4

Analytic D-Modules and the de Rham Functor

Although our objectives in this book are algebraic D-modules (D-modules on smooth algebraic varieties), we have to consider the corresponding analytic D-modules (D-modules on the underlying complex manifolds with classical topology) in defining their solution (and de Rham) complexes. In this chapter after giving a brief survey of the general theory of analytic D-modules which are partially parallel to the theory of algebraic D-modules given in earlier chapters we present fundamental properties on the solution and the de Rham complexes. In particular, we give a proof of Kashiwara's constructibility theorem for analytic holonomic D-modules. We note that we also include another shorter proof of this important result in the special case of algebraic holonomic D-modules due to Beilinson–Bernstein. Therefore, readers who are interested only in the theory of algebraic D-modules can skip reading Sections 4.4 and 4.6 of this chapter.

4.1 Analytic D-modules

The aim of this section is to give a brief account of the theory of D-modules on complex manifolds. The proofs are occasionally similar to the algebraic cases and are omitted. Readers can refer to the standard textbooks such as Björk [Bj2] and Kashiwara [Kas18] for details.

Let X be a complex manifold. It is regarded as a topological space via the classical topology, and its dimension is denoted by d_X. We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X, and by Θ_X, Ω_X^p the sheaves of $\mathcal{O}_X$-modules consisting of holomorphic vector fields and holomorphic differential forms of degree p, respectively ($0 \leq p \leq d_X$). We also set $\Omega_X = \Omega_X^{d_X}$. The sheaf D_X of holomorphic differential operators on X is defined as the subring of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and Θ_X. In terms of a local coordinate $\{x_i\}_{1 \leq i \leq n}$ on a open subset U of X we have

$$D_X|_U = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_U \partial^{\alpha},$$

where

$$\partial_i = \frac{\partial}{\partial x_i} \quad (1 \le i \le n), \qquad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \quad (\alpha = (\alpha_1, \dots, \alpha_n)).$$

We have the order filtration $F = \{F_l D_X\}_{l \ge 0}$ of D_X given by

$$F_l D_X|_U = \sum_{|\alpha| \le l} \mathcal{O}_U \partial_x^\alpha \qquad \big(|\alpha| = \sum_i \alpha_i\big),$$

where U and $\{x_i\}$ are as above. It satisfies properties parallel to those in Proposition 1.1.3, and D_X turns out to be a filtered ring. The associated graded ring $\operatorname{gr} D_X$ is a sheaf of commutative algebras over $\mathcal{O}_X$, which is canonically regarded as a subalgebra of $\pi_* \mathcal{O}_{T^*X}$, where $\pi : T^*X \to X$ denotes the cotangent bundle of X.

Note that we have obvious analogies of the contents of Section 1.2, 1.3. In particular, we have an equivalence

$$\Omega_X \otimes_{\mathcal{O}_X} (\bullet) : \operatorname{Mod}(D_X) \longrightarrow \operatorname{Mod}(D_X^{\mathrm{op}})$$

between the categories $\operatorname{Mod}(D_X)$, $\operatorname{Mod}(D_X^{\mathrm{op}})$ of left and right D_X-modules, respectively. Moreover, for a morphism $f : X \to Y$ of complex manifolds we have a $(D_X, f^{-1}D_Y)$-bimodule $D_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y$ and an $(f^{-1}D_Y, D_X)$-bimodule $D_{Y\leftarrow X} = \Omega_X \otimes_{\mathcal{O}_X} D_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{\otimes -1}$. We say that a D_X-module is an integrable connection on X if it is locally free over $\mathcal{O}_X$ of finite rank.

Notation 4.1.1. We denote by $\operatorname{Conn}(X)$ the category of integrable connections on the complex manifold X.

We have an analogy of Theorem 1.4.10. In particular, $\operatorname{Conn}(X)$ is an abelian category.

The following result is fundamental in the theory of analytic D-modules.

Theorem 4.1.2.
(i) *D_X is a coherent sheaf of rings.*
(ii) *For any $x \in X$ the stalk $D_{X,x}$ is a noetherian ring with left and right global dimensions* $\dim X$.

The statement (i) follows from the corresponding fact for $\mathcal{O}_X$ due to Oka, and (ii) is proved similarly to the algebraic case.

We can define the notion of a good filtration on a coherent D_X-module as in Section 2.1. We remark that in our analytic situation a good filtration on a coherent D_X-module exists only locally. In fact, there is an example of a coherent D_X-module which does not admit a global good filtration. Nevertheless, this local existence of a good filtration is sufficient for many purposes. For example, we can define the characteristic variety $\operatorname{Ch}(M)$ of a coherent D_X-module M as follows. For an open subset U of X such that $M|_U$ admits a good filtration F we have a coherent $\mathcal{O}_{T^*U}$-module $\operatorname{gr}^F(M|_U) := \mathcal{O}_{T^*U} \otimes_{\pi_U^{-1}\operatorname{gr} D_U} \pi_U^{-1} \operatorname{gr}^F M|_U$, where $\pi_U : T^*U \to U$ denotes the projection. Then the characteristic variety $\operatorname{Ch}(M)$ is defined to be the closed subvariety of T^*X such that $\operatorname{Ch}(M) \cap T^*U = \operatorname{supp}(\operatorname{gr}^F(M|_U))$ for any U and F as above. It is shown to be well defined by Proposition D.1.3.

As in the algebraic case we have the following.

Theorem 4.1.3. *For any coherent D_X-module M its characteristic variety* $\mathrm{Ch}(M)$ *is involutive with respect to the canonical symplectic structure of the cotangent bundle T^*X. In particular, for any irreducible component Λ of* $\mathrm{Ch}(M)$, *we have that* $\dim \Lambda \geq \dim X$.

We say that a coherent D_X-module M is holonomic if it satisfies

$$\dim \mathrm{Ch}(M) \leq \dim X.$$

Notation 4.1.4.

(i) We denote by $\mathrm{Mod}_c(D_X)$ (resp. $\mathrm{Mod}_h(D_X)$) the category of coherent (resp. holonomic) D_X-modules.

(ii) We denote by $D^b_c(D_X)$ (resp. $D^b_h(D_X)$) the subcategory of $D^b(D_X)$ consisting of $M^{\cdot} \in D^b(D_X)$ satisfying $H^i(M^{\cdot}) \in \mathrm{Mod}_c(D_X)$ (resp. $\mathrm{Mod}_h(D_X)$) for any i.

As in Section 2.6 we can define the duality functor $\mathbb{D}_X : D^b_c(D_X) \to D^b_c(D_X)^{\mathrm{op}}$ satisfying $\mathbb{D}_X^2 \simeq \mathrm{Id}$ by

$$\mathbb{D}_X M^{\cdot} = R\mathcal{H}om_{D_X}(M^{\cdot}, D_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}[d_X]).$$

All of the arguments in Section 2.6 are also valid for analytic D-modules. In particular, $\mathbb{D}_X$ induces $\mathbb{D}_X : \mathrm{Mod}_h(D_X) \to \mathrm{Mod}_h(D_X)^{\mathrm{op}}$.

Let $f : X \to Y$ be a morphism of complex manifolds. The functors

$$Lf^* : D^b(D_Y) \to D^b(D_X) \qquad (M^{\cdot} \mapsto D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}M^{\cdot}),$$
$$f^{\dagger} : D^b(D_Y) \to D^b(D_X) \qquad (M^{\cdot} \mapsto Lf^*M^{\cdot}[d_X - d_Y])$$

are called the inverse image functors. Note that the boundedness of $Lf^*M^{\cdot}$ follows from Theorem 4.1.2 (ii). The notion that $f : X \to Y$ is non-characteristic with respect to a coherent D_Y-module M is defined similarly to the algebraic case, and we have the following analogy of Theorems 2.4.6 and 2.7.1.

Theorem 4.1.5. *Let $f : X \to Y$ be a morphism of complex manifolds and let M be a coherent D_Y-module. Assume that f is non-characteristic with respect to M.*

(i) $H^j(Lf^*M) = 0$ *for* $^{\forall} j \neq 0$.

(ii) $H^0(Lf^*M)$ *is a coherent D_Y-module.*

(iii) $\mathrm{Ch}(H^0(Lf^*M)) \subset \rho_f \varpi_f^{-1}(\mathrm{Ch}(M))$.

(iv) $\mathbb{D}_X(Lf^*M) \simeq Lf^*(\mathbb{D}_Y M)$.

Here, $\rho_f : X \times_Y T^*Y \to T^*X$ and $\varpi_f : X \times_Y T^*Y \to T^*Y$ are the canonical morphisms.

The proof is more or less the same as that for Theorem 2.4.6, 2.7.1.

For a morphism $f : X \to Y$ of complex manifolds we can also define the direct image functor

$$\int_f : D^b(D_X) \to D^b(D_Y) \qquad (M^{\cdot} \mapsto Rf_*(D_{Y\leftarrow X} \otimes^L_{D_X} M^{\cdot})).$$

The fact that $\int_f$ preserves the boundedness can be proved as follows. By decomposing f into a composite of a closed embedding and a projection we may assume that f is either a closed embedding or a projection. The case of a closed embedding is easy. Assume that $f : X = Y \times Z \to Y$ is a projection. We may assume that $M^{\cdot} = M \in \mathrm{Mod}(D_X)$. As in the algebraic case we have $\int_f M = Rf_*(DR_{X/Y}M)$, where $DR_{X/Y}M$ is the relative de Rham complex defined similarly to the algebraic case. Then the assertion follows from the well-known fact that $R^i f_*(K) = 0$ unless $0 \leq i \leq 2 \dim Z$ for any sheaf K on X (see, e.g., [KS2, Proposition 3.2.2]).

We have the following analogy of Theorem 2.5.1, Theorem 2.7.2.

Theorem 4.1.6. *Let $f : X \to Y$ be a proper morphism of complex manifolds. Assume that a coherent D_X-module M admits a good filtration locally on Y.*

(i) $\int_f M \in D^b_c(D_Y)$.
(ii) $\int_f \mathbb{D}_X M \simeq \mathbb{D}_Y \int_f M$.

The proof of this result is rather involved and omitted (see Kashiwara [Kas18]). In the situation where $f : X \to Y$ comes from a proper morphism of smooth algebraic varieties and M is associated to an algebraic coherent D-module (in the sense of Section 4.7 below) the statements (i) and (ii) in Theorem 4.1.6 follow from Theorem 2.5.1, Theorem 2.7.2, respectively, in view of Proposition 4.7.2 (ii) below. We also point out that if f is a projective morphism of complex manifolds, the proof of Theorem 4.1.6 is more or less the same as that of Theorem 2.5.1, 2.7.2.

In the algebraic case holonomicity is preserved under the inverse and direct images; however, in our analytic situation this is true for inverse images but not for general direct images.

Theorem 4.1.7. *Let $f : X \to Y$ be a morphism of complex manifolds, and let M be a holonomic D_Y-module. Then we have $Lf^*M \in D^b_h(D_X)$.*

Theorem 4.1.8. *Let $f : X \to Y$ be a proper morphism of complex manifolds. Assume that a holonomic D_X-module M admits a good filtration locally on Y. Then we have $\int_f M \in D^b_h(D_Y)$.*

Theorem 4.1.7 is proved using the theory of b-functions (see Kashiwara [Kas7]), and Theorem 4.1.8 can be proved using $\mathrm{Ch}(\int_f M) \subset \varpi_f \rho_f^{-1}(\mathrm{Ch}(M))$ and some results from symplectic geometry. The proofs are omitted. We note that in both theorems if we only consider the situation where f comes from a morphism of smooth algebraic varieties and M is associated to an algebraic holonomic D-module, then they are consequences of the corresponding facts on algebraic D-modules in view of Proposition 4.7.2 below.

Example 4.1.9. Let us give an example so that the holonomicity is not preserved by the direct image with respect to a non-proper morphism of complex manifolds even if it comes from a morphism of smooth algebraic varieties. Set $X = \mathbb{C} \setminus \{0\}$, $Y = \mathbb{C}$ and let x be the canonical coordinate of $Y = \mathbb{C}$. Let $j : X \to Y$ be the embedding. We regard it as a morphism of algebraic varieties. If we regard it as a morphism of complex manifolds, we denote it by $j^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$. Then we have $H^0(\int_j \mathcal{O}_X) = j_*\mathcal{O}_X$ and $H^0(\int_{j^{\mathrm{an}}} \mathcal{O}_{X^{\mathrm{an}}}) = j_*^{\mathrm{an}}\mathcal{O}_{X^{\mathrm{an}}}$. Note that $j_*\mathcal{O}_X = \mathcal{O}_Y[x^{-1}]$ is holonomic, while $j_*^{\mathrm{an}}\mathcal{O}_{X^{\mathrm{an}}}$ contains non-meromorphic functions like $\exp(x^{-1})$ and is much larger than $\mathcal{O}_{Y^{\mathrm{an}}}[x^{-1}]$. The $D_{Y^{\mathrm{an}}}$-module $\mathcal{O}_{Y^{\mathrm{an}}}[x^{-1}]$ is holonomic; however, $j_*^{\mathrm{an}}\mathcal{O}_{X^{\mathrm{an}}}$ is not even a coherent $D_{Y^{\mathrm{an}}}$-module.

