Furthermore, standard constructions and notions such as direct sums, kernels, cokernels and exact sequences make sense within $C(Mod_R)$. This amounts to saying that this is an abelian category. See Rotman section 5.5 for the precise definition. Given complexes $C_\bullet, D_\bullet$, let $Null(C_\bullet, D_\bullet) \subset Hom(C_\bullet, D_\bullet)$ be the subset of morphisms homotopic to 0. This is easily seen to be a subgroup. Let $K(Mod_R)$ denote the category with the same objects as before, but morphisms from $C_\bullet$ to $D_\bullet$ are homotopy classes of morphisms in $C(Mod_R)$, or equivalently cosets $Hom(C_\bullet, D_\bullet)/Null(C_\bullet, D_\bullet)$. This is still an additive category since for example, the set of morphisms form and abelian group, but it is not abelian. Among other problems, the kernel and cokernel of a morphism need not exist in the homotopy category. If f and g are homotopic maps, then the complexes $\ker f$ (resp. $\operatorname{coker} f$) and $\ker g$ (resp. $\operatorname{coker} g$) need not be isomorphic in $K(Mod_R)$. Fortunately, there is a reasonable substitute. Given a morphism $f : A_\bullet \to B_\bullet$, we form a new complex $C(f)_\bullet$ called the mapping cone by

$$C(f)_n = B_n \oplus A_{n-1}$$

with differential

$$d(x, y) = (dx - f(y), dy)$$

This is also analogue of a topological notion, which is explained on pp 18-24 of Weibel's book.

Lemma 2.24. *If f and g are homotopic, then $C(f)_\bullet$ and $C(g)_\bullet$ are isomorphic.*

Proof. Let F be a homotopy from f to g. Then $(x, y) \mapsto (x - F(y), y)$ is morphism of $C(f) \to C(g)$ with inverse $(x, y) \mapsto (x + F(y), y)$. □

The mapping cone can play the role of either the kernel or the cokernel under appropriate conditions. Let us explain the second. Suppose that

$$0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$$

is exact in $C(Mod_R)$, and that for each n there are splittings

$$s_n : C_n \to B_n$$

for g. Note that we do not require that the splittings are compatible with differentials.

Lemma 2.25. *The morphism $C(g)_\bullet \to C_\bullet$ given by $(x, y) \mapsto g(y)$ is an isomorphism in $K(Mod_R)$ with inverse*

$$z \mapsto (s_n(z), s_{n-1}d(z) - ds_n(z))$$

We omit the proof, which is a long calculation. Under these conditions, we see that the connecting map $H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ is induced by the projection $C(g)_\bullet \to A_{\bullet-1}$.