Introduction to differential forms

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1 1-forms

A differential 1-form (or simply a differential or a 1-form) on an open subset of $\mathbb{R}^2$ is an expression $F(x,y)dx + G(x,y)dy$ where F, G are $\mathbb{R}$-valued functions on the open set. A very important example of a differential is given as follows: If $f(x,y)$ is C^1 $\mathbb{R}$-valued function on an open set U, then its total differential (or exterior derivative) is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial g}dy$$

It is a differential on U.

In a similar fashion, a differential 1-form on an open subset of $\mathbb{R}^3$ is an expression $F(x,y,z)dx + G(x,y,z)dy + H(x,y,z)dz$ where F, G, H are $\mathbb{R}$-valued functions on the open set. If $f(x,y,z)$ is a C^1 function on this set, then its total differential is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

At this stage, it is worth pointing out that a differential form is very similar to a vector field. In fact, we can set up a correspondence:

$$F\mathbf{i} + G\mathbf{j} + H\mathbf{k} \leftrightarrow Fdx + Gdy + Hdz$$

Under this set up, the gradient ∇f corresponds to df. Thus it might seem that all we are doing is writing the previous concepts in a funny notation. However, the notation is very suggestive and ultimately quite powerful. Suppose that that x, y, z depend on some parameter t, and f depends on x, y, z, then the chain rule says

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Thus the formula for df can be obtained by canceling dt.

2 Exactness in $\mathbb{R}^2$

Suppose that $Fdx + Gdy$ is a differential on $\mathbb{R}^2$ with C^1 coefficients. We will say that it is *exact* if one can find a C^2 function $f(x,y)$ with $df = Fdx + Gdy$

Most differential forms are not exact. To see why, note that the above equation is equivalent to

$$F = \frac{\partial f}{\partial x},\ G = \frac{\partial f}{\partial y}.$$

Therefore if f exists then

$$\frac{\partial F}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial G}{\partial x}$$

But this equation would fail for most examples such as ydx. We will call a differential *closed* if $\frac{\partial F}{\partial y}$ and $\frac{\partial G}{\partial x}$ are equal. So we have just shown that if a differential is to be exact, then it had better be closed.

Exactness is a very important concept. You've probably already encountered it in the context of differential equations. Given an equation

$$\frac{dy}{dx} = F(x, y)$$

we can rewrite it as

$$Fdx - dy = 0$$

If the differential on the left is exact and equal to say, df, then the curves $f(x, y) = c$ give solutions to this equation.

These concepts arise in physics. For example given a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ representing a force, one would like find a function $P(x, y)$ called the potential energy, such that $\mathbf{F} = -\nabla P$. The force is called *conservative* (see section 11) if it has a potential energy function. In terms of differential forms, $\mathbf{F}$ is conservative precisely when $F_1dx + F_2dy$ is exact.

3 Line integrals

Now comes the real question. Given a differential $Fdx + Gdy$, when is it exact? Or equivalently, how can we tell whether a force is conservative or not? Checking that it's closed is easy, and as we've seen, if a differential is not closed, then it can't be exact. The amazing thing is that the converse statement is often (although not always) true:

THEOREM 4 *If $F(x, y)dx + G(x, y)dy$ is a closed form on all of $\mathbb{R}^2$ with C^1 coefficients, then it is exact.*

To prove this, we would need solve the equation $df = Fdx + Gdy$. In other words, we need to undo the effect of d and this should clearly involve some kind of integration process. To define this, we first have to choose a piecewise C^1 parametric curve $\mathbf{c} : [a, b] \to \mathbb{R}^2$. In other words, we let x and y depend on some parameter t running from a to b.

DEFINITION 5

$$\int_{\mathbf{c}} Fdx + Gdy = \int_a^b [F(x(t), y(t))\frac{dx}{dt} + G(x(t), y(t))\frac{dy}{dt}]dt$$

Although we've done everything at once, it is often easier, in practice, to do this in steps. First change the variables from x and y to expresions in t, then replace dx by $\frac{dx}{dt}dt$ etc. Then integrate with respect to t.

While we're at it, we can also define a line integral in $\mathbb{R}^3$. Suppose that $Fdx + Gdy + Hdz$ is a differential form with C^1 coeffients. Let $c : [a, b] \to \mathbb{R}^3$ be a piecewise C^1 parametric curve, then

DEFINITION 6

$$\int_{\mathbf{c}} Fdx+Gdy+Hdz = \int_a^b [F(x(t), y(t), z(t))\frac{dx}{dt}+G(x(t), y(t), z(t))\frac{dy}{dt}+H(x(t), y(t), z(t))\frac{dz}{dt}]dt$$

[Many examples done in class.]