For a closed submanifold X of a complex manifold Y we denote by $\mathrm{Mod}_c^X(D_Y)$ (resp. $\mathrm{Mod}_h^X(D_Y)$) the category of coherent (resp. holonomic) D_Y-modules whose support is contained in X. Kashiwara's equivalence also holds in the analytic situation.

Theorem 4.1.10. *Let $i : X \hookrightarrow Y$ be a closed embedding of complex manifolds. Then the functor $\int_i$ induces equivalences*

$$\mathrm{Mod}_c(D_X) \xrightarrow{\sim} \mathrm{Mod}_c^X(D_Y),$$
$$\mathrm{Mod}_h(D_X) \xrightarrow{\sim} \mathrm{Mod}_h^X(D_Y)$$

of categories.

The proof is more or less the same as that of the corresponding result on algebraic D-modules.

4.2 Solution complexes and de Rham functors

Let X be a complex manifold. For $M^\cdot \in D^b(D_X)$ we set

$$\begin{cases} DR_X\, M^\cdot := \Omega_X \otimes^L_{D_X} M^\cdot \\ \mathrm{Sol}_X\, M^\cdot := R\mathcal{H}om_{D_X}(M^\cdot, \mathcal{O}_X). \end{cases}$$

We call $DR_X\, M^\cdot \in D^b(\mathbb{C}_X)$ (resp. $\mathrm{Sol}_X\, M^\cdot \in D^b(\mathbb{C}_X)$) the *de Rham complex* (resp. the *solution complex*) of $M^\cdot \in D^b(D_X)$. Then $DR_X(\bullet)$ and $\mathrm{Sol}_X(\bullet)$ define functors

$$DR_X : D^b(D_X) \longrightarrow D^b(\mathbb{C}_X),$$
$$\mathrm{Sol}_X : D^b(D_X) \longrightarrow D^b(\mathbb{C}_X)^{\mathrm{op}}.$$

As we have explained in the introduction, a motivation for introducing the solution complexes $\mathrm{Sol}_X\, M^\cdot = R\mathcal{H}om_{D_X}(M^\cdot, \mathcal{O}_X)$ came from the theory of linear partial differential equations. In fact, for a coherent D_X-module M the sheaf $\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ (on X) is the sheaf of holomorphic solutions to the system of linear PDEs corresponding to M.

By (an analogue in the analytic situation of) Proposition 2.6.14 we have the following.

Proposition 4.2.1. *For $M^\cdot \in D^b_c(D_X)$ we have*

$$DR_X(M^\cdot) \simeq R\mathcal{H}om_{D_X}(\mathcal{O}_X, M^\cdot)[d_X] \simeq \mathrm{Sol}_X(\mathbb{D}_X M^\cdot)[d_X].$$

Hence properties of Sol_X can be deduced from those of DR_X. The functor DR_X has the advantage that it can be computed using a resolution of the right D_X-module Ω_X. In fact, similar to Lemma 1.5.27 we have a locally free resolution

$$0 \to \Omega_X^0 \otimes_{\mathcal{O}_X} D_X \to \cdots \to \Omega_X^{d_X} \otimes_{\mathcal{O}_X} D_X \to \Omega_X \to 0$$

of the right D_X-module Ω_X. It follows that for $M \in \mathrm{Mod}(D_X)$ the object $DR_X(M)[-d_X]$ of the derived category is represented by the complex

$$\Omega_X^{\cdot} \otimes_{\mathcal{O}_X} M = \left[\Omega_X^0 \otimes_{\mathcal{O}_X} M \to \cdots \to \Omega_X^{d_X} \otimes_{\mathcal{O}_X} M\right],$$

where

$$d^p : \Omega_X^p \otimes_{\mathcal{O}_X} M \longrightarrow \Omega_X^{p+1} \otimes_{\mathcal{O}_X} M$$

is given by

$$d^p(\omega \otimes s) = d\omega \otimes s + \sum_i dx_i \wedge \omega \otimes \partial_i s \qquad (\omega \in \Omega_X^p, s \in M)$$

($\{x_i, \partial_i\}$ is a local coordinate system of X).

Let us consider the case where M is an integrable connection of rank m (a coherent D_X-module which is locally free of rank m over $\mathcal{O}_X$). In this case the 0th cohomology sheaf $L := H^0(\Omega_X^{\cdot} \otimes_{\mathcal{O}_X} M) \simeq \mathcal{H}om_{D_X}(\mathcal{O}_X, M)$ of $\Omega_X^{\cdot} \otimes_{\mathcal{O}_X} M$ coincides with the kernel of the sheaf homomorphism

$$d^0 = \nabla : M \simeq \Omega_X^0 \otimes_{\mathcal{O}_X} M \longrightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} M,$$

which is the sheaf

$$M^{\nabla} = \{s \in M \mid \nabla s = 0\} = \{s \in M \mid \Theta_X s = 0\}$$

of *horizontal sections* of the integrable connection M. It is a locally free $\mathbb{C}_X$-module of rank m by the classical Frobenius theorem.

Definition 4.2.2. We call a locally free $\mathbb{C}_X$-module of finite rank a *local system* on X.

Notation 4.2.3. We denote by $\mathrm{Loc}(X)$ the category of local systems on X.

Using the local system $L = M^{\nabla}$ we have a D_X-linear isomorphism $\mathcal{O}_X \otimes_{\mathbb{C}_X} L \simeq M$. Conversely, for a local system L we can define an integrable connection M by $M = \mathcal{O}_X \otimes_{\mathbb{C}_X} L$ and $\nabla = d \otimes \mathrm{id}_L : \mathcal{O}_X \otimes_{\mathbb{C}_X} L \simeq \Omega_X^0 \otimes_{\mathcal{O}_X} M \to \Omega_X^1 \otimes_{\mathbb{C}_X} L \simeq \Omega_X^1 \otimes_{\mathcal{O}_X} M$ such that $M^{\nabla} \simeq L$. As a result, the category of integrable connections on X is equivalent to that of local systems on X.

$$\boxed{\text{integrable connections on } X} \longleftrightarrow \boxed{\text{local systems on } X}$$

Under the identification $\mathcal{O}_X \otimes_{\mathbb{C}_X} L \simeq M$, the differentials in the complex $\Omega_X^{\cdot} \otimes_{\mathcal{O}_X} M$ are written explicitly by

$$d \otimes \mathrm{id}_L : \Omega_X^p \otimes_{\mathbb{C}_X} L \longrightarrow \Omega_X^{p+1} \otimes_{\mathbb{C}_X} L.$$

Therefore, the higher cohomology groups $H^i(\Omega_X^{\cdot} \otimes_{\mathcal{O}_X} M)$ $(i \geq 1)$ of the complex $\Omega_X^{\cdot} \otimes_{\mathcal{O}_X} M$ vanish by the holomorphic Poincaré lemma, and we get finally a quasi-isomorphism $\Omega_X^{\cdot} \otimes_{\mathcal{O}_X} M \simeq L = M^{\nabla}$ for an integrable connection M. We have obtained the following.

Theorem 4.2.4. *Let M be an integrable connection of rank m on a complex manifold X. Then $H^i(DR_X(M)) = 0$ for $i \neq -d_X$, and $H^{-d_X}(DR_X(M))$ is a local system on X. Moreover, we have an equivalence*

$$H^{-d_X}(DR_X(\bullet)) : \mathrm{Conn}(X) \xrightarrow{\sim} \mathrm{Loc}(X)$$

of categories.

Theorem 4.2.5. *Let $f : X \to Y$ be a morphism of complex manifolds. For $M^{\cdot} \in D^b(D_X)$ we have an isomorphism*

$$Rf_* DR_X\ M^{\cdot} \simeq DR_Y \int_f M^{\cdot}$$

in $D^b(\mathbb{C}_Y)$

Proof. By

$$\begin{aligned} DR_Y \int_f M^{\cdot} &= \Omega_Y \otimes_{D_Y}^L Rf_*(D_{Y\leftarrow X} \otimes_{D_X}^L M^{\cdot}) \\ &\simeq Rf_*\big(f^{-1}\Omega_Y \otimes_{f^{-1}D_Y}^L D_{Y\leftarrow X} \otimes_{D_X}^L M^{\cdot}\big), \\ Rf_* DR_X\ M &= Rf_*(\Omega_X \otimes_{D_X}^L M). \end{aligned}$$

It is sufficient to show $\Omega_X \simeq f^{-1}\Omega_Y \otimes_{f^{-1}D_Y}^L D_{Y\leftarrow X}$. This follows easily from Lemma 1.3.4. □

4.3 Cauchy–Kowalevski–Kashiwara theorem

The following classical theorem due to Cauchy–Kowalevski is one of the most fundamental results in the theory of PDEs.

Theorem 4.3.1 (Cauchy–Kowalevski). *Let X be an open subset of $\mathbb{C}^n$ with a local coordinate $\{z_i, \partial_i\}_{1\leq i\leq n}$, and let Y be the hypersurface of X defined by $Y = \{z_1 = 0\}$. Let $P \in D_X$ be a differential operator of order $m \geq 0$ on X such that Y is non-characteristic with respect to P (this notion is defined similarly to the algebraic case, see Example 2.4.4). Then for any holomorphic function $v \in \mathcal{O}_X$ defined on an open neighborhood of Y and any m-tuple $(u_0, \dots, u_{m-1}) \in \mathcal{O}_Y^{\oplus m}$ of holomorphic* *functions on Y, there exists a unique holomorphic solution $u \in \mathcal{O}_X$ defined on an open neighborhood of Y to the Cauchy problem*

$$\begin{cases} Pu = v, \\ \partial_1^j u|_Y = u_j \quad (j = 0, 1, \dots, m-1). \end{cases}$$

For X, Y, P as in Theorem 4.3.1 let $f : Y \to X$ be the inclusion and set $M = D_X/D_X P$. By Theorem 4.1.5 we have $H^i(Lf^*M) = 0$ for $i \neq 0$. Set $M_Y = H^0(Lf^*M)$. Then Theorem 4.3.1 implies in particular that the natural morphism

$$\begin{array}{rcl} f^{-1}\mathcal{H}om_{D_X}(M, \mathcal{O}_X) & & \\ \simeq \{u \in \mathcal{O}_X|_Y \mid Pu = 0\} & \longrightarrow & \mathcal{O}_Y^{\oplus m} \simeq \mathcal{H}om_{D_Y}(M_Y, \mathcal{O}_Y). \\ \Psi & & \Psi \\ u & \longmapsto & (u|_Y, \partial_1 u|_Y, \dots, \partial_1^{m-1} u|_Y) \end{array}$$

obtained by taking the first m-traces of $u \in \mathcal{O}_X|_Y$ is an isomorphism (see Example 2.4.4).

In this section we will give a generalization of this result due to Kashiwara. We first note that results in Section 2.4 for algebraic D-modules can be formulated in the framework of analytic D-modules and proved similarly to the algebraic case. Let $f : Y \to X$ be a morphism of complex manifolds. For any coherent D_X-module M we can construct a canonical morphism

$$\begin{aligned} f^{-1}\mathcal{H}om_{D_X}(M, \mathcal{O}_X) &\longrightarrow \mathcal{H}om_{f^{-1}D_X}(f^{-1}M, f^{-1}\mathcal{O}_X) \\ &\longrightarrow \mathcal{H}om_{D_Y}(\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}M, \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{O}_X) \\ &\simeq \mathcal{H}om_{D_Y}(\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}M, \mathcal{O}_Y), \end{aligned}$$

which extends the above trace map in the classical case. The corresponding morphism

$$\begin{aligned} & f^{-1}\operatorname{Sol}_X(M) \left(= f^{-1}R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)\right) \\ & \qquad \longrightarrow \operatorname{Sol}_Y(Lf^*M) \left(= R\mathcal{H}om_{D_Y}(Lf^*M, \mathcal{O}_Y)\right) \end{aligned}$$

in the derived category $D^b(\mathbb{C}_Y)$ can be also constructed similarly. The following theorem is a vast generalization of the Cauchy–Kowalevski theorem.

