

Holonomic D -Modules

In this chapter we study functorial behaviors of holonomic systems and show that any simple object in the abelian category of holonomic D_X -modules is a minimal extension of an integrable connection on a locally closed smooth subvariety Y of X .

3.1 Basic results

Recall that the dimension of the characteristic variety $\text{Ch}(M)$ of a coherent D_X -module M ($\neq 0$) satisfies the inequality $\dim \text{Ch}(M) \geq \dim X$ and that a coherent D_X -module M is called holonomic if $\dim \text{Ch}(M) = \dim X$ or $M = 0$.

Notation 3.1.1. We denote by $\text{Mod}_h(D_X)$ the full subcategory of $\text{Mod}_c(D_X)$ consisting of holonomic D_X -modules.

The next proposition implies that $\text{Mod}_h(D_X)$ is a thick abelian subcategory of $\text{Mod}_c(D_X)$.

Proposition 3.1.2.

(i) For an exact sequence

$$0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$$

in $\text{Mod}_c(D_X)$ we have

$$N \in \text{Mod}_h(D_X) \iff M, L \in \text{Mod}_h(D_X).$$

(ii) Any holonomic D_X -module has finite length. In other words, the category $\text{Mod}_h(D_X)$ is **artinian**.

Proof. The statement (i) is a consequence of $\text{Ch}(N) = \text{Ch}(M) \cup \text{Ch}(L)$.

The statement (ii) is proved using the characteristic cycle as follows. For a holonomic D_X -module M consider its characteristic cycle

$$\mathbf{CC}(M) = \sum_{C \in I(\text{Ch}(M))} m_C(M) C.$$

Note that $\dim C = d_X$ for any $C \in I(\text{Ch}(M))$. Define the *total multiplicity* of M by

$$m(M) := \sum_{C \in I(\text{Ch}(M))} m_C(M).$$

By Theorem 2.2.3 the total multiplicity is additive in the sense that we have $m(M) = m(L) + m(N)$ for any short exact sequence

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

in $\text{Mod}_h(D_X)$. Moreover, we have $m(M) = 0 \iff \text{Ch}(M) = \emptyset \iff M = 0$ by the definition of characteristic varieties. Hence the assertion follows by induction on $m(M)$. \square

Notation 3.1.3. We denote by $D_h^b(D_X)$ the full subcategory of $D_c^b(D_X)$ consisting of objects $M \in D_c^b(D_X)$ whose cohomology groups are holonomic, that is, $H^i(M) \in \text{Mod}_h(D_X)$ for $\forall i \in \mathbb{Z}$.

We easily see the following from Propositions 3.1.2 and B.4.7.

Corollary 3.1.4. $D_h^b(D_X)$ is a full triangulated subcategory of $D_c^b(D_X)$.

Remark 3.1.5. It is known that

$$D^b(\text{Mod}_h(D_X)) \xrightarrow{\sim} D_h^b(D_X)$$

(see Beilinson [Bei]).

The following result is the first important step in the study of holonomic D -modules. Namely, we can say “A holonomic D -module is generically an integrable connection.”

Proposition 3.1.6. *Let M be a holonomic D_X -module. Then there exists an open dense subset $U \subset X$ such that $M|_U$ is coherent over \mathcal{O}_U . In other words, $M|_U$ is an integrable connection on U .*

Proof. Let $T_X^*X \subset T^*X$ be the zero section of T^*X and set $S := \text{Ch}(M) \setminus T_X^*X$. If $S = \emptyset$, then M itself is coherent over \mathcal{O}_X by Proposition 2.2.5. Assume that $S \neq \emptyset$. Since S is conic, the dimension of each fiber of $\pi|_S : S \rightarrow \pi(S)$ ($\pi : T^*X \rightarrow X$) is ≥ 1 and hence $\dim \pi(S) < \dim S \leq \dim X$. Therefore, there exists an open subset $U \subset X$ such that $X \setminus \pi(S) \supset U \neq \emptyset$. In this case we have $\text{Ch}(M|_U) \setminus T_U^*U = \emptyset$ and hence $M|_U$ is coherent over \mathcal{O}_U by Proposition 2.2.5. \square

The following result, which can be proved by duality, is also important.

Proposition 3.1.7. *Let $M \in \text{Mod}_{qc}(D_X)$. For an open subset $U \subset X$ suppose that we are given a holonomic submodule N of $M|_U$. Then there exists a holonomic submodule \tilde{N} of M such that $\tilde{N}|_U = N$.*

Proof. By Corollary 1.4.17 we may assume that M is coherent and $M|_U = N$. Set $L = H^0(\mathbb{D}_X M)$. By Corollary 2.6.8 (ii) we have $\text{codim Ch}(L) \geq d_X$ and hence L is a holonomic D_X -module. Moreover, its dual $\tilde{N} = \mathbb{D}_X L$ is also holonomic by Corollary 2.6.8 (vi). By $L = H^0(\mathbb{D}_X M) \simeq \tau^{\geq 0}(\mathbb{D}_X M)$ we have a distinguished triangle

$$K^\cdot \longrightarrow \mathbb{D}_X M \longrightarrow L \xrightarrow{+1},$$

where $K^\cdot = \tau^{\leq -1}(\mathbb{D}_X M)$. By applying \mathbb{D}_X we obtain

$$\tilde{N} \longrightarrow M \longrightarrow \mathbb{D}_X K^\cdot \xrightarrow{+1}.$$

Since the duality functors commute with restrictions to open subsets, we have

$$\tilde{N}|_U = \mathbb{D}_U(L|_U) = \mathbb{D}_U^2(M|_U) = M|_U = N.$$

It remains to show that the canonical morphism $\tilde{N} \rightarrow M$ is injective. For this we have only to show $H^{-1}(\mathbb{D}_X K^\cdot) = 0$. In fact, we will show that

$$H^i(\mathbb{D}_X(\tau^{\geq -k} K^\cdot)) = 0 \quad (i < 0, k > 0) \quad (3.1.1)$$

(note that $\tau^{\geq -k} K^\cdot \simeq K^\cdot$ for $k \gg 0$). Let us first show

$$H^i(\mathbb{D}_X(H^{-k}(K^\cdot)[k])) = 0 \quad (i < 0, k > 0). \quad (3.1.2)$$

For $k > 0$ we have $H^{-k}(K^\cdot) \simeq H^{-k}(\mathbb{D}_X M)$ and hence $\text{codim Ch}(H^{-k}(K^\cdot)) \geq d_X - k$ by Corollary 2.6.8 (ii). Hence the assertion is a consequence of Corollary 2.6.8 (i). Now we prove (3.1.1) by induction on k . If $k = 1$, then we have $\tau^{\geq -k} K^\cdot = H^{-k}(K^\cdot)[k]$, and hence the assertion follows from (3.1.2). Assume $k \geq 2$. By applying \mathbb{D}_X to the distinguished triangle

$$H^{-k}(K^\cdot)[k] \longrightarrow \tau^{\geq -k} K^\cdot \longrightarrow \tau^{\geq -(k-1)} K^\cdot \xrightarrow{+1}$$

we obtain a distinguished triangle

$$\mathbb{D}_X(\tau^{\geq -(k-1)} K^\cdot) \longrightarrow \mathbb{D}_X(\tau^{\geq -k} K^\cdot) \longrightarrow \mathbb{D}_X(H^{-k}(K^\cdot)[k]) \xrightarrow{+1}.$$

Hence the assertion follows from (3.1.2) and the hypothesis of induction. \square

3.2 Functors for holonomic D -modules

3.2.1 Stability of holonomicity

We first note the following, which is an obvious consequence of Corollary 2.6.8.