The notion of exactness extends to $\mathbb{R}^3$ automatically: a form is exact if it equals df for a C^2 function. One of the most important properties of exactness is its path independence:

PROPOSITION 7 *If ω is exact and $\mathbf{c}_1$ and $\mathbf{c}_2$ are two parametrized curves with the same endpoints (or more acurately the same starting point and ending point), then*

$$\int_{\mathbf{c}_1} \omega = \int_{\mathbf{c}_2} \omega$$

It's quite easy to see why this works. If $\omega = df$ and $\mathbf{c}_1 : [a, b] \to \mathbb{R}^3$ then

$$\int_{\mathbf{c}_1} df = \int_a^b \frac{df}{dt}dt$$

by the chain rule. Now the fundamental theorem of calculus shows that the last integral equals $f(\mathbf{c}_1(b)) - f(\mathbf{c}_1(a))$, which is to say the value of f at the endpoint minus its value at the starting point. A similar calculation shows that the integral over $\mathbf{c}_2$ gives same answer.

Now we can describe the basic idea for proving theorem 4. If $Fdx + Gdy$ is a closed form on $\mathbb{R}^2$, set

$$f(x, y) = \int_{\mathbf{c}} Fdx + Gdy$$

where the curve is indicated below:

Then $df = Fdx + Gdy$. [Details in class, see also Marsden and Tromba].

The theorem will generally fail if we replace $\mathbb{R}^2$ by a subset. For example,

$$-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

is C^1 1-form on the open set $\{(x,y)\,|\,(x,y) \neq (0,0)\}$ which is closed but not exact.

8 Work

Line integrals have many important uses. One very direct application in physics comes from the idea of work. If you pick up a rock off the ground, or perhaps roll it up a ramp, it takes energy. The energy expended is called work. If you're moving the rock in straight line for a short distance, then the displacement can be represented by a vector $\mathbf{d} = (\Delta x, \Delta y, \Delta z)$ and the force of gravity by a vector $\mathbf{F} = (F_1, F_2, F3)$. Then the work done is simply

$$-\mathbf{F}\cdot\mathbf{d} = -(F_1\Delta x + F_2\Delta y + F_3\Delta z).$$

On the other hand, if you decide to shoot a rocket up into space, then you would have to take into account that the trajectory $\mathbf{c}$ may not be straight nor can the force $\mathbf{F}$ be assumed to be constant (it's a vector field). However as the notation suggests, for the work we would now need to calculate the integral

$$-\int_{\mathbf{c}} F_1dx + F_2dy + F_3dz$$

One often writes this as

$$-\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s}$$

(think of $d\mathbf{s}$ as the "vector" (dx, dy, dz).)

9 2-forms

Recall that the cross product is an operation on vector fields satisfying:

$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ (anticommutative law)

$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ (distributive law)

Geometrically $\mathbf{u} \times \mathbf{v}$ represents the vector whose length is the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$ with direction determined by right hand rule.

We'll introduce an operation on 1-forms called the *wedge product* (written as $\wedge$) which is analogous to the cross product. One important difference is that while the cross product of two vectors is again a vector, the wedge product results a new kind of expression called a 2-form. The wedge product will be both anticommutative and distributive like the cross product:

$$\alpha \wedge \beta = -\beta \wedge \alpha$$

$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$$

A typical 2-form looks like this:

$$F(x,y,z)dx \wedge dy + G(x,y,z)dy \wedge dz + H(x,y,z)dz \wedge dx$$

where F, G and H are functions defined on an open subset of $\mathbb{R}^3$. The real significance of 2-forms will come later when we do surface integrals. A 2-form will be an expression that can be integrated over a surface in the same way that a 1-form can be integrated over a curve.

10 "d" of a 1-form and the curl

Given a 1-form $F(x,y,z)dx + G(x,y,z)dy + H(x,y,z)dz$. We want to define its derivative $d\omega$ which will be a 2-form. The rules we use to evaluate it are:

$$d(\alpha + \beta) = d\alpha + d\beta$$

$$d(f\alpha) = (df) \wedge \alpha + f d\alpha$$

$$d(dx) = d(dy) = d(dz) = 0$$

where α and β are 1-forms and f is a function. Putting these together yields a formula

$$d(F dx + G dy + H dz) = (G_x - F_y)dx \wedge dy + (H_y - G_z)dy \wedge dz + (F_z - H_x)dz \wedge dx$$

where $F_x = \frac{\partial F}{\partial x}$ and so on.