Theorem 4.3.2 (Kashiwara [Kas1]). *Let $f : Y \to X$ be a morphism of complex manifolds. Assume that f is non-characteristic for a coherent D_X-module M. Then we have*

$$f^{-1}\operatorname{Sol}_X(M) \xrightarrow{\sim} \operatorname{Sol}_Y(Lf^*M). \tag{4.3.1}$$

Proof. As in the proof of Theorem 2.4.6 we can reduce the problem to the case when Y is a hypersurface in X. Since the problem is local, we may assume that X and Y are as in Theorem 4.3.1. By an analogue (in the analytic situation) of Lemma 2.4.7 we have an exact sequence of coherent D_X-modules

$$0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$$

where $L = \bigoplus_{i=1}^{r} D_X/D_X P_i$ and Y is non-characteristic with respect to each P_i. By the classical Cauchy–Kowalevski theorem (Theorem 4.3.1) we have an isomorphism

$$f^{-1} R\mathcal{H}om_{D_X}(L, \mathcal{O}_X) \xrightarrow{\sim} R\mathcal{H}om_{D_Y}(L_Y, \mathcal{O}_Y).$$

for L. Now consider the commutative diagram

$$\begin{array}{ccccc} 0 \longrightarrow f^{-1}\mathcal{H}om_{D_X}(M, \mathcal{O}_X) & \longrightarrow & f^{-1}\mathcal{H}om_{D_X}(L, \mathcal{O}_X) \longrightarrow \\ \mathbf{A}\downarrow & & \wr\downarrow \\ 0 \longrightarrow \mathcal{H}om_{D_Y}(M_Y, \mathcal{O}_Y) & \longrightarrow & \mathcal{H}om_{D_Y}(L_Y, \mathcal{O}_Y) \longrightarrow \end{array}$$

$$\begin{array}{ccccc} f^{-1}\mathcal{H}om_{D_X}(K, \mathcal{O}_X) & \longrightarrow & f^{-1}\mathcal{E}xt^1_{D_X}(M, \mathcal{O}_X) & \longrightarrow & f^{-1}\mathcal{E}xt^1_{D_X}(L, \mathcal{O}_X) \\ \mathbf{B}\downarrow & & \downarrow & & \wr\downarrow \\ \mathcal{H}om_{D_Y}(K_Y, \mathcal{O}_Y) & \longrightarrow & \mathcal{E}xt^1_{D_Y}(M_Y, \mathcal{O}_Y) & \longrightarrow & \mathcal{E}xt^1_{D_Y}(L_Y, \mathcal{O}_Y) \end{array}$$

with exact rows. We see from this that the morphism **A** is injective. It implies that the canonical morphism

$$f^{-1}\mathcal{H}om_{D_X}(N, \mathcal{O}_X) \to \mathcal{H}om_{D_Y}(N_Y, \mathcal{O}_Y)$$

is injective for any coherent D_X-module N with respect to which Y is non-characteristic. In particular, the morphism **B** is injective because Y is non-characteristic with respect to K. Hence by the five lemma, the morphism **A** is an isomorphism. Consequently **B** is also an isomorphism by applying the same argument to K instead of M. Repeating this argument we finally obtain the quasi-isomorphism

$$f^{-1} R\mathcal{H}om_{D_X}(M, \mathcal{O}_X) \xrightarrow{\sim} R\mathcal{H}om_{D_Y}(M_Y, \mathcal{O}_Y).$$

This completes the proof. □

By Theorem 4.1.5 (iv), Proposition 4.2.1, and Theorem 4.3.2 we have the following.

Corollary 4.3.3. *Let $f : Y \to X$ be a morphism of complex manifolds. Assume that f is non-characteristic for a coherent D_X-module M. Then we have*

$$DR_Y(Lf^* M) \simeq f^{-1} DR_X(M)[d_Y - d_X].$$

4.4 Cauchy problems and micro-supports

Theorem 4.3.2 has been extended into several directions. For example, we refer to [DS1], [Is]. Indeed, the methods used in the proof of Theorem 2.4.6 (see also

Theorem 4.1.5) and Theorem 4.3.2 have many interesting applications. We can prove various results for general systems of linear PDEs by reducing the problems to those for single equations. Let us give an example. Denote by $X_{\mathbb{R}}$ the underlying real manifold of X. Then we have a natural isomorphism $T^*X_{\mathbb{R}} \simeq (T^*X)_{\mathbb{R}}$. For a point $p \in T_x^*X_{\mathbb{R}}$ take a real-valued C^1-function $\phi : X_{\mathbb{R}} \longrightarrow \mathbb{R}$ such that $d\phi(x) = p$ (here $d\phi$ is the real differential of ϕ) and denote by $\partial\phi(x) \in T_x^*X$ its holomorphic part. Then by this identification $T^*X_{\mathbb{R}} \simeq (T^*X)_{\mathbb{R}}$. The point $p \in T_x^*X_{\mathbb{R}}$ corresponds to $\partial\phi(x) \in T_x^*X$. As for a more intrinsic construction of the isomorphism $T^*X_{\mathbb{R}} \simeq (T^*X)_{\mathbb{R}}$, see Kashiwara–Schapira [KS2, Section 11.1].

Theorem 4.4.1. *Let $\phi : X \longrightarrow \mathbb{R}$ be a real-valued C^∞-function on X such that $S = \{z \in X \mid \phi(z) = 0\} \subset X$ is a real smooth hypersurface and $\Omega = \{z \in X \mid \phi(z) < 0\} \subset X$ is Stein. Identifying T^*X with $T^*X_{\mathbb{R}}$ as above, assume that a coherent D_X-module M satisfies the condition $\mathrm{Ch}(M) \cap T_S^*(X_{\mathbb{R}}) \subset T_X^*X$. Then for $S_+ = \{z \in X \mid \phi(z) \geq 0\}$ we have*

$$\left[\mathrm{R}\Gamma_{S_+} R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)\right]_S \simeq 0.$$

Proof. Since we have

$$\left[\mathrm{R}\Gamma_{S_+} R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)\right]_S \simeq R\mathcal{H}om_{D_X}(M, H^1\left[\mathrm{R}\Gamma_{S_+}(\mathcal{O}_X)\right]_S[-1])$$

and $H^1\left[\mathrm{R}\Gamma_{S_+}(\mathcal{O}_X)\right]_S \simeq \left[\Gamma_\Omega(\mathcal{O}_X)/\mathcal{O}_X\right]_S$, the assertion for single equations $M = D_X/D_X P$ is just an interpretation of the classical result in Theorem 4.4.2 below. The general case can be proved by reducing the problem to the case of single equations in the same way as in the proof of Theorem 4.3.2. □

Theorem 4.4.2. *Let $\phi : X \longrightarrow \mathbb{R}$ be a real-valued C^∞-function on X such that $S = \{z \in X \mid \phi(z) = 0\} \subset X$ is a real smooth hypersurface and set $\Omega = \{z \in X \mid \phi(z) < 0\}$. For a differential operator $P \in D_X$ assume the condition: $\sigma(P)(z; \partial\phi(z)) \neq 0$ for any $z \in S$. Then we have the following:*

(i) (*Zerner* [Z]) *Let f be a holomorphic function on Ω such that Pf extends holomorphically across $S = \partial\Omega$ in a neighborhood of $z \in S$. Then f is also holomorphic in a neighborhood of z.*
(ii) (*Bony–Schapira* [BS]) *For any $z \in S = \partial\Omega$ the morphism $P : \Gamma_\Omega(\mathcal{O}_X)_z \longrightarrow \Gamma_\Omega(\mathcal{O}_X)_z$ is surjective.*

Corollary 4.4.3. *Let M, ϕ, S as in Theorem* 4.4.1. *Then we have an isomorphism*

$$\mathcal{H}om_{D_X}(M, \mathcal{O}_X)_{\bar{\Omega}} \xrightarrow{\sim} \Gamma_\Omega \mathcal{H}om_{D_X}(M, \mathcal{O}_X).$$

That is, any holomorphic solution to M on Ω extends across S as a holomorphic solution to M.

Proof. Consider the cohomology long exact sequence associated to the distinguished triangle

$$\mathrm{R}\Gamma_{S_+}(R\mathcal{H}om_{D_X}(M,\mathcal{O}_X)) \longrightarrow R\mathcal{H}om_{D_X}(M,\mathcal{O}_X)$$
$$\longrightarrow \mathrm{R}\Gamma_{\Omega}(R\mathcal{H}om_{D_X}(M,\mathcal{O}_X)) \xrightarrow{+1}$$

and apply Theorem 4.4.1. □

It is well known that Theorem 4.4.1 is true for arbitrary real-valued C^∞-function $\phi : X \longrightarrow \mathbb{R}$ such that $S = \{z \in X \mid \phi(z) = 0\}$ is smooth. Namely, we do not have to assume that $\Omega = \{z \in X \mid \phi(z) < 0\}$ is Stein. For the proof of this generalization of Theorem 4.4.1, see Kashiwara–Schapira [KS2, Theorem 11.3.3]. This remarkable result was a motivation for introducing the notion of micro-supports in Kashiwara–Schapira [KS1], [KS2].

Definition 4.4.4. Let X be a real C^∞-manifold and $F^\cdot \in D^b(\mathbb{C}_X)$. We define a closed $\mathbb{R}_{>0}$-invariant subset $\mathrm{SS}(F^\cdot)$ of T^*X as follows:

$p_0 = (x_0, \xi_0) \notin T^*X$

$\Longleftrightarrow$ There exists an open neighborhood U of p_0 in T^*X such that for any $x \in X$ and any C^∞-function $\phi : X \longrightarrow \mathbb{R}$ satisfying $\phi(x) = 0$ and $(x, \operatorname{grad}\phi(x)) \in U$ we have $\mathrm{R}\Gamma_{\{\phi \geq 0\}}(F^\cdot)_x \simeq 0$.

We call $\mathrm{SS}(F^\cdot)$ the *micro-support* of $F^\cdot$.

Note that the notion of micro-supports was recently generalized to that of truncated micro-supports in [KFS]. As we see in the next theorem, using micro-supports we can reconstruct the characteristic variety of a coherent D_X-module M from its solution complex $R\mathcal{H}om_{D_X}(M,\mathcal{O}_X)$.

Theorem 4.4.5 (Kashiwara–Schapira [KS1]). *Let X be a complex manifold and M a coherent D_X-module. Then under the natural identification $(T^*X)_\mathbb{R} \simeq T^*X_\mathbb{R}$, we have*

$$\mathrm{Ch}(M) = \mathrm{SS}(R\mathcal{H}om_{D_X}(M,\mathcal{O}_X)).$$

The inclusion $\mathrm{Ch}(M) \supset \mathrm{SS}(R\mathcal{H}om_{D_X}(M,\mathcal{O}_X))$ is just an interpretation of Theorem 4.4.1 and its generalization in Kashiwara–Schapira [KS2, Theorem 11.3.3]. The proof of the inverse inclusion is much more difficult and requires the theory of microdifferential operators. See [KS1, Theorem 10.1.1]. Combining Theorem 4.4.1 (or its generalization in [KS2, Theorem 11.3.3]) with Kashiwara's non-characteristic deformation lemma (Theorem C.3.6 in Appendix C), we obtain various global extension theorems for holomorphic solution complexes $R\mathcal{H}om_{D_X}(M,\mathcal{O}_X)$ as in the following theorem.

Theorem 4.4.6. *Let X be a complex manifold, $\{\Omega_t\}_{t\in\mathbb{R}}$ a family of relatively compact Stein open subsets of X such that $\partial\Omega_t$ is a C^∞-hypersurface in $X_\mathbb{R}$ for any $t \in \mathbb{R}$, and M a coherent D_X-module. Identifying $(T^*X)_\mathbb{R}$ with $T^*X_\mathbb{R}$ assume the following conditions:*

(i) *For any pair $s < t$ of real numbers, $\Omega_s \subset \Omega_t$.*

(ii) *For any* $t \in \mathbb{R}$, $\Omega_t = \bigcup_{s<t} \Omega_s$.
(iii) *For any* $t \in \mathbb{R}$, $\bigcap_{s>t}(\Omega_s \setminus \Omega_t) = \partial\Omega_t$ *and* $\mathrm{Ch}(M) \cap T^*_{\partial\Omega_t}(X_{\mathbb{R}}) \subset T^*_X X$.

Then we have an isomorphism

$$\mathrm{R}\Gamma\Big(\bigcup_{s\in\mathbb{R}} \Omega_s, R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)\Big) \xrightarrow{\sim} \mathrm{R}\Gamma(\Omega_t, R\mathcal{H}om_{D_X}(M, \mathcal{O}_X))$$

for any $t \in \mathbb{R}$.

This result will be effectively used in the proof of Kashiwara's constructibility theorem later.

4.5 Constructible sheaves

In this section we recall basic facts concerning constructible sheaves on analytic spaces and algebraic varieties. For the details of this subject we refer to Dimca [Di], Goresky–MacPherson [GM2], Kashiwara–Schapira [KS2], Schürmann [Schu], and Verdier [V1].

For a morphism $f : X \to Y$ of analytic spaces we have functors

$$\begin{aligned} f^{-1} &: \mathrm{Mod}(\mathbb{C}_Y) \to \mathrm{Mod}(\mathbb{C}_X), \\ f_* &: \mathrm{Mod}(\mathbb{C}_X) \to \mathrm{Mod}(\mathbb{C}_Y), \\ f_! &: \mathrm{Mod}(\mathbb{C}_X) \to \mathrm{Mod}(\mathbb{C}_Y). \end{aligned}$$

The functor f^{-1} is exact, and the functors f_*, $f_!$ are left exact. By taking their derived functors we obtain functors

$$\begin{aligned} f^{-1} &: D^b(\mathbb{C}_Y) \to D^b(\mathbb{C}_X), \\ Rf_* &: D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_Y), \\ Rf_! &: D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_Y) \end{aligned}$$

for derived categories, where $D^b(\mathbb{C}_X) = D^b(\mathrm{Mod}(\mathbb{C}_X))$. We also have a functor

$$f^! : D^b(\mathbb{C}_Y) \to D^b(\mathbb{C}_X)$$

which is right adjoint to $Rf_!$.

Let X be an analytic space. The tensor product gives a functor

$$(\bullet) \otimes_{\mathbb{C}} (\bullet) : D^b(\mathbb{C}_X) \times D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_X)$$

sending $(K^\cdot, L^\cdot)$ to $K^\cdot \otimes_{\mathbb{C}} L^\cdot$.