Proposition 3.2.1. *The duality functor \mathbb{D}_X induces isomorphisms*

$$\mathbb{D}_X : \text{Mod}_h(D_X) \xrightarrow{\simeq} \text{Mod}_h(D_X)^{\text{op}},$$

$$\mathbb{D}_X : D_h^b(D_X) \xrightarrow{\simeq} D_h^b(D_X)^{\text{op}}.$$

The following is also obvious by $\mathrm{Ch}(M \boxtimes N) = \mathrm{Ch}(M) \times \mathrm{Ch}(N)$.

Proposition 3.2.2. *The external tensor product \boxtimes induces the functors*

$$\begin{aligned} (\bullet) \boxtimes (\bullet) &: \mathrm{Mod}_h(D_X) \times \mathrm{Mod}_h(D_Y) \rightarrow \mathrm{Mod}_h(D_{X \times Y}), \\ (\bullet) \boxtimes (\bullet) &: D_h^b(D_X) \times D_h^b(D_Y) \rightarrow D_h^b(D_{X \times Y}). \end{aligned}$$

Recall that for a morphism $f : X \rightarrow Y$ of smooth algebraic varieties we have functors

$$\begin{aligned} \int_f &: D_{qc}^b(D_X) \longrightarrow D_{qc}^b(D_Y), \\ f^\dagger &: D_{qc}^b(D_Y) \longrightarrow D_{qc}^b(D_X). \end{aligned}$$

Moreover, if f is proper (resp. smooth), \int_f (resp. f^\dagger) preserves the coherency and we have the functors

$$\int_f : D_c^b(D_X) \rightarrow D_c^b(D_Y) \quad (\text{resp. } f^\dagger : D_c^b(D_Y) \rightarrow D_c^b(D_X)).$$

However, neither \int_f nor f^\dagger preserves the coherency for general morphisms f . A surprising fact, which we will show in this section, is that the holonomicity is nevertheless preserved by these functors for any morphism $f : X \rightarrow Y$. Namely, we have the following.

Theorem 3.2.3. *Let $f : X \rightarrow Y$ be a morphism of smooth algebraic varieties.*

- (i) \int_f sends $D_h^b(D_X)$ to $D_h^b(D_Y)$.
- (ii) f^\dagger sends $D_h^b(D_Y)$ to $D_h^b(D_X)$.

Corollary 3.2.4. *The internal tensor product $\otimes_{\mathcal{O}_X}^L$ induces the functor*

$$(\bullet) \otimes_{\mathcal{O}_X}^L (\bullet) : D_h^b(D_X) \times D_h^b(D_X) \rightarrow D_h^b(D_X).$$

Proof. This follows from Proposition 3.2.2 and Theorem 3.2.3 (ii) noting that $(\bullet) \otimes_{\mathcal{O}_X}^L (\bullet) = L\Delta_X^* \circ ((\bullet) \boxtimes (\bullet))$, where $\Delta_X : X \rightarrow X \times X$ is the diagonal embedding. \square

The proof of Theorem 3.2.3 will be completed in the next subsection. In the rest of this subsection we reduce it to that of Theorem 3.2.3 (i) in the case when f is the projection $\mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$.

Lemma 3.2.5. *Let $i : X \rightarrow Y$ be a closed embedding. Then for $M \in D_c^b(D_X)$ we have*

$$M \in D_h^b(D_X) \iff \int_i M \in D_h^b(D_Y).$$

Proof. Since \int_i is exact, we may assume that $M = M' \in \text{Mod}_c(D_X)$. Let

$$T^*Y \xleftarrow{\varpi} X \times_Y T^*Y \xrightarrow{\rho} T^*X$$

be the canonical morphisms. Then we have

$$\text{Ch}\left(\int_i M\right) = \varpi\rho^{-1}(\text{Ch}(M)),$$

by Lemma 2.3.5. Since ϖ is a closed embedding and ρ is a smooth surjective morphism with one-dimensional fibers, we have

$$\dim \text{Ch}\left(\int_i M\right) = \dim \text{Ch}(M) + 1,$$

form which we obtain the desired result. □

Next we reduce the proof of Theorem 3.2.3 (i) to the case when f is the projection $\mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$. In order to prove Theorem 3.2.3 (i) it is sufficient to show $\int_f M \in D_h^b(D_Y)$ for $M \in \text{Mod}_h(D_X)$. By considering the decomposition of f into a composite of a closed embedding and a projection we may assume that f is either a closed embedding or a projection. The case of a closed embedding has already been dealt with in Lemma 3.2.5, and hence we can only consider the case when f is the projection $X = Z \times Y \rightarrow Y$. Since the problem is local on Y , we may assume that Y is affine. Take a finite affine open covering $Z = \bigcup_{i=0}^r Z_i$ of Z such that $Z \setminus Z_i$ is a divisor on Z for each i , and set $X_i = Z_i \times Y$. Then $X = \bigcup_{i=0}^r X_i$ is an affine open covering of X . For $0 \leq i_0 < \dots < i_k \leq r$ let $j_{i_0, \dots, i_k} : X_{i_0, \dots, i_k} = \bigcap_{p=0}^k X_{i_p} \rightarrow X$ be the embedding (note that X_{i_0, \dots, i_k} is affine by the choice of Z_i 's). Then M is quasi-isomorphic to the Čech complex

$$\dots \rightarrow 0 \rightarrow C^0(M) \rightarrow C^1(M) \rightarrow \dots \rightarrow C^r(M) \rightarrow 0 \rightarrow \dots$$

with

$$C^k(M) = \bigoplus_{i_0 < \dots < i_k} j_{i_0, \dots, i_k}^*(M|_{X_{i_0, \dots, i_k}})$$