A 2-form can be converted to a vector field by replacing $dx \wedge dy$ by $\mathbf{k} = \mathbf{i} \times \mathbf{j}$, $dy \wedge dz$ by $\mathbf{i} = \mathbf{j} \times \mathbf{k}$ and $dz \wedge dx$ by $\mathbf{j} = \mathbf{k} \times \mathbf{i}$. If we start with a vector field $\mathbf{V} = F\mathbf{i} + G\mathbf{j} + H\mathbf{k}$, replace it by a 1-form $F dx + G dy + H dz$, apply d, then convert it back to a vector field, we end up with the curl of $\mathbf{V}$

$$\nabla \times \mathbf{V} = (H_y - G_z)\mathbf{i} + (G_x - F_y)\mathbf{k} + (F_z - H_x)\mathbf{j}$$

(In practice, one often writes this as a determinant

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial dx} & \frac{\partial}{\partial dy} & \frac{\partial}{\partial dz} \\ F & G & H \end{vmatrix}.$$

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11 Exactness in $\mathbb{R}^3$ and conservation of energy

A C^1 1-form $\omega = Fdx + Gdy + Hdz$ is called exact if there is a C^2 function (called a potential) such that $\omega = df$. ω is called closed if $d\omega = 0$, or equivalently if

$$F_y = G_x,\ F_z = H_x,\ G_z = H_y$$

Then exact 1-forms are closed.

THEOREM 12 *If $\omega = Fdx + Gdy + Hdz$ is a closed form on $\mathbb{R}^3$ with C^1 coefficients, then ω is exact. In fact if $f(x_0, y_o, z_0) = \int_C \omega$, where C is any piecewise C^1 curve connecting $(0,0,0)$ to (x_0, y_0, z_0), then $df = \omega$.*

This can be rephrased in the language of vector fields. If $\mathbf{F} = F\mathbf{i} + G\mathbf{j} + H\mathbf{k}$ is C^1 vector field representing a force, then it is called conservative if there is a C^2 real valued function P, called potential energy, such that $\mathbf{F} = -\nabla P$. The theorem implies that a force $\mathbf{F}$, which is C^1 on all of $\mathbb{R}^3$, is conservative if and only if $\nabla \times \mathbf{F} = 0$. $P(x, y, z)$ is given the work done by moving a particle of unit mass along a path connecting $(0,0,0)$ to (x, y, z).

To appreciate the importance of this concept, recall from physics that the kinetic energy of a particle of constant mass m and velocity

$$\mathbf{v} = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})$$

is

$$K = \frac{1}{2}m||\mathbf{v}||^2 = \frac{1}{2}m\mathbf{v} \cdot \mathbf{v}.$$

Also one of Newton's laws says

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}$$

If $\mathbf{F}$ is conservative, then we can replace it by $-\nabla P$ above, move it to the other side, and then dot both sides by $\mathbf{v}$ to obtain

$$m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \mathbf{v} \cdot \nabla P = 0$$

which simplifies[1] to

$$\frac{d}{dt}(K + P) = 0.$$

This implies that the total energy $K + P$ is constant.

[1]This takes a bit of work that I'm leaving as an exercise. It's probably easier to work backwards. You'll need the product rule for dot products and the chain rule.

13 "d" of a 2-form and divergence

A 3-form is simply an expression $f(x,y,z)dx \wedge dy \wedge dz$. These are things that will eventually get integrated over solid regions. The important thing for the present is an operation which takes 2-forms to 3-forms once again denoted by "d".

$$d(F dy \wedge dz + G dz \wedge dx + H dx \wedge dy) = (F_x + G_y + H_z) dx \wedge dy \wedge dz$$

It's probably easier to understand the pattern after converting the above 2-form to the vector field $\mathbf{V} = F\mathbf{i} + G\mathbf{j} + H\mathbf{k}$. Then the coeffiecient of $dx \wedge dy \wedge dz$ is the divergence

$$\nabla \cdot \mathbf{V} = F_x + G_y + H_z$$

So far we've applied d to functions to obtain 1-forms, and then to 1-forms to get 2-forms, and finally to 2-forms. The real power of this notation is contained in the following simple-looking formula

PROPOSITION 14 $d^2 = 0$

What this means is that given a C^2 real valued function defined on an open subset of $\mathbb{R}^3$, then $d(df) = 0$, and given a 1-form $\omega = F dx + G dy + H dz$ with C^2 coefficents defined on an open subset of $\mathbb{R}^3$, $d(d\omega) = 0$. Both of these are quite easy to check:

$$d(df) = (f_{yx} - f_{xy}) dx \wedge dy + (f_{zy} - f_{yz}) dy \wedge dz + (f_{xz} - f_{zx}) dz \wedge dx = 0$$

$$d(d\omega) = [G_{xz} - F_{yz} + H_{yx} - G_{zx} + F_{zy} - H_{xy}] dx \wedge dy \wedge dz = 0$$

In terms of standard vector notation this is equivalent to

$$\nabla \times (\nabla f) = 0$$

$$\nabla \cdot (\nabla \mathbf{V}) = 0$$