Definition 4.5.1. Let X and Y be analytic spaces. For $K^\cdot \in D^b(\mathbb{C}_X)$ and $L^\cdot \in D^b(\mathbb{C}_Y)$ we define $K^\cdot \boxtimes_{\mathbb{C}} L^\cdot \in D^b(\mathbb{C}_{X\times Y})$ by

$$K^\cdot \boxtimes_{\mathbb{C}} L^\cdot = p_1^{-1} K^\cdot \otimes_{\mathbb{C}_{X\times Y}} p_2^{-1} L^\cdot,$$

where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are projections.

Definition 4.5.2. For an analytic space X we set

$$\omega_X^{\cdot} = a_X^! \mathbb{C} \in D^b(\mathbb{C}_X),$$

where $a_X : X \to \mathrm{pt}$ is the unique morphism from X to the one-point space pt. We call it the *dualizing complex* of X.

When X is a complex manifold, ω_X is isomorphic to $\mathbb{C}_X[2\dim X]$. The *Verdier dual* $\mathbf{D}_X(F^{\cdot})$ of $F^{\cdot} \in D^b(\mathbb{C}_X)$ is defined by

$$\mathbf{D}_X(F^{\cdot}) := R\mathcal{H}om_{\mathbb{C}_X}(F^{\cdot}, \omega_X^{\cdot}) \in D^b(\mathbb{C}_X).$$

It defines a functor

$$\mathbf{D}_X : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_X)^{\mathrm{op}}.$$

Recall that a locally finite partition $X = \bigsqcup_{\alpha\in A} X_\alpha$ of an analytic space X by locally closed analytic subsets X_α ($\alpha \in A$) is called a *stratification* of X if, for any $\alpha \in A$, X_α is smooth (hence a complex manifold) and $\overline{X_\alpha} = \sqcup_{\beta\in B} X_\beta$ for a subset B of A. Each complex manifold X_α for $\alpha \in A$ is called a *stratum* of the stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$.

Definition 4.5.3. Let X be an analytic space. A $\mathbb{C}_X$-module F is called a *constructible sheaf* on X if there exists a stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of X such that the restriction $F|_{X_\alpha}$ is a local system on X_α for $^\forall\alpha \in A$.

Notation 4.5.4. For an analytic space X we denote by $D_c^b(X)$ the full subcategory of $D^b(\mathbb{C}_X)$ consisting of bounded complexes of $\mathbb{C}_X$-modules whose cohomology groups are constructible.

Example 4.5.5. On the complex plane $X = \mathbb{C}$ let us consider the ordinary differential equation $(x\frac{d}{dx} - \lambda)u = 0$ ($\lambda \in \mathbb{C}$). Denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X and define a subsheaf $F \subset \mathcal{O}_X$ of holomorphic solutions to this ordinary equation by

$$F = \left\{u \in \mathcal{O}_X \;\middle|\; \left(x\frac{d}{dx} - \lambda\right)u = 0\right\}.$$

Then the sheaf F is constructible with respect to the stratification $X = (\mathbb{C} - \{0\}) \sqcup \{0\}$ of X. Indeed, the restriction $F|_{\mathbb{C}-\{0\}} \simeq \mathbb{C}x^\lambda$ of F to $\mathbb{C} - \{0\}$ is a locally free sheaf of rank one over $\mathbb{C}_{\mathbb{C}-\{0\}}$ and the stalk at $0 \in X = \mathbb{C}$ is calculated as follows:

$$F_0 \simeq \begin{cases} \mathbb{C} & \lambda = 0, 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

For an algebraic variety X we denote the underlying analytic space by X^{an}. For a morphism $f : X \to Y$ of algebraic varieties we denote the corresponding morphism for analytic spaces by $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$. A locally finite partition $X = \bigsqcup_{\alpha\in A} X_\alpha$ of an algebraic variety X by locally closed subvarieties X_α ($\alpha \in A$) is called a stratification of X if for any $\alpha \in A$ X_α is smooth and $\overline{X_\alpha} = \sqcup_{\beta\in B} X_\beta$ for a subset B of A. A stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of an algebraic variety X induces a stratification $X^{\mathrm{an}} = \bigsqcup_{\alpha\in A} X_\alpha^{\mathrm{an}}$ of the corresponding analytic space X^{an}.

Definition 4.5.6. Let X be an algebraic variety. A $\mathbb{C}_{X^{\mathrm{an}}}$-module F is called an *algebraically constructible sheaf* if there exists a stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X such that $F|_{X_\alpha^{\mathrm{an}}}$ is a locally constant sheaf on X_α^{an} for $^\forall \alpha \in A$.

Notation 4.5.7.

(i) For an algebraic variety X, we denote by $D_c^b(X)$ the full subcategory of $D^b(\mathbb{C}_{X^{\mathrm{an}}})$ consisting of bounded complexes of $\mathbb{C}_{X^{\mathrm{an}}}$-modules whose cohomology groups are algebraically constructible (note that $D_c^b(X)$ is not a subcategory of $D^b(\mathbb{C}_X)$ but of $D^b(\mathbb{C}_{X^{\mathrm{an}}})$).

(ii) For an algebraic variety X we write $\omega^\cdot_{X^{\mathrm{an}}}$ and $\mathbf{D}_{X^{\mathrm{an}}} : D^b(\mathbb{C}_{X^{\mathrm{an}}}) \to D^b(\mathbb{C}_{X^{\mathrm{an}}})^{\mathrm{op}}$ simply as $\omega^\cdot_X$ and $\mathbf{D}_X$, respectively, by abuse of notation.

(iii) For a morphism $f : X \to Y$ of algebraic varieties we write $(f^{\mathrm{an}})^{-1}$, $(f^{\mathrm{an}})^!$, Rf_*^{an}, $Rf_!^{\mathrm{an}}$ as f^{-1}, $f^!$, Rf_*, $Rf_!$, respectively.

Theorem 4.5.8.

(i) *Let X be an algebraic variety or an analytic space. Then we have $\omega^\cdot_X \in D_c^b(X)$. Moreover, the functor $\mathbf{D}_X$ preserves the category $D_c^b(X)$ and $\mathbf{D}_X \circ \mathbf{D}_X \simeq \mathrm{Id}$ on $D_c^b(X)$.*

(ii) *Let $f : X \to Y$ be a morphism of algebraic varieties or analytic spaces. Then the functors f^{-1}, and $f^!$ induce*

$$f^{-1},\ f^! : D_c^b(Y) \longrightarrow D_c^b(X),$$

and we have

$$f^! = \mathbf{D}_X \circ f^{-1} \circ \mathbf{D}_Y$$

on $D_c^b(Y)$.

(iii) *Let $f : X \to Y$ be a morphism of algebraic varieties or analytic spaces. We assume that f is proper in the case where f is a morphism of analytic spaces. Then the functors Rf_*, $Rf_!$ induce*

$$Rf_*,\ Rf_! : D_c^b(X) \longrightarrow D_c^b(Y),$$

and we have

$$Rf_! = \mathbf{D}_Y \circ Rf_* \circ \mathbf{D}_X$$

on $D_c^b(X)$.

(iv) *Let X be an algebraic variety or an analytic space. Then the functor $(\bullet) \otimes_{\mathbb{C}} (\bullet)$ induces*

$$(\bullet) \otimes_{\mathbb{C}} (\bullet) : D_c^b(X) \times D_c^b(X) \longrightarrow D_c^b(X).$$

Proposition 4.5.9.

(i) *Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be a morphism of algebraic varieties or analytic spaces. Then we have*

$$(f_1 \times f_2)^{-1}(L_1^\cdot \boxtimes_{\mathbb{C}} L_2^\cdot) \simeq f_1^{-1}L_1^\cdot \boxtimes_{\mathbb{C}} f_2^{-1}L_2^\cdot \qquad (L_i^\cdot \in D^b(Y_i)), \qquad (4.5.1)$$

$$(f_1 \times f_2)^!(L_1^\cdot \boxtimes_{\mathbb{C}} L_2^\cdot) \simeq f_1^!L_1^\cdot \boxtimes_{\mathbb{C}} f_2^!L_2^\cdot \qquad (L_i^\cdot \in D_c^b(Y_i)). \qquad (4.5.2)$$

(ii) *Let* $f_i : X_i \to Y_i$ $(i = 1, 2)$ *be a morphism of algebraic varieties or analytic spaces. We assume that* f *is proper in the case where* f *is a morphism of analytic spaces. Then we have*

$$R(f_1 \times f_2)_!(K_1^\cdot \boxtimes_{\mathbb{C}} K_2^\cdot) \simeq Rf_{1!}K_1^\cdot \boxtimes_{\mathbb{C}} Rf_{2!}K_2^\cdot \qquad (K_i^\cdot \in D^b(X_i)), \quad (4.5.3)$$

$$R(f_1 \times f_2)_*(K_1^\cdot \boxtimes_{\mathbb{C}} K_2^\cdot) \simeq Rf_{1*}K_1^\cdot \boxtimes_{\mathbb{C}} Rf_{2*}K_2^\cdot \qquad (K_i^\cdot \in D_c^b(X_i)). \quad (4.5.4)$$

(iii) *Let* X_1, X_2 *be analytic spaces. Then we have*

$$\mathbf{D}_{X_1 \times X_2}(K_1^\cdot \boxtimes_{\mathbb{C}} K_2^\cdot) \simeq \mathbf{D}_{X_1}(K_1^\cdot) \boxtimes_{\mathbb{C}} \mathbf{D}_{X_2}(K_2^\cdot) \qquad (K_i^\cdot \in D_c^b(X_i)).$$

Proof. Note that (4.5.1) follows easily from the definition, and (4.5.3) is a consequence of the projection formula (see Proposition C.2.6). Hence in view of Theorem 4.5.8 we have only to show (iii). Let $p_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projections. Then we have

$$\begin{aligned}\mathbf{D}_{X_1 \times X_2}(K_1^\cdot \boxtimes_{\mathbb{C}} K_2^\cdot) &\simeq R\mathcal{H}om(p_1^{-1}K_1^\cdot \otimes_{\mathbb{C}} p_2^{-1}K_2^\cdot, \omega_{X_1 \times X_2}^\cdot) \\ &\simeq R\mathcal{H}om(p_1^{-1}K_1^\cdot, R\mathcal{H}om(p_2^{-1}K_2^\cdot, \omega_{X_1 \times X_2}^\cdot)) \\ &\simeq R\mathcal{H}om(p_1^{-1}K_1^\cdot, \mathbf{D}_{X_1 \times X_2} p_2^{-1}K_2^\cdot) \\ &\simeq R\mathcal{H}om(p_1^{-1}K_1^\cdot, p_2^!\mathbf{D}_{X_2}K_2^\cdot) \\ &\simeq \mathbf{D}_{X_1}K_1^\cdot \boxtimes_{\mathbb{C}} \mathbf{D}_{X_2}K_2^\cdot,\end{aligned}$$

where the last isomorphism is a consequence of [KS1, Proposition 3.4.4]. □

Definition 4.5.10. Let X be an algebraic variety or an analytic space. An object $F^\cdot \in D_c^b(X)$ is called a *perverse sheaf* if we have

$$\dim \operatorname{supp}(H^j(F^\cdot)) \le -j, \qquad \dim \operatorname{supp}(H^j(\mathbf{D}_X F^\cdot)) \le -j$$

for any $j \in \mathbb{Z}$. We denote by $\mathrm{Perv}(\mathbb{C}_X)$ the full subcategory of $D_c^b(X)$ consisting of perverse sheaves.

We will present a detailed account of the theory of perverse sheaves in Chapter 8.

4.6 Kashiwara's constructibility theorem

In this section we prove some basic properties of holomorphic solutions to holonomic D-modules. If M is a holonomic D_X-module on a complex manifold X, its holomorphic solution complex $\mathrm{Sol}_X(M) = R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)$ possesses very rigid structures. Namely, all the cohomology groups of $\mathrm{Sol}_X(M)$ are constructible sheaves on X. In other words, we have $\mathrm{Sol}_X(M) \in D_c^b(\mathbb{C}_X) = D_c^b(X)$. This is the famous *constructibility theorem*, due to Kashiwara [Kas3]. In particular, we obtain

$$\dim H^j \,\mathrm{Sol}_X(M)_z < +\infty$$

for ${}^{\forall} j \in \mathbb{Z}$ and ${}^{\forall} z \in X$. Moreover, in his Ph.D. thesis [Kas3], Kashiwara essentially proved that $\mathrm{Sol}_X(M)[d_X]$ satisfies the conditions of perverse sheaves, although the theory of perverse sheaves did not exist at that time. Let us give a typical example. Let Y be a complex submanifold of X with codimension $d = d_X - d_Y$. Then for the holonomic D_X-module $M = \mathcal{B}_{Y|X}$ (see Example 1.6.4), the complex

$$\mathrm{Sol}_X(M)[d_X] \simeq (\mathbb{C}_Y[-d])[d_X] = \mathbb{C}_Y[d_Y]$$

is a perverse sheaf on X. Before giving the proof of Kashiwara's results, let us recall the following fact. It was shown by Kashiwara that for any holonomic D_X-module there exists a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X such that $\mathrm{Ch}(M) \subset \bigsqcup_{\alpha \in A} T^*_{X_\alpha} X$. This follows from the geometric fact that $\mathrm{Ch}(M)$ is a $\mathbb{C}^\times$-invariant Lagrangian analytic subset of T^*X (see Theorem E.3.9). Let us fix such a stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ for a holonomic system M.