(note $j_{i_0, \dots, i_k}^*(M|_{X_{i_0, \dots, i_k}}) \simeq \int_{j_{i_0, \dots, i_k}} j_{i_0, \dots, i_k}^* M$). Hence it is sufficient to show $\int_f \circ j_{i_0, \dots, i_k} j_{i_0, \dots, i_k}^* M (= \int_f \int_{j_{i_0, \dots, i_k}} j_{i_0, \dots, i_k}^* M) \in D_h^b(D_Y)$ for any (i_0, \dots, i_k) . Therefore, we may assume from the beginning that X and Y are affine. Fix closed embeddings $\alpha : X \hookrightarrow \mathbb{C}^n$, $\beta : Y \hookrightarrow \mathbb{C}^m$, and consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \downarrow \beta \\ X \times Y & & \mathbb{C}^m \\ \downarrow \alpha \times \beta & & \downarrow \\ \mathbb{C}^{n+m} & \xrightarrow{p} & \mathbb{C}^m \end{array}$$

where g is the graph embedding associated to f and p is the projection. By Lemma 3.2.5 $\int_f M \in D_h^b(D_Y)$ if and only if $\int_\beta \int_f M \in D_h^b(D_{\mathbb{C}^m})$. Note that

$$\int_\beta \int_f M = \int_{\beta \circ f} M = \int_p \int_{(\alpha \times \beta) \circ g} M.$$

Since $(\alpha \times \beta) \circ g$ is a closed embedding, we have

$$\int_{(\alpha \times \beta) \circ g} M \in \text{Mod}_h(D_{\mathbb{C}^{n+m}})$$

by Lemma 3.2.5, and hence the problem is reduced to the case when f is the projection $\mathbb{C}^{n+m} \rightarrow \mathbb{C}^m$. Since $\mathbb{C}^{n+m} \rightarrow \mathbb{C}^m$ is a composite of morphisms $\mathbb{C}^k \rightarrow \mathbb{C}^{k-1}$, the problem is finally reduced to the case when f is the projection $\mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$.

Let us show that Theorem 3.2.3 (i) implies Theorem 3.2.3 (ii). So we assume that Theorem 3.2.3 (i) holds and show $f^\dagger M \in D_h^b(D_X)$ for any $M \in \text{Mod}_h(D_Y)$. By decomposing f into a composite of a closed embedding and a projection we may further assume that f is either a closed embedding or a projection. Consider first the case where f is the projection $X = Z \times Y \rightarrow Y$. Then f^* is an exact functor and the complex $f^\dagger M = f^* M[\dim Z]$ is concentrated in the degree $-\dim Z$. Moreover, we have $f^* M \simeq \mathcal{O}_Z \boxtimes M$ and it is holonomic by

$$\text{Ch}(\mathcal{O}_Z \boxtimes M) = \text{Ch}(\mathcal{O}_Z) \times \text{Ch}(M) = T_Z^* \times \text{Ch}(M),$$

and hence $f^\dagger M \in D_h^b(D_X)$. Let us consider the case of a closed embedding $i : X \hookrightarrow Y$. Let $j : U := Y \setminus X \hookrightarrow Y$ be the corresponding open embedding. Then by the results in Section 1.7 there exists a distinguished triangle

$$\int_i i^\dagger M \longrightarrow M \longrightarrow \int_j j^\dagger M \xrightarrow{+1}.$$

We have $j^\dagger M = M|_U \in \text{Mod}_h(D_U)$, and hence (i) implies $\int_j j^\dagger M \in D_h^b(D_Y)$. Therefore, we see by the above distinguished triangle that $\int_i i^\dagger M \in D_h^b(D_Y)$. This implies $i^\dagger M \in D_h^b(D_X)$ by Lemma 3.2.5. Theorem 3.2.3 (ii) is verified assuming Theorem 3.2.3 (i).

3.2.2 Holonomicity of modules over Weyl algebras

In the last subsection the proof Theorem 3.2.3 (i), (ii) was reduced to that of (i) in the case when f is the projection $p : \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-1}$. The aim of this subsection is to prove it using the theory of D -modules on \mathbb{C}^n .

Set

$$D_n := \Gamma(\mathbb{C}^n, D_{\mathbb{C}^n}) = \bigoplus_{\alpha, \beta} \mathbb{C} x^\alpha \partial^\beta,$$

where $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\partial^\beta = \partial_1^{\beta_1} \partial_2^{\beta_2} \cdots \partial_n^{\beta_n}$ for $\beta = (\beta_1, \dots, \beta_n)$. The algebra D_n is called the *Weyl algebra*. Since \mathbb{C}^n is affine, we have equivalences of categories

$$\mathrm{Mod}_{qc}(D_{\mathbb{C}^n}) \xrightarrow{\sim} \mathrm{Mod}(D_n),$$

$$\mathrm{Mod}_c(D_{\mathbb{C}^n}) \xrightarrow{\sim} \mathrm{Mod}_f(D_n)$$

given by $M \mapsto \Gamma(\mathbb{C}^n, M)$. For $N \in \mathrm{Mod}(D_n)$ we denote the corresponding $D_{\mathbb{C}^n}$ -module by \tilde{N} . A D_n -module N is called holonomic if \tilde{N} is a holonomic $D_{\mathbb{C}^n}$ -module.

Let N be a D_n -module. We define its *Fourier transform* \widehat{N} as follows. As an additive group \widehat{N} is the same as N , and the action of the generators x_i, ∂_i of D_n on \widehat{N} is given by

$$x_i \circ s := -\partial_i s, \quad \partial_i \circ s := x_i s.$$

It is easily checked that \widehat{N} is a left D_n -module with respect to this action \circ . This definition of the Fourier transform \widehat{N} is motivated by the classical Fourier transform. The Fourier transform induces equivalences of categories

$$\widehat{(\bullet)} : \mathrm{Mod}(D_n) \xrightarrow{\sim} \mathrm{Mod}(D_n),$$

$$\widehat{(\bullet)} : \mathrm{Mod}_f(D_n) \xrightarrow{\sim} \mathrm{Mod}_f(D_n).$$

The corresponding equivalences for the categories of $D_{\mathbb{C}^n}$ -modules are also denoted by

$$\widehat{(\bullet)} : \mathrm{Mod}_{qc}(D_{\mathbb{C}^n}) \xrightarrow{\sim} \mathrm{Mod}_{qc}(D_{\mathbb{C}^n}),$$

$$\widehat{(\bullet)} : \mathrm{Mod}_c(D_{\mathbb{C}^n}) \xrightarrow{\sim} \mathrm{Mod}_c(D_{\mathbb{C}^n}).$$

Proposition 3.2.6. *Let $p : \mathbb{C}^n (= \mathbb{C} \times \mathbb{C}^{n-1}) \rightarrow \mathbb{C}^{n-1}$ be the projection and let $i : \mathbb{C}^{n-1} (= \{0\} \times \mathbb{C}^{n-1}) \hookrightarrow \mathbb{C}^n (= \mathbb{C} \times \mathbb{C}^{n-1})$ be the embedding. For $M \in \mathrm{Mod}_{qc}(D_{\mathbb{C}^n})$ we have*

$$H^k \left(\widehat{\int_p M} \right) \simeq H^k(Li^* \widehat{M})$$

for any k .

