Chapter 7

Applications to commutative algebra

Refs.

1. Eisenbud, Commutative algebra
2. Rotman, Intro. to homological algebra
3. Weibel, Intro. to homological algebra

7.1 Global dimensions

For this section, R is a not necessarily commutative ring. Given an R-module M, we say that the projective dimension $pd(M) \leq n \in \mathbb{N} \cup \{\infty\}$ if there exists a projective resolution

$$\dots P_1 \to P_0 \to M \to 0,$$

such that $P_i = 0$ for $i > n$. We say $pd(M) = n$, if $pd(M) \leq n$ and not for anything smaller. Clearly $pd(M) = 0$ if and only if M is projective. We have to following test

Proposition 7.2. *The following are equivalent*

1. $pd(M) \leq n$
2. *For all* N, $Ext_R^{n+1}(M, N) = 0$
3. *For all* $m > n$ *and for all* N, $Ext_R^m(M, N) = 0$.

Proof. This was a (slightly misstated) homework problem; or see Rotman prop 8.6. □

We define the (left) global dimension of R to be

$$\operatorname{gldim}(R) = \sup pd(M)$$

From the previous proposition, we deduce

Corollary 7.3. $\operatorname{gldim}(R) \leq n$ *if and only if* $Ext^{n+1}(M,N) = 0$ *for all* M *and* N*.*

Example 7.4. *If* R *is a field, or more generally a division ring, all modules are free by standard arguments in linear algebra. Therefore* $\operatorname{gldim} R = 0$*.*

Example 7.5. *Let* $R = \mathbb{Z}[t]/(t^2-1)$*. A previous homework problem showed that group cohomology of* $H^i(Z/2, Z/2) = Ext^i(\mathbb{Z}, \mathbb{Z}/2) \neq 0$ *for all* i*. Therefore* $\operatorname{gldim} R = \infty$*.*

We can define the injective dimension $id(M)$ of a module as the length of the shortest injective resolution

$$0 \to M \to I^0 \to I^1 \ldots I^n \to 0$$

This is the same the projective dimension in Mod_R^{op}. Therefore, we obtain a result dual to the previous proposition.

Proposition 7.6. *The following are equivalent*

1. $id(M) \leq n$
2. *For all* N*,* $Ext_R^{n+1}(N,M) = 0$
3. *For all* $m > n$ *and for all* N*,* $Ext_R^m(N,M) = 0$*.*

By combining this with the previous corollary, we obtain

Corollary 7.7. $\operatorname{gldim} R = \sup id(M)$*.*

Proposition 7.8. $\operatorname{gldim} R = \sup pd(M)$ *as* M *varies over finitely generated left modules.*

Proof. Let $n = \sup pd(M)$ over finitely generated modules. Given a module N, it suffices to show that $id(N) \leq n$. Let

$$0 \to N \to E^0 \to E^1 \ldots E^{n-1} \to C \to 0 \tag{7.1}$$

be a resolution with E^i injective. If we can show that C is injective, then we are done. We can use Baer's criterion, which says that C is injective if any homomorphism from a left ideal $I \to C$ extends to a homomorphism from $R \to C$. From the sequence

$$Hom(R,C) \to Hom(I,C) \to Ext^1(R/I,C)$$

we see that it suffices to prove that $Ext^1(R/I, C) = 0$. By breaking up (7.1) into short exact sequences

$$0 \to N \to E^0 \to E^0/N \to 0$$

etc., we find that

$$Ext^n(R/I, E^0) \to Ext^n(R/I, E^0/N) \to Ext^{n+1}(R/I, N) \to Ext^{n+1}(R/I, E^0) = 0$$

etc. Therefore we get isomorphisms

$$Ext^{n+1}(R/I, N) \cong Ext^n(R/I, E^0/N) \ldots \cong Ext^1(R/I, C)$$

Since R/I is finitely generated, $Ext^{n+1}(R/I, N) = 0$. □

7.9 Global dimension of commutative rings

From now on, let us assume that R is commutative and noetherian.