Proposition 4.6.1. *Set* $F^\cdot = R\mathcal{H}om_{D_X}(M, \mathcal{O}_X) \in D^b(\mathbb{C}_X) = D^b(X)$. *Then for* ${}^{\forall} j \in \mathbb{Z}$ *and* ${}^{\forall} \alpha \in A$, $H^j(F^\cdot)|_{X_\alpha}$ *is a locally constant sheaf on* X_α.

Proof. Let us fix a stratum X_{α_0}. The problem being local, we may assume

$$X_{\alpha_0} = \mathbb{C}^{n-d} = \{z_1 = \cdots = z_d = 0\} \subset X = \mathbb{C}^n_z.$$

It is enough to show that for ${}^{\forall} j \in \mathbb{Z}$ and $z_0 \in X_{\alpha_0}$ there exists a small open ball $B(z_0; \varepsilon)$ in X_{α_0} centered at z_0 such that the restriction map

$$\Gamma(\overline{B(z_0; \varepsilon)}, H^j(F^\cdot)) \longrightarrow H^j(F^\cdot)_z$$

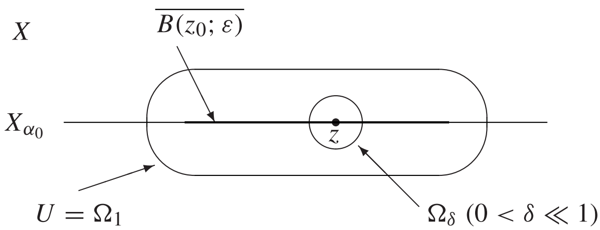
is an isomorphism for ${}^{\forall} z \in B(z_0; \varepsilon)$. First, let us treat the case when $j = 0$. Since the geometric normal structure of the Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ is locally constant along the stratum X_{α_0} (the Whitney condition (b)), by Theorem 4.4.6 for each $z \in B(z_0; \varepsilon)$ we can choose a sufficiently small open neighborhood U of $\overline{B(z_0; \varepsilon)}$ in X so that we have a quasi-isomorphism

$$\mathrm{R}\Gamma(U, F^\cdot) \longrightarrow F_z^\cdot. \tag{4.6.1}$$

Indeed, by $\mathrm{SS}(F^\cdot) = \mathrm{Ch}(M) \subset \bigsqcup_{\alpha \in A} T^*_{X_\alpha} X$ and the Whitney condition (b) (see Definition E.3.7) we can find a family of increasing open subsets $\{\Omega_t \subset X\}_{t \in (0,1]}$ of X such that

$$\begin{cases} \text{(i) } \Omega_1 = U, \quad \bigcap_{t \in (0,1]} \Omega_t = \{z\} \\ \text{(ii) } \partial\Omega_t \text{ is a real } C^\infty\text{-hypersurface in } X \text{ and } T^*_{\partial\Omega_t}(X) \cap \mathrm{Ch}(M) \subset T^*_X X \end{cases}$$

(see the figure below).



Since $H^j(F^\cdot) = 0$ for $j < 0$, it follows from the quasi-isomorphism (4.6.1) that

$$\Gamma(U, H^0(F^\cdot)) \xrightarrow{\sim} H^0(F^\cdot)_z.$$

If we take an inductive limit of the left-hand side by shrinking U, we get the desired isomorphism

$$\Gamma(\overline{B(z_0; \varepsilon)}, H^0(F^\cdot)) \xrightarrow{\sim} H^0(F^\cdot)_z.$$

This shows that $H^0(F^\cdot)|_{X_{\alpha_0}}$ is a locally constant sheaf on X_{α_0} in a neighborhood of $z_0 \in X_{\alpha_0}$. To prove the corresponding assertion for $H^1(F^\cdot)|_{X_{\alpha_0}}$ at the given point $z_0 \in X_\alpha$, first choose a sufficiently small open ball $B(z_0; \varepsilon)$ in X_{α_0} centered at z_0 so that we have a quasi-isomorphism

$$\mathrm{R}\Gamma(\overline{B(z_0; \varepsilon)}, F^\cdot) \xrightarrow{\sim} F_z^\cdot$$

for $^\forall z \in B(z_0; \varepsilon)$. Next setting $K = \overline{B(z_0; \varepsilon)}$ and fixing $z \in B(z_0; \varepsilon)$ consider the morphism of distinguished triangles

$$\begin{array}{ccccccc}
\mathrm{R}\Gamma(K, H^0(F^\cdot)) & \longrightarrow & \mathrm{R}\Gamma(K, F^\cdot) & \longrightarrow & \mathrm{R}\Gamma(K, \tau^{\geqslant 1}F^\cdot) & \xrightarrow{+1} & \\
\downarrow & & \wr\downarrow & & \downarrow & & \\
H^0(F^\cdot)_z & \longrightarrow & F_z^\cdot & \longrightarrow & \tau^{\geqslant 1}F_z^\cdot & \xrightarrow{+1} & .
\end{array}$$

Then the leftmost vertical arrow is a quasi-isomorphism, because $H^0(F^\cdot)|_{X_{\alpha_0}}$ is a locally constant sheaf on X_{α_0} and K is contractible. Therefore, the rightmost vertical arrow is also a quasi-isomorphism:

$$\mathrm{R}\Gamma(\overline{B(z_0; \varepsilon)}, \tau^{\geqslant 1}F^\cdot) \xrightarrow{\sim} \tau^{\geqslant 1}F_z^\cdot.$$

Taking $H^1(\bullet)$ of both sides, we finally get

$$\Gamma(\overline{B(z_0; \varepsilon)}, H^1(F^\cdot)) \xrightarrow{\sim} H^1(F^\cdot)_z.$$

By repeating this argument, we can finally show that for all $j \in \mathbb{Z}$, $H^j(F^\cdot)|_{X_{\alpha_0}}$ is a locally constant sheaf on X_α for $^\forall \alpha \in A$. This completes the proof. □

Proposition 4.6.2. *Let M be a holonomic D_X-module. Then for $^\forall j \in \mathbb{Z}$ and $^\forall z \in X$ the stalk $H^j[R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)]_z$ at z is a finite-dimensional vector space over $\mathbb{C}$.*

Proof. Let $X = \bigsqcup_{\alpha \in A} X_\alpha$ be a Whitney stratification of X such that $\mathrm{Ch}(M) \subset \bigsqcup_{\alpha \in A} T^*_{X_\alpha} X$. Let us prove our assertion for $z \in X_\alpha$. By the Whitney condition (b) of the stratification $\bigsqcup_{\alpha \in A} X_\alpha$ we can take a small positive number $\delta > 0$ such that

$$T^*_{\partial(B(z;\varepsilon))} X \cap \mathrm{Ch}(M) \subset T^*_X X$$

for $0 < {}^\forall \varepsilon < \delta$. Here $B(z; \varepsilon)$ is an open ball in X centered at z with radius ε. Therefore, by the non-characteristic deformation lemma (see Theorem 4.4.6) we have

$$\mathrm{R}\Gamma(B(z;\varepsilon_1), R\mathcal{H}om_{D_X}(M,\mathcal{O}_X)) \xrightarrow{\sim} \mathrm{R}\Gamma(B(z;\varepsilon_2), R\mathcal{H}om_{D_X}(M,\mathcal{O}_X))$$

for $0 < {}^\forall\varepsilon_2 < {}^\forall\varepsilon_1 < \delta$. Since the open balls $B(z;\varepsilon_i)$ $(i = 1,2)$ are Stein, this quasi-isomorphism can be represented by the morphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X(B(z;\varepsilon_1))^{N_0} & \xrightarrow{P_1\times} & \mathcal{O}_X(B(z;\varepsilon_1))^{N_1} & \xrightarrow{P_2\times} & \cdots \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X(B(z;\varepsilon_2))^{N_0} & \xrightarrow{P_1\times} & \mathcal{O}_X(B(z;\varepsilon_2))^{N_1} & \xrightarrow{P_2\times} & \cdots \end{array}$$

between complexes, where P_i is an $N_i \times N_{i-1}$ matrix of differential operators. Since the vertical arrows are compact maps of Fréchet spaces, the resulting cohomology groups

$$H^i(B(z;\varepsilon_1), R\mathcal{H}om_{D_X}(M,\mathcal{O}_X)) \xrightarrow{\sim} H^i(B(z;\varepsilon_2), R\mathcal{H}om_{D_X}(M,\mathcal{O}_X))$$

are finite dimensional by a standard result in functional analysis. □

By Proposition 4.2.1, 4.6.1 and 4.6.2 we obtain Kashiwara's constructibility theorem:

Theorem 4.6.3. *Let M be a holonomic D-module on a complex manifold X. Then* $\mathrm{Sol}_X(M) = R\mathcal{H}om_{D_X}(M,\mathcal{O}_X)$ *and* $DR_X(M) = \Omega_X \otimes^L_{D_X} M$ *are objects in the category* $D^b_c(X)$.

For a holonomic D_X-module M we saw that $\mathrm{Sol}_X(M)[d_X]$ and $DR_X(M)$ were constructible sheaves on X. Next we will prove moreover that they are dual to each other:

$$DR_X(M) \xrightarrow{\sim} \mathbf{D}_X(\mathrm{Sol}_X(M)[d_X]),$$

where $\mathbf{D}_X : D^b_c(X) \xrightarrow{\sim} D^b_c(X)$ is the Verdier duality functor. For this purpose, recall that for a point $z \in X$ the complex $\mathrm{R}\Gamma_{\{z\}}(\mathcal{O}_X)|_z$ satisfies

$$H^j\big(\mathrm{R}\Gamma_{\{z\}}(\mathcal{O}_X)|_z\big) \simeq 0 \quad \text{for } {}^\forall j \neq d_X$$

and that $\mathcal{B}^\infty_{\{z\}|X} = H^{d_X}(\mathrm{R}\Gamma_{\{z\}}(\mathcal{O}_X)|_z)$ is an (FS) type (Fréchet–Schwartz type) topological vector space. Regarding $\mathcal{B}^\infty_{\{z\}|X}$ as the space of Sato's hyperfunctions supported by the point $z \in X$, we see that the (DFS) type (dual F-S type) topological vector space $(\mathcal{O}_X)_z$ is a topological dual of $\mathcal{B}^\infty_{\{z\}|X}$. The following results were also obtained in Kashiwara [Kas3].

Proposition 4.6.4. *Let M be a holonomic D_X-module. Then*

(i) *For ${}^\forall z \in X$ and ${}^\forall i \in \mathbb{Z}$, $\mathcal{E}xt^i_{D_X}(M, \mathcal{B}^\infty_{\{z\}|X})$ is a finite-dimensional vector space over $\mathbb{C}$.*
(ii) *For ${}^\forall z \in X$ and ${}^\forall i \in \mathbb{Z}$, the vector spaces $H^{-i}(DR_X(M)_z)$ and $\mathcal{E}xt^i_{D_X}(M, \mathcal{B}^\infty_{\{z\}|X})$ are dual to each other.*

(iii) $DR_X(M) \xrightarrow{\sim} \mathbf{D}_X(\mathrm{Sol}_X(M)[d_X])$.

Proof. (i) Since $R\mathcal{H}om_{D_X}(M, \mathcal{B}^\infty_{\{z\}|X}) = \mathrm{R}\Gamma_{\{z\}} R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)[d_X]$ and the functor $\mathrm{R}\Gamma_{\{z\}}(\bullet)$ preserves the constructibility, the result follows.