Proof. Set $N = \Gamma(\mathbb{C}^n, M)$. Since p is an affine morphism, we have

$$\int_p M \simeq Rp_*(DR_{\mathbb{C}^n/\mathbb{C}^{n-1}}(M)) \simeq [p_*M \xrightarrow{\partial_1} p_*M],$$

and hence

$$\Gamma \left(\mathbb{C}^{n-1}, H^k \left(\int_p M \right) \right) \simeq \begin{cases} \mathrm{Ker}[N \xrightarrow{\partial_1} N] & (k = -1), \\ \mathrm{Coker}[N \xrightarrow{\partial_1} N] & (k = 0), \\ 0 & (k \neq 0, -1). \end{cases}$$

Therefore, we have

$$\Gamma\left(\mathbb{C}^{n-1}, \widehat{H^k\left(\int_p M\right)}\right) \simeq \begin{cases} \text{Ker}[\widehat{N} \xrightarrow{x_1} \widehat{N}] & (k = -1), \\ \text{Coker}[\widehat{N} \xrightarrow{x_1} \widehat{N}] & (k = 0), \\ 0 & (k \neq 0, -1), \end{cases}$$

$$\simeq \Gamma(\mathbb{C}^{n-1}, H^k(Li^*\widehat{M}))$$

from which we obtain the desired result. □

In proving Theorem 3.2.3 we also need the following results.

Proposition 3.2.7. *A coherent $D_{\mathbb{C}^n}$ -module M is holonomic if and only if \widehat{M} is as well.*

Proposition 3.2.8. *Let $j : (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{n-1} \rightarrow \mathbb{C}^n$ be the embedding. If M is a holonomic $D_{\mathbb{C}^n}$ -module, then so is $H^0(\int_j j^\dagger M)$ (since $(\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{n-1}$ is an affine open subset of \mathbb{C}^n we have $H^k(\int_j j^\dagger M) = 0$ for $k \neq 0$).*

Let us complete the proof of Theorem 3.2.3 assuming Propositions 3.2.7 and 3.2.8. By Propositions 3.2.6 and 3.2.7 and the arguments in the last subsection it is sufficient to show $i^\dagger M \in D_h^b(D_{\mathbb{C}^{n-1}})$ for $M \in \text{Mod}_h(D_{\mathbb{C}^n})$, where $i : \mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$ is as in Proposition 3.2.6. Let $j : (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$ be as in Proposition 3.2.8. By the distinguished triangle

$$\int_i i^\dagger M \longrightarrow M \longrightarrow \int_j j^\dagger M \xrightarrow{+1}$$

we obtain an exact sequence

$$0 \longrightarrow H^0\left(\int_i i^\dagger M\right) \longrightarrow M \longrightarrow H^0\left(\int_j j^\dagger M\right) \longrightarrow H^1\left(\int_i i^\dagger M\right) \longrightarrow 0.$$

Since $H^0(\int_j j^\dagger M)$ is holonomic by Proposition 3.2.8, we obtain $\int_i i^\dagger M \in D_h^b(D_{\mathbb{C}^n})$ (note $H^k(\int_i i^\dagger M) = 0$ for $k \neq 0, 1$). Hence we have $i^\dagger M \in D_h^b(D_{\mathbb{C}^{n-1}})$ by Lemma 3.2.5.

The rest of this subsection is devoted to proving Proposition 3.2.7 and Proposition 3.2.8.

In addition to the usual order filtration F , the Weyl algebra D_n has another filtration B defined by

$$B_i D_n := \sum_{|\alpha|+|\beta|\leq i} \mathbb{C}x^\alpha \partial^\beta \subset D_n.$$

We call it the *Bernstein filtration* of the Weyl algebra D_n . The graded algebra $\text{gr}^B D_n$ associated to the Bernstein filtration B is commutative and isomorphic to the polynomial ring $\mathbb{C}[x, \xi]$ ($x = (x_1, x_2, \dots, x_n)$, $\xi = (\xi_1, \xi_2, \dots, \xi_n)$), as in the case of the usual order filtration. For a D_n -module M we can also define good filtrations F on it with respect to the Bernstein filtration B . Any finitely generated D_n -module has a good filtration. The Bernstein filtration has the advantage that for any good filtration F of a finitely generated D_n -module M each $F_i M$ is *finite dimensional* over \mathbb{C} . Therefore, we can apply results on Hilbert polynomials to the associated graded $\text{gr}^B D_n$ -module.

Proposition 3.2.9.

(i) Let F be a good filtration on a non-zero module $M \in \text{Mod}_f(D_n)$ with respect to the Bernstein filtration. Then there exists a unique polynomial $\chi(M, F; T) \in \mathbb{Q}[T]$ such that

$$\chi(M, F; i) = \dim_{\mathbb{C}} F_i M \quad (i \gg 0).$$

(ii) If the degree of $\chi(M, F; T)$ is d , then the coefficient of the degree d (the highest degree) part of $\chi(M, F; T)$ is $m/d!$ for some integer $m > 0$. These two integers d and m do not depend on the choice of the good filtration F . They depend only on M itself.

Proof. By $\dim_{\mathbb{C}} F_i M = \sum_{k \leq i} \dim_{\mathbb{C}} \text{gr}_k^F M$ most of the statements are well known in algebraic geometry [Ha2, Chapter 1]. Let us show that d and m are independent of the choice of a good filtration. Let F and F' be good filtrations of a finitely generated D_n -module M . By Proposition D.1.3 there exists $i_0 > 0$ satisfying

$$F'_{i-i_0} M \subset F_i M \subset F'_{i+i_0} M,$$

and hence

$$\chi(M, F'; i - i_0) \leq \chi(M, F; i) \leq \chi(M, F'; i + i_0)$$

for $i \gg 0$. The desired result easily follows from this. □

We call $d = d_B(M)$ the *dimension* of M , and $m = m_B(M)$ the *multiplicity* of M .

Proposition 3.2.10. *Let*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence of finitely generated D_n -modules.

