Chapter 4

Cohomology of groups

Refs.

1. Brown, Cohomology of groups
2. Rotman, Intro to homological algebra
3. Weibel, An intro to homological algebra

4.1 Group cohomology

Given a group G, a left $\mathbb{Z}G$-module will simply be called a G-module. It is the same thing as an abelian group with an action by G. Let $\mathbb{Z}$ stand for the group of integers with trivial G-action. Fix a G-module A. We define the 0th cohomology by

$$H^0(G, A) = Hom_{\mathbb{Z}G}(\mathbb{Z}, A)$$

Lemma 4.2. *$H^0(G, A)$ is isomorphic to the subgroup of invariant elements $A^G = \{a \in A \mid \forall g \in G, ga = a\}$.*

Proof. The image of $1 \in \mathbb{Z}$ under an element of $H^0(G, A) = Hom_{\mathbb{Z}G}(\mathbb{Z}, A)$ lies in A^G, and conversely. $\square$

Corollary 4.3. *The functor $A \mapsto A^G$ is left exact, i.e. given a sequence of G-modules*

$$0 \to A \to B \to C \to 0$$

we obtain an exact sequence

$$0 \to A^G \to B^G \to C^G$$

The answer to the question of what comes next is higher cohomology

$$H^n(G, A) = Ext^n_{\mathbb{Z}G}(\mathbb{Z}, A)$$

Using the properties of Ext established earlier

Theorem 4.4. *Given a short exact sequence of G-modules, as above, we have a long exact sequence*

$$\dots H^n(G,A) \to H^n(G,B) \to H^n(G,C) \to H^{n+1}(G,A)\dots$$

extending the previous sequence.

Example 4.5. *If $G = \{1\}$ is trivial, then $H^n(G,A) = 0$ for any A and $n > 0$. This because $\mathbb{Z}G = \mathbb{Z}$ and $\mathbb{Z}$ is projective over it.*

Nontrivial examples will have to wait.

4.6 Bar resolution

Group cohomology can be computed using an explicit projective resolution, that we now define. The map

$$\varepsilon : \mathbb{Z}G \to \mathbb{Z}$$

defined by

$$\sum n_g g = \sum n_g$$

is a surjective G-module homomorphism. We will extend this to a free resolution of $\mathbb{Z}$. Let B_n is the free abelian group generated by the $(n+1)$ fold product $G^{n+1} = G \times G \times \dots G$. This is a G-module, where $g \in G$ acts by $g(g_0,\dots,g_n) = (gg_0,\dots,gg_n)$. We define a maps $\varepsilon : B_0 \to \mathbb{Z}$ as above, and

$$d : B_n \to B_{n-1}$$

by

$$d(g_0,\dots,g_n) = \sum (-1)^i (g_0,\dots,\hat{g}_i,\dots g_n)$$

Lemma 4.7. $d^2 = 0$*.*

Proof. The calculation is similar to what we did for the simplicial chain complex. □

Proposition 4.8. *The complex*

$$\dots B_1 \to B_0 \to \mathbb{Z}_0$$

is exact.

Proof. Set $B_{-1} = \mathbb{Z}$ and $d_0 = \epsilon$. We have to show that the extended complex $B_\bullet$ is acyclic. Let $h : B_n \to B_{n+1}$ be defined by

$$h(g_0,\dots,g_n) = (1, g_0,\dots,g_n), \quad n \geq 0$$

$$h(1) = 1, \quad n = -1$$

One sees that

$$(dh + hd)(g_0,\dots,g_n) = (g_0,\dots,g_n)$$

so that h is a contracting homotopy. □

Since B_n is a free $\mathbb{Z}G$-module with basis $(1, g_1, \ldots, g_n)$, we obtain

Corollary 4.9. *$B_\bullet$ is a free resolution of $\mathbb{Z}$*

$B_\bullet$ is called the bar or standard resolution. The first name comes from the bar notation

$$[g_1|\ldots,|g_n] = (1, g_1, g_1g_2, g_1g_2g_3, \ldots)$$

With this notation, the differential is given by

$$d[g_1,\ldots,|g_n] = g_1[g_2|\ldots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\ldots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\ldots] + (-1)^n[g_1|\ldots|g_{n-1}]$$

It is often convenient to work with a smaller complex called the normalized bar resolution

$$\bar{B}_\bullet = \frac{B_\bullet}{\{[g_1|g_2|\ldots] \mid \ldots] \mid \exists i, g_i = 1\}}$$

Proposition 4.10. *$\bar{B}_\bullet$ is also a free resolution of $\mathbb{Z}$.*

Proof. Rotman, theorem 9.38. □

4.11 Low degree cohomology

Recall

$$H^n(G, A) = Ext^n_{\mathbb{Z}G}(\mathbb{Z}, A)$$

We have already seen what this means when $n = 0$. Let us look at the next few cases. A derivation or crossed homomorphism is a map $f : G \to A$ satisfying $f(gh) = f(g) + gf(h), g, h \in G$. If $a \in A$, then $f(g) = ga - a$ is an example of a derivation, called an inner derivation.