(ii) Let us take a locally free resolution

$$0 \longrightarrow D_X^{N_k} \longrightarrow \cdots \xrightarrow{\times P_2} D_X^{N_1} \xrightarrow{\times P_1} D_X^{N_0} \longrightarrow M \longrightarrow 0$$

of M on an open neighborhood of $z \in X$, where P_i is a $N_i \times N_{i-1}$ matrix of differential operators acting on the right of $D_X^{N_i}$. Then we get

$$\begin{aligned} DR_X(M) &= \left[0 \longrightarrow (\Omega_X)^{N_k} \longrightarrow \cdots \xrightarrow{\times P_2} (\Omega_X)^{N_1} \xrightarrow{\times P_1} (\Omega_X)^{N_0} \longrightarrow 0\right], \\ R\mathcal{H}om_{D_X}&(M, \mathcal{B}^\infty_{\{z\}|X}) \\ &= \left[0 \longrightarrow \mathcal{B}^{\infty}_{\{z\}|X}{}^{N_0} \xrightarrow{P_1 \times} \mathcal{B}^{\infty}_{\{z\}|X}{}^{N_1} \xrightarrow{P_2 \times} \cdots \longrightarrow \mathcal{B}^{\infty}_{\{z\}|X}{}^{N_k} \longrightarrow 0\right]. \end{aligned}$$

Taking a local coordinate and identifying Ω_X with $\mathcal{O}_X$, we also have

$$DR_X(M)_z = \left[\quad \cdots \xrightarrow{P_2^* \times} (\mathcal{O}_X)_z^{N_1} \xrightarrow{P_1^* \times} (\mathcal{O}_X)_z^{N_0} \longrightarrow 0\right],$$

where P_i^* is a formal adjoint of P_i. Because $(\mathcal{O}_X)_z$ and $\mathcal{B}^\infty_{\{z\}|X}$ are topological dual to each other, and both $DR_X(M)_z$ and $R\mathcal{H}om_{D_X}(M, \mathcal{B}^\infty_{\{z\}|X})$ have finite-dimensional cohomology groups, we obtain the duality isomorphism

$$\left[H^{-i}(DR_X(M)_z)\right]^* \simeq \mathcal{E}xt^i_{D_X}(M, \mathcal{B}^\infty_{\{z\}|X}).$$

(iii) Since $\mathbb{C}_X \simeq R\mathcal{H}om_{D_X}(\mathcal{O}_X, \mathcal{O}_X)$, we have a natural morphism

$$\begin{aligned} DR_X(M) &= R\mathcal{H}om_{D_X}(\mathcal{O}_X, M)[d_X] \\ &\xrightarrow{\exists} R\mathcal{H}om_{\mathbb{C}_X}\big(R\mathcal{H}om_{D_X}(M, \mathcal{O}_X),\ R\mathcal{H}om_{D_X}(\mathcal{O}_X, \mathcal{O}_X)\big)[d_X] \\ &\simeq R\mathcal{H}om_{\mathbb{C}_X}\big(\mathrm{Sol}_X(M)[d_X],\ \mathbb{C}_X\big)[2d_X] \\ &\simeq \mathbf{D}_X(\mathrm{Sol}_X(M)[d_X]). \end{aligned}$$

Our task is to prove $DR_X(M)_z \simeq \mathbf{D}_X(\mathrm{Sol}_X(M)[d_X])_z$ for ${}^\forall z \in X$. Indeed, by (ii), we get the following chain of isomorphisms for $i_{\{z\}} : \{z\} \hookrightarrow X$:

$$\begin{aligned} \mathbf{D}_X(\mathrm{Sol}_X(M)[d_X])_z &= i_{\{z\}}^{-1}\mathbf{D}_X(\mathrm{Sol}_X(M)[d_X]) \\ &\simeq \mathbf{D}_{\{\mathrm{pt}\}} i_{\{z\}}^!(\mathrm{Sol}_X(M)[d_X]) \\ &\simeq \left[R\mathcal{H}om_{D_X}(M,\ \mathrm{R}\Gamma_{\{z\}}(\mathcal{O}_X)[d_X])\right]^* \\ &\simeq \left[R\mathcal{H}om_{D_X}(M,\ \mathcal{B}^\infty_{\{z\}|X})\right]^* \\ &\simeq DR_X(M)_z. \end{aligned}$$

This completes the proof. □

Corollary 4.6.5. *Let M be a holonomic D_X-module and $\mathbb{D}_X M$ its dual. Then we have isomorphisms*

$$\begin{cases} \mathbf{D}_X(DR_X(M)) \simeq DR_X(\mathbb{D}_X M) \\ \mathbf{D}_X(\mathrm{Sol}_X(M)[d_X]) \simeq \mathrm{Sol}_X(\mathbb{D}_X M)[d_X]. \end{cases}$$

Proof. The results follow immediately from Proposition 4.6.4 and the formula $DR_X(\mathbb{D}_X M) \simeq \mathrm{Sol}_X(M)[d_X]$. □

Theorem 4.6.6. *Let X be a complex manifold and M a holonomic D-module on it. Then $\mathrm{Sol}_X(M)[d_X] = R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)[d_X]$ and $DR_X(M) = \Omega_X \otimes^L_{D_X} M$ are perverse sheaves on X.*

Proof. By $DR_X(M) \simeq \mathrm{Sol}_X(\mathbb{D}_X M)[d_X]$, it is sufficient to prove that $F^{\cdot} = \mathrm{Sol}_X(M)[d_X]$ is a perverse sheaf for any holonomic D_X-module M. Moreover, since we have $\mathbf{D}_X(\mathrm{Sol}_X(M)[d_X]) \simeq \mathrm{Sol}_X(\mathbb{D}_X M)[d_X]$ by Corollary 4.6.5, we have only to prove that $\dim \mathrm{supp}(H^j(F^{\cdot})) \leq -j$ for $^\forall j \in \mathbb{Z}$. Let us take a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X such that $\mathrm{Ch}(M) \subset \bigsqcup_{\alpha \in A} T^*_{X_\alpha} X$ and set $i_{X_\alpha} : X_\alpha \hookrightarrow X$ for $\alpha \in A$. Then by Proposition 4.6.1 the complex $i^{-1}_{X_\alpha} F^{\cdot}$ of sheaves on X_α has locally constant cohomology groups for $^\forall \alpha \in A$. For $j \in \mathbb{Z}$ set $Z = \mathrm{supp}\, H^j(F^{\cdot})$. Then Z is a union of connected components of strata X_α's. We need to prove $\dim Z = d_Z \leq -j$. Choose a smooth point z of Z contained in a stratum X_α such that $\dim X_\alpha = \dim Z$ and take a germ of complex submanifold Y of X at z which intersects with Z transversally at $z \in Z$ ($\dim Y = d_Y = d_X - d_Z$). We can choose the pair (z, Y) so that Y is non-characteristic for M, because for the Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ we have the estimate $\mathrm{Ch}(M) \subset \bigsqcup_{\alpha \in A} T^*_{X_\alpha} X$. Therefore, by the Cauchy–Kowalevski–Kashiwara theorem (Theorem 4.3.2), we obtain

$$\begin{aligned} F^{\cdot}|_Y &= R\mathcal{H}om_{D_X}(M, \mathcal{O}_X)|_Y[d_X] \\ &\simeq R\mathcal{H}om_{D_Y}(M_Y, \mathcal{O}_Y)[d_X]. \end{aligned}$$

Our assumption $H^j(F^{\cdot})_z \neq 0$ implies $\mathcal{E}xt^{j+d_X}_{D_Y}(M_Y, \mathcal{O}_Y)_z \neq 0$. On the other hand, by Theorem 4.1.2 and

$$R\mathcal{H}om_{D_Y}(M_Y, \mathcal{O}_Y) \simeq R\mathcal{H}om_{D_Y}(M_Y, D_Y) \otimes^L_{D_Y} \mathcal{O}_Y,$$

we have $\mathcal{E}xt^i_{D_Y}(M_Y, \mathcal{O}_Y) = 0$ for $^\forall i > d_Y$. Hence we must have $j + d_X \leq d_Y \iff d_Z = d_X - d_Y \leq -j$. This completes the proof. □

Let M be a holonomic D_X-module as before. Then Kashiwara's constructibility theorem implies that for any point $x \in X$ the *local Euler–Poincaré index*

$$\chi_x[\mathrm{Sol}_X(M)] := \sum_{i \in \mathbb{Z}} (-1)^i \dim \mathcal{E}xt^i_{D_X}(M, \mathcal{O}_X)_x$$

of $\mathrm{Sol}_X(M)$ at x is a finite number (an integer). An important problem is to express this local Euler–Poincaré index in terms of geometric invariants of M. This problem was solved by Kashiwara [Kas8] and its solution has many applications in various fields of mathematics.

Let us briefly explain this result. First take a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X such that $\mathrm{Ch}(M) \subset \bigsqcup_{\alpha \in A} \overline{T^*_{X_\alpha} X}$. Next denote by $m_\alpha \in \mathbb{Z}_{\geq 0}$ the multiplicity of the coherent $\mathcal{O}_{T^*X}$-module $\mathrm{gr}^F M = \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathrm{gr}^F D_X} \pi^{-1}(\mathrm{gr}^F M)$ along $T^*_{X_\alpha} X$, where F is a good filtration of M and $\pi : T^*X \to X$ is the projection. Then the characteristic cycle $\mathbf{CC}(M)$ of the (analytic) holonomic D_X-module M is defined by

$$\mathbf{CC}(M) := \sum_{\alpha \in A} m_\alpha [\overline{T^*_{X_\alpha} X}].$$

This is a Lagrangian cycle in T^*X. Finally, for an analytic subset $S \subset X$ denote by $Eu_S : S \to \mathbb{Z}$ the *Euler obstruction* of S, which is introduced by Kashiwara [Kas2], [Kas8] and MacPherson [Mac] independently. Recall that for any Whitney stratification of S the Euler obstruction Eu_S is a locally constant function on each stratum (and on the regular part of S, the value of Eu_S is one). Then we have

Theorem 4.6.7 (Kashiwara [Kas2], [Kas8]). *For any $x \in X$ the local Euler–Poincaré index $\chi_x[\mathrm{Sol}_X(M)]$ of the solution complex $\mathrm{Sol}_X(M)$ at x is given by*

$$\chi_x[\mathrm{Sol}_X(M)] = \sum_{x \in \overline{X_\alpha}} (-1)^{c_\alpha} m_\alpha \cdot Eu_{\overline{X_\alpha}}(x),$$

where c_α is the codimension of the stratum X_α in X.

Kashiwara's local index theorem for holonomic D-modules was a starting point of intensive activities in the last decades. The global index theorem was obtained by Dubson [Du] and its generalization to real constructible sheaves was proved by Kashiwara [Kas11] (see also Kashiwara–Schapira [KS2] for the details). As for further developments of the theory of index theorems, see, for example, [BMM], [Gi], [Gui], [SS], [SV], [Tk1]. Note also that Euler obstructions play a central role in the study of characteristic classes of singular varieties (see [Mac], [Sab1]).

4.7 Analytic *D*-modules associated to algebraic *D*-modules

Recall that for an algebraic variety X we denote by X^{an} the corresponding analytic space. We have a morphism $\iota = \iota_X : X^{\mathrm{an}} \to X$ of topological spaces, and a morphism $\iota^{-1}\mathcal{O}_X \to \mathcal{O}_{X^{\mathrm{an}}}$ of sheaves of rings. In other words we have a morphism $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$ of ringed spaces.

Assume that X is a smooth algebraic variety. Then X^{an} is a complex manifold, and we have a canonical morphism

$$\iota^{-1} D_X \to D_{X^{\mathrm{an}}}$$

of sheaves of rings satisfying

$$D_{X^{\mathrm{an}}} \simeq \mathcal{O}_{X^{\mathrm{an}}} \otimes_{\iota^{-1}\mathcal{O}_X} \iota^{-1} D_X \simeq \iota^{-1} D_X \otimes_{\iota^{-1}\mathcal{O}_X} \mathcal{O}_{X^{\mathrm{an}}}.$$

Hence we obtain a functor

$$(\bullet)^{\mathrm{an}} : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_{X^{\mathrm{an}}})$$

sending $M \in \mathrm{Mod}(D_X)$ to $M^{\mathrm{an}} := D_{X^{\mathrm{an}}} \otimes_{\iota^{-1}D_X} \iota^{-1} M \in \mathrm{Mod}(D_{X^{\mathrm{an}}})$. Since $D_{X^{\mathrm{an}}}$ is faithfully flat over $\iota^{-1} D_X$, this functor is exact and extends to a functor

$$(\bullet)^{\mathrm{an}} : D^b(D_X) \to D^b(D_{X^{\mathrm{an}}})$$

between derived categories. Note that $(\bullet)^{\mathrm{an}}$ induces

$$(\bullet)^{\mathrm{an}} : \mathrm{Mod}_c(D_X) \to \mathrm{Mod}_c(D_{X^{\mathrm{an}}}), \qquad (\bullet)^{\mathrm{an}} : D^b(D_X) \to D^b(D_{X^{\mathrm{an}}}).$$

We will sometimes write $(M^{\cdot})^{\mathrm{an}} = D_{X^{\mathrm{an}}} \otimes_{D_X} M^{\cdot}$ by abuse of notation.

The following is easily verified.

Proposition 4.7.1. *For $M^{\cdot} \in D^b_c(D_X)$ we have $(\mathbb{D}_X M^{\cdot})^{\mathrm{an}} \simeq \mathbb{D}_{X^{\mathrm{an}}} (M^{\cdot})^{\mathrm{an}}$.*

Proposition 4.7.2. *Let $f : X \to Y$ be a morphism of smooth algebraic varieties.*

(i) *For $M^{\cdot} \in D^b(D_Y)$ we have $(f^{\dagger} M^{\cdot})^{\mathrm{an}} \simeq (f^{\mathrm{an}})^{\dagger} (M^{\cdot})^{\mathrm{an}}$.*

(ii) *For $M^{\cdot} \in D^b(D_X)$ we have a canonical morphism $(\int_f M^{\cdot})^{\mathrm{an}} \to \int_{f^{\mathrm{an}}} (M^{\cdot})^{\mathrm{an}}$. This morphism is an isomorphism if f is proper and $M^{\cdot} \in D^b_c(D_X)$.*

Proof. The proof of (i) is easy and omitted.