(i) *We have $d_B(M) = \text{Max}\{d_B(L), d_B(N)\}$.*

(ii) *We have*

$$m_B(M) = \begin{cases} m_B(L) + m_B(N) & (d_B(L) = d_B(N)), \\ m_B(L) & (d_B(L) > d_B(N)), \\ m_B(N) & (d_B(L) < d_B(N)). \end{cases}$$

Proof. Take a good filtration F on M . With respect to the induced filtrations on L and N we have an exact sequence

$$0 \longrightarrow \text{gr}^F L \longrightarrow \text{gr}^F M \longrightarrow \text{gr}^F N \longrightarrow 0$$

of graded $\text{gr}^B D_n$ -modules. The desired result follows from this. □

Proposition 3.2.11. *For a non-zero finitely generated D_n -module M we have*

$$\dim \text{Ch}(\tilde{M}) = d_B(M).$$

Proof. Set $j(M) := \text{Min}\{i \mid \text{Ext}_{D_n}^i(M, D_n) \neq 0\}$. By applying Theorem D.4.3 to the two filtrations F and B of D_n we have

$$\dim \text{Ch}(\widetilde{M}) = 2n - j(M) = \dim \text{supp}(\widetilde{\text{gr}}^F M),$$

where F is a good filtration on M with respect to the Bernstein filtration and $\widetilde{\text{gr}}^F M$ denotes the corresponding coherent $\mathcal{O}_{\mathbb{C}^{2n}}$ -module. It is well known in algebraic geometry that we have $\dim \text{supp}(\widetilde{\text{gr}}^F M) = d_B(M)$ [Ha2, Chapter 1]. \square

By Proposition 3.2.10 a coherent $D_{\mathbb{C}^n}$ -module \widetilde{M} associated to M is holonomic if and only if $d_B(M) = n$. We can use the following estimate as a useful criterion for the holonomicity of M .

Proposition 3.2.12. *Let M be a (not necessarily finitely generated) non-zero D_n -module. We assume that M has a filtration F bounded from below (with respect to the Bernstein filtration B of D_n) such that there exist constants c, c' satisfying the condition*

$$\dim_{\mathbb{C}} F_i M \leq \frac{c}{n!} i^n + c' i^{n-1}$$

for any i . Then M is holonomic and $m_B(M) \leq c$.

Proof. We first show that any finitely generated non-zero D_n -submodule N of M is holonomic and satisfies $m_B(N) \leq c$. Take a good filtration G on N . By Proposition D.1.3 we have

$$G_i N \subset N \cap F_{i+i_0} M \subset F_{i+i_0} M \quad (\forall i)$$

for some i_0 , and hence

$$\chi(N, G; i) \leq \frac{c}{n!} (i + i_0)^n + c' (i + i_0)^{n-1}.$$

It follows that $d_B(N) \leq n$. By $N \neq 0$ and $d_B(N) = \dim \text{Ch}(N)$ we obtain $d_B(N) = n$ and $m_B(N) \leq c$.

It remains to show that M is finitely generated. It is sufficient to show that any increasing sequence

$$0 \neq N_1 \subset N_2 \subset \dots \subset M$$

of finitely generated submodules of M is stationary. We have shown that N_i is holonomic and satisfies $m_B(N_i) \leq c$. Moreover, we have

$$m_B(N_1) \leq m_B(N_2) \leq m_B(N_3) \leq \dots \leq c$$

by Proposition 3.2.10, and hence the sequence $\{m_B(N_i)\}$ is stationary. This implies the desired result by Proposition 3.2.10. \square

Now we are ready to give proofs of Proposition 3.2.7 and Proposition 3.2.8.

Proof of Proposition 3.2.7. Set $N = \Gamma(\mathbb{C}^n, M)$. By the definition of B and the Fourier transform we have $d_B(N) = d_B(\widehat{N})$. Hence by Proposition 3.2.11 we have $\dim \text{Ch}(M) = \dim \text{Ch}(\widehat{M})$. This implies the desired result. \square

Proof of Proposition 3.2.8. Set $N = \Gamma(\mathbb{C}^n, M)$. Note that $\Gamma(\mathbb{C}^n, H^0(\int_j j^\dagger M))$ is isomorphic to the localization $N_{x_1} = \mathbb{C}[x, x_1^{-1}] \otimes_{\mathbb{C}[x]} N$. Hence it is sufficient to show that N_{x_1} is holonomic. Take a good filtration F of N and define $F_i N_{x_1}$ to be the image of $F_{2i} N \ni s \mapsto x_1^{-i} s \in N_{x_1}$. It is easily checked that this defines a filtration of N_{x_1} with respect to the Bernstein filtration. Moreover, we have

$$\begin{aligned} \dim_{\mathbb{C}} F_i N_{x_1} &\leq \dim_{\mathbb{C}} F_{2i} N \\ &= \frac{m_B(M)}{n!} (2i)^n + O(i^{n-1}) \\ &= \frac{m_B(M)2^n}{n!} i^n + O(i^{n-1}), \end{aligned}$$

and hence N_{x_1} is holonomic by Proposition 3.2.12. \square

3.2.3 Adjunction formulas

Let $f : X \rightarrow Y$ be a morphism of smooth algebraic varieties.

Definition 3.2.13. We define new functors by

$$\begin{aligned} \int_{f!} &:= \mathbb{D}_Y \int_f \mathbb{D}_X : D_h^b(D_X) \longrightarrow D_h^b(D_Y), \\ f^\star &:= \mathbb{D}_X f^\dagger \mathbb{D}_Y : D_h^b(D_Y) \longrightarrow D_h^b(D_X). \end{aligned}$$

Theorem 3.2.14. For $M \in D_h^b(D_X)$ and $N \in D_h^b(D_Y)$ we have natural isomorphisms

$$\begin{aligned} R\mathcal{H}om_{D_Y} \left(\int_{f!} M, N \right) &\xrightarrow{\sim} Rf_* R\mathcal{H}om_{D_X}(M, f^\dagger N), \\ Rf_* R\mathcal{H}om_{D_X}(f^\star N, M) &\xrightarrow{\sim} R\mathcal{H}om_{D_Y} \left(N, \int_f M \right). \end{aligned}$$

Proof. We have

$$\begin{aligned} Rf_* R\mathcal{H}om_{D_X}(M, f^\dagger N) &\simeq Rf_* \left(\left(\Omega_X \otimes_{\mathcal{O}_X}^L \mathbb{D}_X M \right) \otimes_{D_X}^L f^\dagger N \right) [-d_X] \\ &\simeq Rf_* \left(\left(\Omega_X \otimes_{\mathcal{O}_X}^L \mathbb{D}_X M \right) \otimes_{D_X}^L D_{X \rightarrow Y} \otimes_{f^{-1}D_Y}^L f^{-1} N \right) [-d_Y] \\ &\simeq Rf_* \left(\left(\Omega_X \otimes_{\mathcal{O}_X}^L \mathbb{D}_X M \right) \otimes_{D_X}^L D_{X \rightarrow Y} \right) \otimes_{D_Y}^L N[-d_Y] \end{aligned}$$

$$\begin{aligned} &\simeq \left(\Omega_Y \otimes_{\mathcal{O}_Y}^L \int_f \mathbb{D}_X M^\cdot \right) \otimes_{D_Y}^L N^\cdot[-d_Y] \\ &\simeq \left(\Omega_Y \otimes_{\mathcal{O}_Y}^L \mathbb{D}_Y \int_{f!} M^\cdot \right) \otimes_{D_Y}^L N^\cdot[-d_Y] \\ &\simeq R\mathcal{H}om_{D_Y} \left(\int_{f!} M^\cdot, N^\cdot \right). \end{aligned}$$

The first isomorphism is established. The second isomorphism follows from the first by duality. \square

By applying $H^0(R\Gamma(Y, \bullet))$ to the isomorphisms in Theorem 3.2.14, we obtain the following.