Theorem 7.10. *The global dimension*

$$\text{gldim } R = \sup_{p \in \text{Spec } R} \text{gldim } R_p$$

Proof. Suppose that M is finitely generated. Earlier we proved that

$$Ext^i_R(M, N)_p = Ext^i_{R_p}(M_p, N_p) \tag{7.2}$$

Therefore $Ext^i_R(M, N) = 0$ for $i > \sup \text{gldim}\, R_p$. This proves $\text{gldim}\, R \leq \sup \text{gldim}\, R_p$. So it remains to prove the opposite inequality. For this we need

Lemma 7.11. *Any (finitely generated) R_p-module is isomorphic to M_p for some (finitely generated) R-module M.*

Proof. If $\mathcal{M}$ is a finitely generated R_p-module, we can find a presentation

$$R_p^n \xrightarrow{A} R_p^m \to \mathcal{M} \to 0$$

We can find a matrix B over R such that $A = \frac{1}{f}B$ with $f \notin p$. It follows that $\mathcal{M} = M_p$ where $M = R^m/BR^n$.

If $\mathcal{M}$ is not finitely generated then we can take $M = \mathcal{M}$ regarded as an R-module. □

From the lemma along with (7.2), we obtain $\text{gldim}\, R_p \leq \text{gldim}\, R$.

□

To understand global dimensions of commutative noetherian rings, we can, by the previous theorem, focus on the case where R commutative noetherian local ring. We now assume this. Let m be the unique maximal ideal, and $k = R/m$ the residue field.

Proposition 7.12. *If M is a finitely generated R-module, then*

$$pd(M) \leq n \Leftrightarrow Tor^R_{n+1}(M,k) = 0$$

Proof. If $pd(M) \leq n$, then M has a projective resolution $P_\bullet$ of length $\leq n$, therefore $Tor_{n+1}(M,k) = H_i(P_\bullet \otimes k) = 0$.

For the converse, by dimension shifting, it is enough to consider $n = 0$. That is assuming $Tor_1(M,k) = 0$, we have to show that M is projective. Choose set of generators $m_i \in M$ which reduce to a basis of $M \otimes k$. Consider the map $R^n \to M$ sending the ith basis vector of R^n to m_i. We have an exact sequence

$$0 \to K \otimes R^n \to M \to 0$$

After tensoring with k, we obtain

$$0 = Tor_1(M,k) \to K \otimes k \to k^n \to M \otimes k \to 0$$

The last map is bijective by construction, therefore $K \otimes k = 0$. This implies $K = 0$ by Nakayama's lemma. □

The ring R is called regular if m can be generated by d elements, where $d = \dim R$ is the Krull dimension (the maximal length of a chain of prime ideals $p_0 \subsetneq p_1 \ldots \subsetneq p_d$). The importance of regularity comes from algebraic geometry. A point $x \in X$ in algebraic variety is nonsingular if and only if the corresponding local ring is regular.

Example 7.13. *If k is a field, then the localization $= k[x_1,\ldots,x_d]_{(x_1,\ldots,x_d)}$ is regular, since the maximal is generated by x_i and $d = \dim R$. In geometric terms, this says that the origin of affine space $\mathbb{A}^d_k$ is nonsingular.*

The following fundamental theorem gives an elegant characterization.