Lemma 4.12. *$H^1(G, A)$ is isomorphic to the quotient of the group of derivations from G to A by the subgroup of inner derivations.*

Proof. We can compute

$$H^1(G, A) = H^1(Hom(B_\bullet, A)) = \frac{\ker Hom(B_1, A) \to Hom(B_2, A)}{\operatorname{im} Hom(B_0, A) \to Hom(B_1, A)}$$

Elements of the numerator, called 1-cocycles, are G-homomorphisms $f : B_1 \to A$ such that

$$f(d[g|h]) = f(g[h] - [gh] + [g]) = 0$$

This means that 1-cocycles are derivations. We have to divide this space of these by the space of 1-coboundaries, which are elements in $\operatorname{im} Hom(B_0, A)$. These can be seen to be inner derivations. □

Corollary 4.13. *If G acts trivially on A, then $H^1(G, A) = Hom_{Groups}(G, A)$.*

Given groups G and N, an extension of G by N is an exact sequence of groups

$$1 \to N \to E \to G \to 1$$

This means that N is a normal subgroup of E, such that $E/N = G$. We will focus on the case where $N = A$ is abelian. The action of E on itself by conjugation stabilizes A, because it is normal. Therefore we have a homomorphism $E \to Aut(A)$. The image of A is trivial because A is abelian, this homomorphism factors through G. Therefore A is naturally a G-module. Conversely, given a G-module, we can ask whether it arises from an extension. The answer is yes.

Proposition 4.14. *The semidirect product $G \ltimes A$ is $G \times A$ as a set. When equipped with a product*

$$(g, a)(h, b) = (gh, a + gb)$$

it becomes a group, fitting into an extension

$$0 \to A \to G \ltimes A \to G \to 1$$

such that the G-module structure on A is the given structure.

Proof. See Rotman theorem 9.5. □

We can now ask what are all possible extensions of G by A, with given G-module structure? Of course, to get a reasonable answer we work up to the equivalence, where two extensions are equivalent if they fit into the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow = & & \downarrow & & \downarrow = & & \\ 1 & \longrightarrow & A & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

Theorem 4.15 (Schreier)**.** *There is a bijection between the set of equivalence classes of extensions of G by A with given G-module structure, and $H^2(G, A)$. The equivalence class of $G \ltimes A$ corresponds to 0.*

Proof. First we compute $H^2(G, A)$ using the normalized bar resolution. It is the group of 2-cocycles modulo the subgroup of 2-coboundaries. A 2-cocycle is a map $f : \bar{B}_2 \to A$ such that $f \circ d = 0$, and a 2-coboundary is a map of the form $k \circ d$ for some $k : \bar{B}_1 \to A$. More explicity a 2-cocycle is given by a map $f : G \times G \to A$ satisfying

$$\begin{aligned} gf(h, \ell) - f(gh, \ell) + f(g, h\ell) - f(g, h) = 0 \\ f(g, 1) = f(1, g) = 0 \end{aligned} \tag{4.1}$$

Classically a 2-cocycle is also called a "factor set". A cocycle f is a 2-coboundary if there exists a function $k : G \to A$, called a 1-cochain, such that $k(1) = 0$ and

$$f(g, h) = gk(h) - k(gh) + k(g) := dk(g, h)$$

We now summarize the basic construction, and refer to Rotman section 9.1.2 for the remaining details. Given a 2-cocycle f, define $E(f) = G \times A$ as a set with product

$$(g, a)(h, b) = (gh, a + gb + f(g, h))$$

Note that $E(0) = G \ltimes A$. Using the identities (4.1), we can see that multiplication is associative

$$[(g, a)(h, b)](\ell, c) = (gh\ell, a + gb + ghc + f(g, h) + f(gh, \ell))$$

$$= (gh\ell, a + gb + ghc + gf(h, \ell) + f(g, h\ell)) = (g, a)[(h, b)(\ell, c)],$$

$(1, 0)$ is the identity, and

$$(g, a)^{-1} = (g^{-1}, -g^{-1}a - g^{-1}f(g, g^{-1}))$$

So $E(f)$ is a group. Furthermore, it fits into an extension

$$0 \to A \to E(f) \xrightarrow{\pi} G \to 1$$

where π is projection on the first factor. If we modify f by adding a coboundary associated to a cochain $k : G \to A$, then $(g, a) \mapsto (g, a + k(g))$ defines an equivalence $E(f) \cong E(f + dk)$. So the equivalence class of the above extension depends only on the cohomology class associated to f. This gives the map from H^2 to the set of equivalence classes of extensions.

Conversely, given an extension

$$0 \to A \to E \xrightarrow{\pi} G \to 1$$

choose a set theoretic map $s : G \to E$ such that $\pi \circ s = id$. Define $f(g, h) = s(x)s(y)s(xy)^{-1}$. Since $\pi(f(g, h)) = 1$, we must have $f(g, h) \in A$. This can be seen to define a 2-cocycle such that $E = E(f)$. A different choice of s will define another cocycle differing from f by a coboundary. This gives the inverse from equivalence classes of extensions to $H^2(G, A)$.