Corollary 3.2.15. *For $M^\cdot \in D_h^b(D_X)$ and $N^\cdot \in D_h^b(D_Y)$ we have natural isomorphisms*

$$\begin{aligned} \mathrm{Hom}_{D_h^b(D_Y)} \left(\int_{f!} M^\cdot, N^\cdot \right) &\xrightarrow{\sim} \mathrm{Hom}_{D_h^b(D_X)}(M^\cdot, f^\dagger N^\cdot), \\ \mathrm{Hom}_{D_h^b(D_X)}(f^\star N^\cdot, M^\cdot) &\xrightarrow{\sim} \mathrm{Hom}_{D_h^b(D_Y)} \left(N^\cdot, \int_f M^\cdot \right). \end{aligned}$$

Namely, $\int_{f!}$ (resp. f^\star) is the left adjoint of f^\dagger (resp. \int_f).

Theorem 3.2.16. *There exists a morphism of functors*

$$\int_{f!} \longrightarrow \int_f : D_h^b(D_X) \longrightarrow D_h^b(D_Y).$$

Moreover, if f is proper, then this morphism is an isomorphism.

Proof. By Hironaka's desingularization theorem [Hi], there exists a smooth completion \tilde{X} of X . Since X is quasi-projective, a desingularization \tilde{X} of the Zariski closure \bar{X} of X in the projective space is such a completion (even if X is not quasi-projective, there exists a smooth completion by a theorem due to Nagata). Therefore, the map $f : X \rightarrow Y$ factorizes as

$$X \xrightarrow{g} X \times Y \xrightarrow{j} \tilde{X} \times Y \xrightarrow{p} Y,$$

where g is the graph embedding associated to f and $p = \mathrm{pr}_Y$ is a projection. In this situation, g and p are proper and j is an open embedding. This implies that we can reduce our problem to the cases of proper morphisms and open embeddings. If f is proper, we have an isomorphism

$$\int_{f!} = \mathbb{D}_Y \int_f \mathbb{D}_X \xrightarrow{\sim} \int_f$$

by Theorem 2.7.2. So let us consider the case when $f = j : X \hookrightarrow Y$ is an open embedding. Let $M^\cdot \in D_h^b(D_X)$. By Corollary 3.2.15 we have

$$\begin{aligned} \mathrm{Hom}_{D_h^b(D_Y)} \left(\int_{j!} M^\cdot, \int_j M^\cdot \right) &\simeq \mathrm{Hom}_{D_h^b(D_X)} \left(M^\cdot, j^\dagger \int_j M^\cdot \right) \\ &\simeq \mathrm{Hom}_{D_h^b(D_X)} (M^\cdot, M^\cdot), \end{aligned}$$

and hence we obtain the desired morphism

$$\int_{j!} M^\cdot \longrightarrow \int_j M^\cdot$$

as the image of $\mathrm{id} \in \mathrm{Hom}_{D_h^b(D_X)} (M^\cdot, M^\cdot)$. \square

3.3 Finiteness property

The aim of this section is to show the following.

Theorem 3.3.1. *The following conditions on $M^\cdot \in D_c^b(D_X)$ are equivalent:*

- (i) $M^\cdot \in D_h^b(D_X)$.
- (ii) *There exists a decreasing sequence*

$$X = X_0 \supset X_1 \supset \cdots \supset X_m \supset X_{m+1} = \emptyset$$

of closed subsets of X such that $X_r \setminus X_{r+1}$ is smooth and all of the cohomology sheaves $H^k(i_r^\dagger M^\cdot)$ are integrable connections, where $i_r : X_r \setminus X_{r+1} \hookrightarrow X$ denotes the embedding.

- (iii) *For any $x \in X$ all of the cohomology groups $H^k(i_x^\dagger M^\cdot)$ are finite dimensional over \mathbb{C} , where $i_x : \{x\} \hookrightarrow X$ denotes the inclusion.*

For the proof we need the following.

Lemma 3.3.2. *Let M be a coherent (but not necessarily holonomic) D_X -module. Then there exists an open dense subset $U \subset X$ such that $M|_U$ is projective over \mathcal{O}_U .*

Proof. Take a good filtration F of M . Then $\mathrm{gr}^F M$ is coherent over $\pi_* \mathcal{O}_{T^*X}$. It follows from a well-known fact on coherent sheaves that there exists an open dense subset $U \subset X$ such that $(\mathrm{gr}^F M)|_U$ is free over $\pi_* \mathcal{O}_{T^*U}$. By shrinking U if necessary we may assume that $(\mathrm{gr}^F M)|_U$ is free over \mathcal{O}_U . This implies that each $(F_i M / F_{i-1} M)|_U$ (and hence each $F_i M|_U$) is projective over \mathcal{O}_U . Consequently $M|_U$ is projective over \mathcal{O}_U . \square

Proof of Theorem 3.3.1. (ii) \Rightarrow (i). Set $U_r = X \setminus X_r$. We will show $M^\cdot|_{U_r} \in D_h^b(D_{U_r})$ by induction on r . Assume $M^\cdot|_{U_r} \in D_h^b(D_{U_r})$. Let $j : U_r \rightarrow U_{r+1}$, $i : X_r \setminus X_{r+1} (= U_{r+1} \setminus U_r) \rightarrow U_{r+1}$ be embeddings. Then we have a distinguished triangle

$$\int_i i^\dagger(M^\cdot|_{U_{r+1}}) \longrightarrow M^\cdot|_{U_{r+1}} \longrightarrow \int_j j^\dagger(M^\cdot|_{U_{r+1}}) \xrightarrow{+1}.$$

By $i^\dagger(M^\cdot|_{U_{r+1}}) = i_r^\dagger M^\cdot \in D_h^b(D_{X_r \setminus X_{r+1}})$ we have $\int_i i^\dagger(M^\cdot|_{U_{r+1}}) \in D_h^b(D_{U_{r+1}})$. On the other hand by $j^\dagger(M^\cdot|_{U_{r+1}}) = M^\cdot|_{U_r} \in D_h^b(D_{U_r})$ we have $\int_j j^\dagger(M^\cdot|_{U_{r+1}}) \in D_h^b(D_{U_{r+1}})$. Hence the above distinguished triangle implies $M^\cdot|_{U_{r+1}} \in D_h^b(D_{U_{r+1}})$.