Theorem 7.14 (Auslander-Buschsbaum-Serre)**.** *A noetherian local ring R is regular if and only if* $\operatorname{gldim} R < \infty$*. The global dimension of R coincides with the Krull dimension.*

Proof. We only prove one direction. Suppose that R is regular, then we can find $d = \dim R$ element $x_1,\ldots,x_d$ generating m. Then $x_1,\ldots,x_d$ is a regular sequence by Eisenbud corollary 10.15. Therefore the Koszul complex $K_\bullet = K(x_1,\ldots,x_d)$ gives a projective resolution of k of length d. It follows that $Tor_i(M,k) = H_i(M \otimes K_\bullet) = 0$ for $i > d$. If M is finitely generated, this implies that $pd(M) \leq d$ be the previous proposition. This implies $\operatorname{gldim} R \leq d$. Since $Tor_d(k,k) = k \neq 0$, we must have equality. □

Corollary 7.15. *The localization of a regular local ring is regular local.*

We define a commutative ring to be regular if all of its local rings are regular. The last corollary says that regular local rings are regular in this sense. Given a field k, a prime ideal $I \subseteq k[x_1,\ldots,x_n]$ defines an affine algebraic variety

$$X = V(I) = \{a \in \mathbb{A}^n_k \mid \forall f \in I, f(a) = 0\}$$

The ring $k[x_1, \dots, x_n]/I$ is called the coordinate ring. X is called nonsingular if the ring is regular.

Corollary 7.16. *A regular ring of finite Krull dimension has finite global dimension. In particular, this is the case for the coordinate ring of a nonsingular affine algebraic variety.*

7.17 Regular local rings are UFDs

We will continue to assume rings are commutative. Recall that an integral domain is a unique factorization domain, or a factorial ring, if every nonzero element is a product of a unit times a product of irreducible elements in an essentially unique way. We give some useful criteria for this property. First recall that a prime ideal p is a minimal prime of an ideal I if it contains I and there are no primes between I and p. The height of a prime ideal p is the length of the longest chain $p_0 \subsetneq p_1 \dots \subsetneq p_n = p$. In algebro-geometric terms minimal primes correspond to irreducible components of $V(I)$, and the height of p is the codimension of the subvariety defined by p.

Proposition 7.18. *Let R be a noetherian domain. The following are equivalent*

1. *R is a UFD .*
2. *Every minimal prime of a principal ideal is itself principal.*
3. *Every height one prime is principal.*

If $x \in R$ generates a prime ideal, and if $R[x^{-1}]$ is a UFD, then so is R.

Proof. This follows from proposition 3.11, corollary 10.6, and lemma 19.20 of Eisenbud. □

We will use the next lemma below.

Lemma 7.19. *If P is a rank one projective module which admits a finite resolution by finitely generated free modules, then P is free.*

Proof. Combine proposition 19.16 and lemma 19.18 from Eisenbud. □

Theorem 7.20 (Auslander-Buchsbaum)**.** *A regular local ring is a UFD.*

Proof. If $x \in R$ generates a prime, then it is enough to show that $R[x^{-1}]$ is a UFD by the previous proposition. Suppose that $q \in \operatorname{Spec} R[x^{-1}]$ is height one, then it is enough to prove that it is principal by the same proposition. If $p \in \operatorname{Spec} R[x^{-1}]$, then $R[x^{-1}]_p$ is a localization of R. By the corollary 7.15, it is regular. Furthermore $\dim R[x^{-1}]_p < \dim R$. So by induction, we can assume that it is a UFD. This shows that each Q_p is principal, and therefore that Q is projective. To show that is principal, it is enough to show that it free. We can find an R-module Q' such that $Q = Q'[x^{-1}]$. Since R has finite global dimension, we can find a finite resolution of $F_\bullet \to Q'$ by finitely generated projective R-modules. Earlier we proved that any finitely generated projective module over a noetherian local ring is free. Therefore $F_\bullet[x^{-1}] \to Q$ is a free resolution. Lemma 7.19 implies that Q is free. $\square$

In algebraic geometry, it is important to study algebraic varieties in terms of their codimension one subvarieties, called Weil divisors. The best behaved among these are the Cartier divisors which are locally defined by a single equation $f = 0$. The above theorem guarantees than on a nonsingular variety (or more generally regular scheme), all Weil divisors are Cartier.