8

Perverse Sheaves

In this chapter we will give a self-contained account of the theory of perverse sheaves and intersection cohomology groups assuming the basic notions concerning constructible sheaves presented in Section 4.5. We also include a survey on the theory of Hodge modules.

8.1 Theory of perverse sheaves

An obvious origin of the theory of perverse sheaves is the Riemann–Hilbert correspondence. Indeed, as we have seen in Section 7.2 one naturally encounters the category of perverse sheaves as the image under the de Rham functor of the category of regular holonomic D-modules. Another origin is the intersection cohomology groups due to Goresky–MacPherson. Perverse sheaves provide the theory of intersection cohomology groups with a sheaf-theoretical foundation.

In this section we present a systematic treatment of the theory of perverse sheaves on analytic spaces or (not necessarily smooth) algebraic varieties based on the language of t-structures.

8.1.1 t-structures

The derived category of an abelian category $\mathcal{C}$ contains $\mathcal{C}$ as a full abelian subcategory; however, it sometimes happens that it also contains another natural full abelian subcategory besides the standard one $\mathcal{C}$. For example, for a smooth algebraic variety (or a complex manifold) X the derived category $D^b_c(X)$ contains the subcategory $\mathrm{Perv}(\mathbb{C}_X)$ of perverse sheaves as a non-standard full abelian subcategory. It is the image of the standard one $\mathrm{Mod}_{rh}(D_X)$ of $D^b_{rh}(D_X)$ by the de Rham functor

$$DR_X : D^b_{rh}(D_X) \xrightarrow{\sim} D^b_c(X).$$

More generally one can consider the following problem: in what situation does a triangulated category contain a natural abelian subcategory? An answer is given by the theory of t-structures due to Beilinson–Bernstein–Deligne [BBD].

In this subsection we give an account of the theory of t-structures. Besides the basic reference [BBD] we are also indebted to Kashiwara–Schapira [KS2, Chapter X].

Definition 8.1.1. Let $\mathbf{D}$ be a triangulated category, and let $\mathbf{D}^{\leqslant 0}$, $\mathbf{D}^{\geqslant 0}$ be its full subcategories. Set $\mathbf{D}^{\leqslant n} = \mathbf{D}^{\leqslant 0}[-n]$ and $\mathbf{D}^{\geqslant n} = \mathbf{D}^{\geqslant 0}[-n]$ for $n \in \mathbb{Z}$. We say that the pair $(\mathbf{D}^{\leqslant 0}, \mathbf{D}^{\geqslant 0})$ defines a *t-structure* on $\mathbf{D}$ if the following three conditions are satisfied:

(T1) $\mathbf{D}^{\leqslant -1} \subset \mathbf{D}^{\leqslant 0}, \mathbf{D}^{\geqslant 1} \subset \mathbf{D}^{\geqslant 0}$.
(T2) For any $X \in \mathbf{D}^{\leqslant 0}$ and any $Y \in \mathbf{D}^{\geqslant 1}$ we have $\mathrm{Hom}_{\mathbf{D}}(X, Y) = 0$.
(T3) For any $X \in \mathbf{D}$ there exists a distinguished triangle

$$X_0 \longrightarrow X \longrightarrow X_1 \xrightarrow{+1}$$

such that $X_0 \in \mathbf{D}^{\leqslant 0}$ and $X_1 \in \mathbf{D}^{\geqslant 1}$.