(i) \Rightarrow (iii). By Theorem 3.2.3 we have $i_x^\dagger M^\cdot \in D_h^b(D_{\{x\}})$. Note that $D_{\{x\}} \simeq \mathbb{C}$. Hence the desired result follows from the fact that objects of $D_h^b(D_{\{x\}}) = D_c^b(D_{\{x\}}) = D_c^b(\text{Mod}(\mathbb{C}))$ are just complexes of vector spaces whose cohomology groups are finite dimensional.

(iii) \Rightarrow (ii). It is sufficient to show that for any closed subset Y of X satisfying $Y \supset \text{supp}(M^\cdot) := \bigcup_k \text{supp}(H^k(M^\cdot))$ there exists a decreasing sequence

$$Y = Y_0 \supset Y_1 \supset \cdots \supset Y_m \supset Y_{m+1} = \emptyset$$

of closed subsets of Y such that $Y_r \setminus Y_{r+1}$ is smooth and all of the cohomology sheaves $H^k(j_r^\dagger M^\cdot)$ are integrable connections, where $j_r : Y_r \setminus Y_{r+1} \hookrightarrow X$ denotes the embedding. We will prove this statement by induction on $\dim Y$. Take an open dense smooth subset V of Y , and let $i : V \rightarrow X$ denote the embedding. By Kashiwara's equivalence we have $i^\dagger M^\cdot \in D_c^b(D_V)$. Hence by Lemma 3.3.2 there exists an open dense subset V' of V such that each cohomology sheaf $H^k(i^\dagger M^\cdot)|_{V'}$ is projective over $\mathcal{O}_{V'}$. Therefore, by shrinking V if necessary we may assume from the beginning that each cohomology sheaf $H^k(i^\dagger M^\cdot)$ is coherent over D_V and projective over \mathcal{O}_V . We first show that $H^k(i^\dagger M^\cdot)$ is an integrable connection. Take $x \in V$ and denote by $j_x : \{x\} \hookrightarrow V$ the embedding. Then we have

$$\mathbb{C} \otimes_{\mathcal{O}_{V,x}} H^k(i^\dagger M^\cdot)_x \simeq H^{k+d_V}(j_x^\dagger i^\dagger M^\cdot) \simeq H^{k+d_V}(i_x^\dagger M^\cdot),$$

where the first isomorphism follows from the fact that $H^k(i^\dagger M^\cdot)_x$ is projective over $\mathcal{O}_{V,x}$. Hence the finite-dimensionality of $H^{k+d_V}(i_x^\dagger M^\cdot)$ implies that the rank of the projective $\mathcal{O}_{V,x}$ -module $H^k(i^\dagger M^\cdot)_x$ is finite. It follows that $H^k(i^\dagger M^\cdot)$ is coherent over \mathcal{O}_V , hence an integrable connection. Now take an open subset U of X such that $V = Y \cap U$, and let $j : U \rightarrow X$ be the embedding. Define N^\cdot by the distinguished triangle

$$N^\cdot \longrightarrow M^\cdot \longrightarrow \int_j j^\dagger M^\cdot \xrightarrow{+1}.$$

We easily see that $\int_j j^\dagger M^\cdot \simeq \int_i i^\dagger M^\cdot \in D_h^b(D_X)$, and hence the above distinguished triangle implies $N^\cdot \in D_c^b(D_X)$. We also easily see that $\text{supp}(N^\cdot) \subset Y \setminus V$. Moreover, for any locally closed smooth subset Z of $Y \setminus V$ we have $i_Z^\dagger M^\cdot \simeq i_Z^\dagger N^\cdot$, where $i_Z : Z \rightarrow X$ denotes the embedding. Indeed, we have $i_Z^\dagger \int_j = 0$ by Proposition 1.7.1 (ii). In particular, for any $x \in Y \setminus V$ we have $H^*(i_x^\dagger M^\cdot) \simeq H^*(i_x^\dagger N^\cdot)$. Hence by applying the hypothesis of induction to N^\cdot there exists a decreasing sequence

$$Y \setminus V = Y_1 \supset \cdots \supset Y_m \supset Y_{m+1} = \emptyset$$

of closed subsets of $Y \setminus V$ such that $Y_r \setminus Y_{r+1}$ is smooth and all of the cohomology sheaves $H^k(j_r^\dagger M)$ are integrable connections, where $j_r : Y_r \setminus Y_{r+1} \hookrightarrow X$ denotes the embedding. Then the decreasing sequence

$$Y = Y_0 \supset Y_1 \supset \cdots \supset Y_m \supset Y_{m+1} = \emptyset$$

satisfies the desired property. □

3.4 Minimal extensions

A non-zero coherent D -module M is called simple if it contains no coherent D -submodules other than M or 0 . Proposition 3.1.2 implies that for any holonomic D -module M there exists a finite sequence

$$M = M_0 \supset M_1 \supset \cdots \supset M_r \supset M_{r+1} = 0$$

of holonomic D -submodules such that M_i/M_{i+1} is simple for each i (Jordan–Hölder series of M). In this section we will give a classification of simple holonomic D -modules. More precisely, we will construct simple holonomic D -modules from integrable connections on locally closed smooth subvarieties using functors introduced in earlier sections, and show that any simple holonomic D -module is of this type. This construction corresponds via the Riemann–Hilbert correspondence to the minimal extension (Deligne–Goresky–MacPherson extension) in the category of perverse sheaves.

Let Y be a (locally closed) smooth subvariety of a smooth algebraic variety X . Assume that the inclusion map $i : Y \hookrightarrow X$ is affine. Then $D_{X \leftarrow Y}$ is locally free over D_Y and $Ri_* = i_*$ (higher cohomology groups vanish). Therefore, for a holonomic D_Y -module M we have $H^j \int_i M = H^j \int_{i!} M = 0$ for $\forall j \neq 0$. Namely, we may regard $\int_i M$ and $\int_{i!} M$ as D_X -modules. These D_X -modules are holonomic by Theorem 3.2.3. By Theorem 3.2.16 we have a morphism

$$\int_{i!} M \longrightarrow \int_i M$$

in $\text{Mod}_h(D_X)$.

Definition 3.4.1. We call the image $L(Y, M)$ of the canonical morphism $\int_{i!} M \longrightarrow \int_i M$ the *minimal extension* of M .

By Proposition 3.1.2 the minimal extension $L(Y, M)$ is a holonomic D_X -module.

Theorem 3.4.2.