Let us show (ii). First note that there exists a canonical morphism

$$(f^{\mathrm{an}})^{-1} D_{Y^{\mathrm{an}}} \otimes_{(f^{\mathrm{an}})^{-1}\iota_Y^{-1} D_Y} \iota_X^{-1} D_{Y \leftarrow X} \to D_{Y^{\mathrm{an}} \leftarrow X^{\mathrm{an}}}.$$

Indeed, it is obtained by applying the side-changing operation to

$$\begin{aligned}
&\iota_X^{-1} D_{X \to Y} \otimes_{(f^{\mathrm{an}})^{-1}\iota_Y^{-1} D_Y} (f^{\mathrm{an}})^{-1} D_{Y^{\mathrm{an}}} \\
&\quad = \iota_X^{-1}\mathcal{O}_X \otimes_{(f^{\mathrm{an}})^{-1}\iota_Y^{-1}\mathcal{O}_Y} (f^{\mathrm{an}})^{-1} D_{Y^{\mathrm{an}}} \\
&\quad \simeq \left(\iota_X^{-1}\mathcal{O}_X \otimes_{(f^{\mathrm{an}})^{-1}\iota_Y^{-1}\mathcal{O}_Y} (f^{\mathrm{an}})^{-1}\mathcal{O}_{Y^{\mathrm{an}}}\right) \otimes_{(f^{\mathrm{an}})^{-1}\mathcal{O}_{Y^{\mathrm{an}}}} (f^{\mathrm{an}})^{-1} D_{Y^{\mathrm{an}}} \\
&\quad \to \mathcal{O}_{X^{\mathrm{an}}} \otimes_{(f^{\mathrm{an}})^{-1}\mathcal{O}_{Y^{\mathrm{an}}}} (f^{\mathrm{an}})^{-1} D_{Y^{\mathrm{an}}} \\
&\quad = D_{X^{\mathrm{an}} \to Y^{\mathrm{an}}}
\end{aligned}$$

(note $\iota_Y \circ f^{\mathrm{an}} = f \circ \iota_X$). Next note that there exists a canonical morphism

$$\iota_Y^{-1} Rf_* K^{\cdot} \to Rf_*^{\mathrm{an}} \iota_X^{-1} K^{\cdot}$$

for any $K^{\cdot} \in D^b(f^{-1}D_Y)$. Indeed, it is obtained as the image of Id for

$$\mathrm{Hom}_{\iota_X^{-1} f^{-1} D_Y}(\iota_X^{-1} K^{\cdot}, \iota_X^{-1} K^{\cdot}) \simeq \mathrm{Hom}_{f^{-1}D_Y}(K^{\cdot}, R\iota_{X*}\iota_X^{-1} K^{\cdot})$$

$$\begin{aligned}
&\to \mathrm{Hom}_{D_Y}(Rf_*K^{\cdot}, Rf_*R\iota_{X*}\iota_X^{-1}K^{\cdot})\\
&\simeq \mathrm{Hom}_{D_Y}(Rf_*K^{\cdot}, R\iota_{Y*}Rf_*^{\mathrm{an}}\iota_X^{-1}K^{\cdot})\\
&\simeq \mathrm{Hom}_{\iota_Y^{-1}D_Y}(\iota_Y^{-1}Rf_*K^{\cdot}, Rf_*^{\mathrm{an}}\iota_X^{-1}K^{\cdot}).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\Big(\int_f M^{\cdot}\Big)^{\mathrm{an}} &= D_{Y^{\mathrm{an}}} \otimes_{\iota_Y^{-1}D_Y} \iota_Y^{-1}Rf_*(D_{Y\leftarrow X} \otimes^L_{D_X} M^{\cdot})\\
&\to D_{Y^{\mathrm{an}}} \otimes_{\iota_Y^{-1}D_Y} Rf_*^{\mathrm{an}}\iota_X^{-1}(D_{Y\leftarrow X} \otimes^L_{D_X} M^{\cdot})\\
&\to Rf_*^{\mathrm{an}}\left((f^{\mathrm{an}})^{-1}D_{Y^{\mathrm{an}}} \otimes^L_{(f^{\mathrm{an}})^{-1}\iota_Y^{-1}D_Y} \iota_X^{-1}D_{Y\leftarrow X} \otimes^L_{\iota_X^{-1}D_X} \iota_X^{-1}M^{\cdot}\right)\\
&\to Rf_*^{\mathrm{an}}\left(D_{Y^{\mathrm{an}}\leftarrow X^{\mathrm{an}}} \otimes^L_{\iota_X^{-1}D_X} \iota_X^{-1}M^{\cdot}\right)\\
&\to Rf_*^{\mathrm{an}}\left(D_{Y^{\mathrm{an}}\leftarrow X^{\mathrm{an}}} \otimes^L_{D_{X^{\mathrm{an}}}} D_{X^{\mathrm{an}}} \otimes^L_{\iota_X^{-1}D_X} \iota_X^{-1}M^{\cdot}\right)\\
&= \int_{f^{\mathrm{an}}} (M^{\cdot})^{\mathrm{an}}.
\end{aligned}$$

It remains to show that $(\int_f M^{\cdot})^{\mathrm{an}} \to \int_{f^{\mathrm{an}}}(M^{\cdot})^{\mathrm{an}}$ is an isomorphism if f is proper. We may assume that f is either a closed embedding or a projection $f : X = Y \times \mathbb{P}^n \to Y$. The case of a closed embedding is easy and omitted. Assume that f is a projection $f : X = Y \times \mathbb{P}^n \to Y$. We may also assume that $M^{\cdot} = M \in \mathrm{Mod}_c(D_X)$. In this case we have

$$\begin{aligned}
\Big(\int_f M\Big)^{\mathrm{an}} &= \mathcal{O}_{Y^{\mathrm{an}}} \otimes_{\iota_Y^{-1}\mathcal{O}_Y} Rf_*(DR_{X/Y}(M)),\\
\int_{f^{\mathrm{an}}} M^{\mathrm{an}} &= Rf_*^{\mathrm{an}}(DR_{X^{\mathrm{an}}/Y^{\mathrm{an}}}(M^{\mathrm{an}})),
\end{aligned}$$

and hence it is sufficient to show that

$$\mathcal{O}_{Y^{\mathrm{an}}} \otimes_{\iota_Y^{-1}\mathcal{O}_Y} Rf_*(DR_{X/Y}(M)^k) \simeq Rf_*^{\mathrm{an}}(DR_{X^{\mathrm{an}}/Y^{\mathrm{an}}}(M^{\mathrm{an}})^k)$$

for each k. Since $DR_{X/Y}(M)^k$ is a quasi-coherent $\mathcal{O}_X$-module satisfying

$$\mathcal{O}_{X^{\mathrm{an}}} \otimes_{\iota_X^{-1}\mathcal{O}_X} DR_{X/Y}(M)^k \simeq DR_{X^{\mathrm{an}}/Y^{\mathrm{an}}}(M^{\mathrm{an}})^k,$$

this follows from the *GAGA-principle*. □

For a smooth algebraic variety X we define functors

$$\begin{aligned}
DR_X &: D^b(D_X) \longrightarrow D^b(\mathbb{C}_{X^{\mathrm{an}}}),\\
\mathrm{Sol}_X &: D^b(D_X) \longrightarrow D^b(\mathbb{C}_{X^{\mathrm{an}}})^{\mathrm{op}}
\end{aligned}$$

by

$$DR_X(M^\cdot) := DR_{X^{\mathrm{an}}}((M^\cdot)^{\mathrm{an}}) = \Omega_{X^{\mathrm{an}}} \otimes^L_{D_{X^{\mathrm{an}}}} (M^\cdot)^{\mathrm{an}},$$
$$\mathrm{Sol}_X(M^\cdot) := \mathrm{Sol}_{X^{\mathrm{an}}}((M^\cdot)^{\mathrm{an}}) = R\mathcal{H}om_{D_{X^{\mathrm{an}}}}((M^\cdot)^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}).$$

Remark 4.7.3. It is not a good idea to consider $\Omega_X \otimes^L_{D_X} M^\cdot$ and $R\mathcal{H}om_{D_X}(\mathcal{O}_X, M^\cdot)$ for a smooth algebraic variety X as the following example suggests. Regard $X = \mathbb{C}$ as an algebraic variety, and set $M = D_X/D_X(\frac{d}{dx} - \lambda)$ for $\lambda \in \mathbb{C} \setminus \mathbb{Z}$. Then we easily see that $DR_X(M) \simeq \mathbb{C}_{X^{\mathrm{an}}}[1]$ and $\mathrm{Sol}_X(M) \simeq \mathbb{C}_{X^{\mathrm{an}}}$, while $\Omega_X \otimes^L_{D_X} M = R\mathcal{H}om_{D_X}(\mathcal{O}_X, M)[1] = 0$. This comes from the fact that the differential equation $\frac{du}{dx} = \lambda u$ has a holomorphic solution $\exp(\lambda x) \in \mathcal{O}_{X^{\mathrm{an}}}$ which does not belong to $\mathcal{O}_X$.

By Proposition 4.2.1 and Proposition 4.7.1 we have the following.

Proposition 4.7.4. *Let X be a smooth algebraic variety. For $M^\cdot \in D^b_c(D_X)$ we have*

$$DR_X(M^\cdot) \simeq R\mathcal{H}om_{D_{X^{\mathrm{an}}}}(\mathcal{O}_{X^{\mathrm{an}}}, (M^\cdot)^{\mathrm{an}})[d_X] \simeq \mathrm{Sol}_X(\mathbb{D}_X M^\cdot)[d_X].$$

By Theorem 4.2.5 and Proposition 4.7.2 we have the following.

Proposition 4.7.5. *Let $f : X \to Y$ be a morphism of smooth algebraic varieties. For $M^\cdot \in D^b_c(D_X)$ there exists a canonical morphism*

$$DR_Y(\int_f M) \to Rf_*(DR_X(M^\cdot)).$$

This morphism is an isomorphism if f is proper.

By Corollary 4.3.3 and Proposition 4.7.2 we have the following.

Proposition 4.7.6. *Let $f : X \to Y$ be a morphism of smooth algebraic varieties. Assume that f is non-characteristic for a coherent D_Y-module M. Then we have*

$$DR_X(Lf^*M) \simeq f^{-1} DR_Y(M)[d_X - d_Y].$$

The following is a special case of Kashiwara's constructibility theorem for analytic holonomic D-modules. Here we present another proof following Bernstein [Ber3] for the convenience of readers who wants a shortcut for algebraic D-modules.

Theorem 4.7.7. *For $M^\cdot \in D^b_h(D_X)$ we have $DR_X(M^\cdot), \mathrm{Sol}_X(M^\cdot) \in D^b_c(X)$.*

Proof. By Proposition 4.7.4 we have only to show the assertion on $DR_X(M^\cdot)$. Moreover, we may assume that $M^\cdot = M \in \mathrm{Mod}_h(D_X)$. By Proposition 3.1.6 M is generically an integrable connection, and hence $DR_X(M)$ is generically a local system up to a shift of degrees. Hence there exists an open dense subset U of X such that $DR_U(M|_U) \in D^b_c(U)$. Therefore, it is sufficient to show the following.

Claim. Let $M \in \mathrm{Mod}_h(D_X)$, and assume that $DR_U(M|_U) \in D^b_c(U)$ for an open dense subset U of X. Then there exists an open dense subset Y of $X \setminus U$ such that $DR_{U\cup Y}(M|_{U\cup Y}) \in D^b_c(U \cup Y)$.

For each irreducible component Z of $X \setminus U$ there exists an étale morphism f from an open subset V of X onto an open subset V' of $\mathbb{A}^n$ such that $V \cap (X \setminus U)$ (resp. $V' \cap \mathbb{A}^{n-k}$) is an open dense subset of Z (resp. $\mathbb{A}^{n-k}$) and $f^{-1}(V' \cap \mathbb{A}^{n-k}) = V \cap (X \setminus U)$, where $0 < k \leq n$ and $\mathbb{A}^{n-k}$ is identified with the subset $\{0\} \times \mathbb{A}^{n-k}$ of $\mathbb{A}^n$ (see Theorem A.5.3). Since f is an étale morphism, $DR_V(M|_V) \in D^b_c(V)$ if and only if $f_*(DR_V(M|_V)) \in D^b_c(V')$. Moreover, we have $f_*(DR_V(M|_V)) = DR_{V'}(\int_f^0(M|_V))$ by Proposition 4.7.5. Hence we may assume from the beginning that X is an open subset of $\mathbb{A}^n$, $X \setminus U = X \cap \mathbb{A}^{n-k}$, and $X \cap \mathbb{A}^{n-k}$ is dense in $\mathbb{A}^{n-k}$. Set $T = X \cap \mathbb{A}^{n-k}$. By shrinking X if necessary we may assume that X is an open subset of $\mathbb{A}^k \times T$.