□

4.16 Applications to finite groups

Given a subgroup $H \subset G$ of a group, a G-module M is naturally also an H-module. $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module. Therefore the bar resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ can be viewed as a resolution by free $\mathbb{Z}H$-modules. This yields a natural map, called restriction

$$Res : H^n(G, M) \to H^n(H, M)$$

for any G-module.

If H has finite index, then we can define map in the opposite direction called corestriction or transfer

$$Cor : H^n(H, M) \to H^n(H, M)$$

The idea is as follows. Choose a set of representatives $g_1, \ldots, g_N$ of G/H. For $n = 0$, we define

$$Cor : M^H = H^0(H, M) \to H^0(G, M) = M^G$$

by

$$Cor(m) = \sum_i g_i m$$

This is independent of the choice of representatives. For higher n, we can use a dimension shifting technique. Embed M into an injective $\mathbb{Z}G$-module E (which we proved to exist) Then

$$H^1(G, M) = H^0(G, E/M)/\operatorname{im} H^0(H, E)$$
$$H^n(G, M) = H^{n-1}(G, E/M), \quad n > 1$$

So the definition of Cor can be reduced to $n = 0$. A detailed construction can be found in Rotman section 9.6, or also in Brown's book.

Lemma 4.17. *If $H \subset G$ is a subgroup of index $N < \infty$, the composition of restriction and corestriction*

$$H^n(G, M) \to H^n(H, M) \to H^n(G, M)$$

is multiplication by N.

Sketch. This can be reduced to the case $n = 0$ by construction. If $m \in M^G$, then the composition of the above maps sends

$$m \mapsto \sum_{i=1}^{N} g_i m = \sum_{i=1}^{N} m = Nm$$

□

Theorem 4.18. *If G is a finite group of order N, then for any G-module M and $n > 0$, we have $NH^n(G, M) = 0$.*

Proof. Let $H = \{1\}$. Then multiplication by N is the same as the composite

$$H^n(G, M) \to H^n(1, M) \to H^n(G, M)$$

But $H^n(1, M) = 0$ when $n > 0$.

□

Theorem 4.19 (Schur-Zassenhaus)**.** *Let G be a finite group of order mn, where $(m, n) = 1$. If K is a normal subgroup of order n, then G is a semidirect product of K by G/K.*

Proof. We prove this when K is abelian. The general case can be reduced to this with additional work. If K is abelian, it suffices to prove that $H^2(G/K, K) = 0$. Let $\mu : K \to K$ be multiplication by m. This is an isomorphism of G-modules, because m is coprime to $|K|$. Therefore this induces an isomorphism $\mu_* : H^2(G/K, K) \to H^2(G/K, K)$. On the other hand μ_* is the same as multiplication by m. But this is zero by the previous theorem.

□

4.20 Topological interpretation

Suppose that X is a topological space. Let G be a group such that each $g \in G$, defines a homeomorphism $g : X \to X$. Also suppose that $g_1(g_2(x)) = (g_1 g_2)(x)$ for $g_1, g_2 \in G$. Then we say that G acts on X. We give the set of orbits X/G the quotient topology. In general, this can be quite wild, even when X is nice. However, it has reasonable properties if the action is fixed point free and proper, which means that every point $x \in X$ has an open neighbourhood U such that $gU \cap U = \emptyset$ when $g \neq 1$. For example, the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ by translation satisfies these conditions, and $\mathbb{R}^n/\mathbb{Z}^n$ is the n-torus.

Theorem 4.21. *If X is contractible, and G has a fixed point free and proper action, then $H^*(X) \cong H^*(G, \mathbb{Z})$.*

Proofs can be found in Brown's or Weibel's books. Since $\mathbb{R}^n$ is contractible, we obtain:

Corollary 4.22. *Group cohomology of $\mathbb{Z}^n$ is given by $H^i(\mathbb{Z}^n, \mathbb{Z}) = H^i(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{Z})$*

Standard methods from algebraic topology allows us to compute the cohomology of the torus.

$$H^i(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{Z}) = \mathbb{Z}^{\binom{n}{i}} \tag{4.2}$$

More canonically, the answer can be expressed as an exterior power.

When $n = 1$, we can prove this algebraically as follows. Let us write $G = \mathbb{Z}$. We can identify $\mathbb{Z}G \cong \mathbb{Z}[t, t^{-1}]$ where $N \in G$ corresponds to t^N. Then

$$0 \to \mathbb{Z}[t, t^{-1}] \stackrel{t-1}{\to} \mathbb{Z}[t, t^{-1}] \to \mathbb{Z} \to 0$$

gives a free resolution. Therefore for any $\mathbb{Z}[t, t^{-1}]$-module M, we can Hom this resolution into it to obtain the complex

$$M \stackrel{t-1}{\to} M$$

So that

$$H^i(G, M) = \begin{cases} \ker(t-1) : M \to M & i = 0 \\ \operatorname{coker}(t-1) : M \to M & i = 1 \\ 0 & i > 1 \end{cases}$$

When $M = \mathbb{Z}$, we obtain $\mathbb{Z}$ in degrees 0 and 1, which is consistent with (4.2). We will return to deal with the the case $n > 1$ later on.