- (i) *Let Y be a locally closed smooth connected subvariety of X such that $i : Y \rightarrow X$ is affine, and let M be a simple holonomic D_Y -module. Then the minimal extension $L(Y, M)$ is also simple, and it is characterized as the unique simple submodule (resp. unique simple quotient module) of $\int_i M$ (resp. of $\int_{i!} M$).*

- (ii) Any simple holonomic D_X -module is isomorphic to the minimal extension $L(Y, M)$ for some pair (Y, M) , where Y is as in (i) and M is a simple integrable connection on Y .
- (iii) Let (Y, M) be as in (ii), and let (Y', M') be another such pair. Then we have $L(Y, M) \simeq L(Y', M')$ if and only if $\bar{Y} = \bar{Y}'$ and $M|_U \simeq M'|_U$ for an open dense subset U of $Y \cap Y'$.

Proof. (i) We choose an open subset $U \subset X$ containing Y such that $k : Y \hookrightarrow U$ is a closed embedding. Let $j : U \hookrightarrow X$ be the embedding, and let $\text{Mod}_{qc}^{\bar{Y}} D_X$ denote the category of \mathcal{O}_X -quasi-coherent D_X -modules whose support is contained in \bar{Y} . We first show the following four results:

- (a) For any $E \in \text{Mod}_{qc}^{\bar{Y}}(D_X)$ we have $H^l i^\dagger E = 0$ ($l \neq 0$). Hence $H^0 i^\dagger = i^\natural : \text{Mod}_{qc}^{\bar{Y}}(D_X) \rightarrow \text{Mod}_{qc}(D_Y)$ is an exact functor.
- (b) For any non-zero holonomic submodule N of $\int_i M$, we have $i^\dagger N \simeq M$.
- (c) $\int_i M$ (resp. $\int_{i!} M$) has a unique simple holonomic submodule (resp. simple holonomic quotient module).
- (d) For a sequence $0 \neq N_1 \subset N_2 \subset \int_i M$ of holonomic submodules of $\int_i M$, we have $i^\dagger(N_2/N_1) = 0$.

For $E \in \text{Mod}_{qc}^{\bar{Y}}(D_X)$ we have $i^\dagger E = k^\dagger j^\dagger E = k^\dagger j^{-1} E$ and $\text{supp } j^{-1} E \subset Y$. Hence (a) is a consequence of Kashiwara's equivalence.

Let N be as in (b). By Corollary 3.2.15 we have

$$\begin{aligned} \text{Hom}_{D_X}\left(N, \int_i M\right) &= \text{Hom}_{D_X}\left(N, \int_j \int_k M\right) \\ &\simeq \text{Hom}_{D_U}\left(j^\star N, \int_k M\right). \end{aligned}$$

Since j is an open embedding, we have $j^\star = j^\dagger = j^{-1}$. Therefore, the inclusion $N \hookrightarrow \int_i M$ induces a non-zero morphism $\varphi : j^\dagger N \rightarrow \int_k M$. Since $\int_k M$ is a simple holonomic D_U -module by Kashiwara's equivalence, φ is surjective. Applying k^\dagger to it, we obtain a surjective morphism $i^\dagger N \rightarrow k^\dagger \int_k M \simeq M$. On the other hand, we have an injective morphism $i^\dagger N \rightarrow i^\dagger \int_i M = M$ because i^\dagger is exact by (a). Hence we must have $i^\dagger N \simeq M$, and (b) is proved.

Suppose there exist two simple holonomic submodules $L \neq L'$ of $\int_i M$. Set $N = L + L' = L \oplus L'$. Then by (b) we have

$$M \simeq i^\dagger N = i^\dagger L \oplus i^\dagger L' = M \oplus M,$$

which is a contradiction. The assertion (c) for $\int_i M$ is proved. Another assertion for $\int_{i!} M$ is easily proved using the duality functor.

By (a) we have

$$i^\dagger N_1 \subset i^\dagger N_2 \subset i^\dagger \int_i M = M, \quad i^\dagger N_2 / i^\dagger N_1 \simeq i^\dagger(N_2/N_1).$$

Hence (b) implies $i^\dagger(N_2/N_1) = 0$, and (d) is proved.

Now let us finish the proof of (i). By (c) there exists a unique simple holonomic submodule L of $\int_i M$. By Corollary 3.2.15 there exist two isomorphisms

$$\begin{aligned} \mathrm{Hom}_{D_X}\left(\int_{i!} M, L\right) &\simeq \mathrm{Hom}_{D_Y}(M, i^\dagger L) \stackrel{(b)}{\simeq} \mathrm{Hom}_{D_Y}(M, M), \\ \mathrm{Hom}_{D_X}\left(\int_{i!} M, \int_i M\right) &\simeq \mathrm{Hom}_{D_Y}\left(M, i^\dagger \int_i M\right) \simeq \mathrm{Hom}_{D_Y}(M, M), \end{aligned}$$

from which we see that the canonical morphism $\int_{i!} M \rightarrow \int_i M$ is non-zero and factorizes as $\int_{i!} M \rightarrow L \hookrightarrow \int_i M$. Since L is a simple module, the image of this morphism should be L . This completes the proof of (i).

(ii) Assume that L is a simple holonomic D_X -module. We take an affine open dense subset $Y (i : Y \hookrightarrow X)$ of an irreducible component of $\mathrm{supp} L$ so that $i^\dagger L$ is an integrable connection on Y (this is possible by Proposition 3.1.6). Set $M = i^\dagger L$. We easily see by Proposition 3.1.7 that M is simple. Moreover, by Corollary 3.2.15 we get an isomorphism

$$\mathrm{Hom}_{D_X}\left(\int_{i!} M, L\right) \simeq \mathrm{Hom}_{D_Y}(M, i^\dagger L) \simeq \mathrm{Hom}_{D_Y}(M, M) \neq 0,$$

from which we see that there exists a non-zero surjective morphism $\int_{i!} M \rightarrow L$. Namely, L is a simple holonomic quotient module of $\int_{i!} M$. Hence we obtain $L = L(Y, M)$ by (i). The assertion (ii) is proved.

The proof for the last part (iii) is easy and left to the readers. □

Proposition 3.4.3. *Let Y be a locally closed smooth subvariety of X such that $i : Y \rightarrow X$ is affine, and let M be an integrable connection on Y . Then we have*

$$\mathbb{D}_X L(Y, M) \simeq L(Y, \mathbb{D}_Y M).$$

Proof. By the exactness of the duality functor we obtain

$$\begin{aligned} \mathbb{D}_X L(Y, M) &\simeq \mathrm{Im}\left(\mathbb{D}_X \int_i M \rightarrow \mathbb{D}_X \int_{i!} M\right) \simeq \mathrm{Im}\left(\int_{i!} \mathbb{D}_Y M \rightarrow \int_i \mathbb{D}_Y M\right) \\ &= L(Y, \mathbb{D}_Y M). \end{aligned}$$

The proof is complete. □