Now we regard $\mathbb{A}^k$ as an open subset of $\mathbb{P}^k$. Set $S = (\mathbb{P}^k \times T) \setminus X$. Then we have $\mathbb{P}^k \times T = S \sqcup U \sqcup T$, $X = U \sqcup T$, and S and T are closed subsets of $\mathbb{P}^k \times T$. Let $p : \mathbb{P}^k \times T \to T$ be the projection and let $j_X : X \to \mathbb{P}^k \times T$, $j_U : U \to \mathbb{P}^k \times T$, $j_S : S \to \mathbb{P}^k \times T$, $j_T : T \to \mathbb{P}^k \times T$ be the embeddings. Set $N^{\cdot} = \int_{j_X} M$, $K^{\cdot} = DR_{\mathbb{P}^k \times T}(N^{\cdot})$. By applying $Rp_*(= Rp_!)$ to the distinguished triangle

$$j_{U!}j_U^{-1}K^{\cdot} \longrightarrow K^{\cdot} \longrightarrow j_{S!}j_S^{-1}K^{\cdot} \oplus j_{T!}j_T^{-1}K^{\cdot} \xrightarrow{+1}$$

we obtain a distinguished triangle

$$R(p \circ j_U)_! j_U^{-1}K^{\cdot} \longrightarrow Rp_*K^{\cdot} \longrightarrow R(p \circ j_S)_! j_S^{-1}K^{\cdot} \oplus R(p \circ j_T)_! j_T^{-1}K^{\cdot} \xrightarrow{+1} .$$

By $j_U^{-1}K^{\cdot} \simeq DR_U(M|_U) \in D^b_c(U)$ we have $R(p \circ j_U)_! j_U^{-1}K^{\cdot} \in D^b_c(T)$. By Proposition 4.7.5 we have

$$Rp_*K^{\cdot} = Rp_*\, DR_{\mathbb{P}^k \times T}(N^{\cdot}) \simeq DR_T(\int_p N^{\cdot}).$$

By $\int_p N^{\cdot} \in D^b_h(D_T)$ there exists an open dense subset Y of T such that $Rp_*K^{\cdot}|_{Y^{\mathrm{an}}} \in D^b_c(Y)$. It follows from the above distinguished triangle that $R(p \circ j_T)_! j_T^{-1}K^{\cdot}|_{Y^{\mathrm{an}}} \in D^b_c(Y)$. By $p \circ j_T = \mathrm{id}$ we have $R(p \circ j_T)_! j_T^{-1}K^{\cdot} \simeq i^{-1}DR_X(M)$, where $i : T \to X$ is the embedding. Thus $i^{-1}DR_X(M)|_{Y^{\mathrm{an}}} \in D^b_c(Y)$. Hence we have $DR_{U \cup Y}(M|_{U \cup Y}) \in D^b_c(U \cup Y)$. The proof is complete. □

The technique used in the proof of Theorem 4.7.7 also allows us to prove the following results.

Proposition 4.7.8. *Let X and Y be smooth algebraic varieties. For $M^{\cdot} \in D^b_c(D_X)$ and $N^{\cdot} \in D^b_c(D_Y)$ we have a canonical morphism*

$$DR_X(M^{\cdot}) \boxtimes_{\mathbb{C}} DR_Y(N^{\cdot}) \to DR_{X \times Y}(M^{\cdot} \boxtimes N^{\cdot}).$$

This morphism is an isomorphism if $M^{\cdot} \in D^b_h(D_X)$ or $N^{\cdot} \in D^b_h(D_Y)$.

Proposition 4.7.9. *Let X be a smooth algebraic variety. For $M^{\cdot} \in D^b_c(D_X)$ we have canonical morphisms*

$$DR_X(\mathbb{D}_X M^{\cdot}) \to \mathbf{D}_X(DR_X(M^{\cdot})),$$
$$\mathrm{Sol}_X(\mathbb{D}_X M^{\cdot})[d_X] \to \mathbf{D}_X(\mathrm{Sol}_X(M^{\cdot})[d_X]).$$

These morphisms are isomorphisms if $M^{\cdot} \in D_h^b(D_X)$.

Proof of Proposition 4.7.8. Let $M^{\cdot} \in D_c^b(D_X)$ and $N^{\cdot} \in D_c^b(D_X)$. By

$$(M^{\cdot} \boxtimes N^{\cdot})^{\mathrm{an}} \simeq D_{X^{\mathrm{an}} \times Y^{\mathrm{an}}} \otimes^L_{D_{X^{\mathrm{an}}} \boxtimes_{\mathbb{C}} D_{Y^{\mathrm{an}}}} ((M^{\cdot})^{\mathrm{an}} \boxtimes_{\mathbb{C}} (N^{\cdot})^{\mathrm{an}})$$

we have

$$DR_{X \times Y}(M^{\cdot} \boxtimes N^{\cdot}) \simeq \Omega_{X^{\mathrm{an}} \times Y^{\mathrm{an}}} \otimes^L_{D_{X^{\mathrm{an}}} \boxtimes_{\mathbb{C}} D_{Y^{\mathrm{an}}}} ((M^{\cdot})^{\mathrm{an}} \boxtimes_{\mathbb{C}} (N^{\cdot})^{\mathrm{an}}).$$

On the other hand we have

$$\begin{aligned} DR_X(M^{\cdot}) \boxtimes_{\mathbb{C}} DR_Y(N^{\cdot}) &\simeq (\Omega_{X^{\mathrm{an}}} \otimes^L_{D_{X^{\mathrm{an}}}} (M^{\cdot})^{\mathrm{an}}) \boxtimes_{\mathbb{C}} (\Omega_{Y^{\mathrm{an}}} \otimes^L_{D_{Y^{\mathrm{an}}}} (N^{\cdot})^{\mathrm{an}}) \\ &\simeq (\Omega_{X^{\mathrm{an}}} \boxtimes_{\mathbb{C}} \Omega_{Y^{\mathrm{an}}}) \otimes^L_{D_{X^{\mathrm{an}}} \boxtimes_{\mathbb{C}} D_{Y^{\mathrm{an}}}} ((M^{\cdot})^{\mathrm{an}} \boxtimes_{\mathbb{C}} (N^{\cdot})^{\mathrm{an}}). \end{aligned}$$

Hence the canonical morphism $\Omega_{X^{\mathrm{an}}} \boxtimes_{\mathbb{C}} \Omega_{Y^{\mathrm{an}}} \to \Omega_{X^{\mathrm{an}} \times Y^{\mathrm{an}}}$ induces a canonical morphism

$$DR_X(M^{\cdot}) \boxtimes_{\mathbb{C}} DR_Y(N^{\cdot}) \to DR_{X \times Y}(M^{\cdot} \boxtimes N^{\cdot}).$$

Let us show that this morphism is an isomorphism if either $M^{\cdot} \in D_h^b(D_X)$ or $N^{\cdot} \in D_h^b(D_Y)$. By symmetry we can only deal with the case $M^{\cdot} \in D_h^b(D_X)$.

We first show it when $M^{\cdot}$ is an integrable connection. In this case we have $(M^{\cdot})^{\mathrm{an}} \simeq \mathcal{O}_{X^{\mathrm{an}}} \otimes_{\mathbb{C}_{X^{\mathrm{an}}}} K$ for a local system K on X^{an} and we have $DR_X(M^{\cdot}) \simeq K[d_X]$. Then we have

$$(M^{\cdot} \boxtimes N^{\cdot})^{\mathrm{an}} \simeq p_1^{-1}K \otimes_{\mathbb{C}_{X^{\mathrm{an}} \times Y^{\mathrm{an}}}} (\mathcal{O}_X \boxtimes N^{\cdot})^{\mathrm{an}} \simeq p_1^{-1}K \otimes_{\mathbb{C}_{X^{\mathrm{an}} \times Y^{\mathrm{an}}}} (p_2^* N^{\cdot})^{\mathrm{an}},$$

where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are projections. Hence we have

$$\begin{aligned} DR_{X \times Y}(M^{\cdot} \boxtimes N^{\cdot}) &\simeq p_1^{-1}K \otimes_{\mathbb{C}_{X^{\mathrm{an}} \times Y^{\mathrm{an}}}} DR_{X \times Y}(p_2^* N^{\cdot}) \\ &\simeq p_1^{-1}K \otimes_{\mathbb{C}_{X^{\mathrm{an}} \times Y^{\mathrm{an}}}} p_2^{-1} DR_Y(N^{\cdot})[d_X] \\ &\simeq DR_X(M^{\cdot}) \boxtimes_{\mathbb{C}} DR_Y(N^{\cdot}) \end{aligned}$$

by Proposition 4.7.6.

Finally, we consider the general case. We may assume that $M^{\cdot} = M \in \mathrm{Mod}_h(D_X)$. Since M is generically an integrable connection, there exists an open subset U of X such that the canonical morphism

$$DR_U(M|_U) \boxtimes_{\mathbb{C}} DR_Y(N^{\cdot}) \to DR_{U \times Y}((M|_U) \boxtimes N^{\cdot})$$

is an isomorphism. Therefore, it is sufficient to show the following.

Claim. Assume that the canonical morphism

$$DR_U(M|_U) \boxtimes_{\mathbb{C}} DR_Y(N^\cdot) \to DR_{U\times Y}((M|_U) \boxtimes N^\cdot)$$

is an isomorphism for an open dense subset U of X. Then there exists an open dense subset Z of $X \setminus U$ such that

$$DR_{U\cup Z}(M|_{U\cup Z}) \boxtimes_{\mathbb{C}} DR_Y(N^\cdot) \to DR_{(U\cup Z)\times Y}((M|_{U\cup Z}) \boxtimes N^\cdot)$$

is an isomorphism.

This can be proved similarly to the claim in Theorem 4.7.7. The details are omitted. □

Proof of Proposition 4.7.9. By Proposition 4.7.4 it is sufficient to show that there exists a canonical morphism

$$\mathrm{Sol}_X(M^\cdot) \to \mathbf{D}_X(DR_X(M^\cdot))[-d_X] \qquad (M^\cdot \in D^b_c(D_X)),$$

which turns out to be an isomorphism for $M^\cdot \in D^b_h(D_X)$.

Let $M^\cdot \in D^b_c(D_X)$. Then we have a canonical morphism

$$\begin{aligned} &R\mathcal{H}om_{D_{X^{\mathrm{an}}}}(\mathcal{O}_{X^{\mathrm{an}}}, (M^\cdot)^{\mathrm{an}}) \otimes_{\mathbb{C}_{X^{\mathrm{an}}}} R\mathcal{H}om_{D_{X^{\mathrm{an}}}}((M^\cdot)^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \\ &\quad \to R\mathcal{H}om_{D_{X^{\mathrm{an}}}}(\mathcal{O}_{X^{\mathrm{an}}}, \mathcal{O}_{X^{\mathrm{an}}}). \end{aligned}$$

By

$$\begin{aligned} R\mathcal{H}om_{D_{X^{\mathrm{an}}}}(\mathcal{O}_{X^{\mathrm{an}}}, (M^\cdot)^{\mathrm{an}}) &\simeq DR_X(M)[-d_X], \\ R\mathcal{H}om_{D_{X^{\mathrm{an}}}}((M^\cdot)^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) &\simeq \mathrm{Sol}_X(M), \\ R\mathcal{H}om_{D_{X^{\mathrm{an}}}}(\mathcal{O}_{X^{\mathrm{an}}}, \mathcal{O}_{X^{\mathrm{an}}}) &\simeq \mathbb{C}_{X^{\mathrm{an}}}, \end{aligned}$$

we obtain

$$DR_X(M) \otimes_{\mathbb{C}_{X^{\mathrm{an}}}} \mathrm{Sol}_X(M) \to \mathbb{C}_{X^{\mathrm{an}}}[d_X].$$

Hence there exists a canonical morphism

$$\mathrm{Sol}_X(M) \to R\mathcal{H}om_{\mathbb{C}_{X^{\mathrm{an}}}}(DR_X(M), \mathbb{C}_{X^{\mathrm{an}}}[d_X])(\simeq \mathbf{D}_X(DR_X(M^\cdot))[-d_X]).$$

Let us show that this morphism is an isomorphism for $M^\cdot \in D^b_h(D_X)$. We may assume that $M^\cdot = M \in \mathrm{Mod}_h(D_X)$. If M is an integrable connection, then we have $M^{\mathrm{an}} \simeq \mathcal{O}_{X^{\mathrm{an}}} \otimes_{\mathbb{C}_{X^{\mathrm{an}}}} K$ for a local system K on X^{an}, and $\mathrm{Sol}_X(M) \simeq \mathcal{H}om_{\mathbb{C}_{X^{\mathrm{an}}}}(K, \mathbb{C}_{X^{\mathrm{an}}})$, $DR_X(M) \simeq K[d_X]$. Hence the assertion is obvious in this case. Let us consider the general case $M \in \mathrm{Mod}_h(D_X)$. Since M is generically an integrable connection, there exists an open subset U of X such that the canonical morphism

$$\mathrm{Sol}_U(M^\cdot|_U) \to \mathbf{D}_U(DR_U(M^\cdot|_U))[-d_X]$$

is an isomorphism. Therefore, it is sufficient to show the following.

Claim. Assume that the canonical morphism

$$\mathrm{Sol}_U(M^{\cdot}|_U) \to \mathbf{D}_U(DR_U(M^{\cdot}|_U))[-d_X]$$

is an isomorphism for an open dense subset U of X. Then there exists an open dense subset Y of $X \setminus U$ such that

$$\mathrm{Sol}_{U\cup Y}(M^{\cdot}|_{U\cup Y}) \to \mathbf{D}_{U\cup Y}(DR_{U\cup Y}(M^{\cdot}|_{U\cup Y}))[-d_X]$$

is an isomorphism.

This can be proved similarly to the claim in Theorem 4.7.7. Details are omitted. □