Example 8.1.2. Let $\mathcal{C}$ be an abelian category and $\mathcal{C}'$ a thick abelian subcategory of $\mathcal{C}$ in the sense of Definition B.4.6. Then by Proposition B.4.7 the full subcategory $D^{\sharp}_{\mathcal{C}'}(\mathcal{C})$ of $D^{\sharp}(\mathcal{C})$ ($\sharp = \emptyset, +, -, b$) consisting of objects $F^{\cdot} \in D^{\sharp}(\mathcal{C})$ satisfying $H^j(F^{\cdot}) \in \mathcal{C}'$ for any j is a triangulated category. We show that $\mathbf{D} = D^{\sharp}_{\mathcal{C}'}(\mathcal{C})$ admits a standard t-structure $(\mathbf{D}^{\leqslant 0}, \mathbf{D}^{\geqslant 0})$ given by

$$\begin{aligned}
\mathbf{D}^{\leqslant 0} &= \{F^{\cdot} \in D^{\sharp}_{\mathcal{C}'}(\mathcal{C}) \mid H^j(F^{\cdot}) = 0 \text{ for } {}^{\forall} j > 0\},\\
\mathbf{D}^{\geqslant 0} &= \{F^{\cdot} \in D^{\sharp}_{\mathcal{C}'}(\mathcal{C}) \mid H^j(F^{\cdot}) = 0 \text{ for } {}^{\forall} j < 0\}.
\end{aligned}$$

Since the condition (T1) is trivially satisfied, we will check (T2) and (T3). For $F^{\cdot} \in \mathbf{D}^{\leqslant 0}$, $G^{\cdot} \in \mathbf{D}^{\geqslant 1}$ and $f \in \mathrm{Hom}_{\mathbf{D}}(F^{\cdot}, G^{\cdot})$ we have a natural commutative diagram

$$\begin{tikzcd}
F^{\cdot} \arrow[r, "f"] & G^{\cdot} \\
\tau^{\leqslant 0}F^{\cdot} \arrow[u, "\wr"'] \arrow[r, "\tau^{\leqslant 0}(f)"] & \tau^{\leqslant 0}G^{\cdot}, \arrow[u]
\end{tikzcd}$$

where $\tau^{\leqslant 0} : \mathbf{D} \longrightarrow \mathbf{D}^{\leqslant 0}$ denotes the usual truncation functor. By $G^{\cdot} \in \mathbf{D}^{\geqslant 1}$ we easily see that $\tau^{\leqslant 0}G^{\cdot}$ is isomorphic in $\mathbf{D}$ to the zero object 0, and hence f is zero. The condition (T2) is thus verified. The remaining condition (T3) follows from the distinguished triangle

$$\tau^{\leqslant 0}F^{\cdot} \longrightarrow F^{\cdot} \longrightarrow \tau^{\geqslant 1}F^{\cdot} \xrightarrow{+1}$$

for $F^{\cdot} \in \mathbf{D}$.

Now let $(\mathbf{D}^{\leqslant 0}, \mathbf{D}^{\geqslant 0})$ be a t-structure of a triangulated category $\mathbf{D}$.

Definition 8.1.3. We call the full subcategory $\mathcal{C} = \mathbf{D}^{\leqslant 0} \cap \mathbf{D}^{\geqslant 0}$ of $\mathbf{D}$ the *heart* (or core) of the t-structure $(\mathbf{D}^{\leqslant 0}, \mathbf{D}^{\geqslant 0})$.

We will see later that hearts of t-structures are abelian categories (Theorem 8.1.9).

Proposition 8.1.4. *Denote by* $\iota : \mathbf{D}^{\leqslant n} \longrightarrow \mathbf{D}$ (*resp.* $\iota' : \mathbf{D}^{\geqslant n} \longrightarrow \mathbf{D}$) *the inclusion. Then there exists a functor* $\tau^{\leqslant n} : \mathbf{D} \longrightarrow \mathbf{D}^{\leqslant n}$ (*resp.* $\tau^{\geqslant n} : \mathbf{D} \longrightarrow \mathbf{D}^{\geqslant n}$) *such that for any* $Y \in \mathbf{D}^{\leqslant n}$ *and any* $X \in \mathbf{D}$ (*resp. for any* $X \in \mathbf{D}$ *and any* $Y \in \mathbf{D}^{\geqslant n}$) *we have an isomorphism*

$$\mathrm{Hom}_{\mathbf{D}^{\leqslant n}}(Y, \tau^{\leqslant n} X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}}(\iota(Y), X)$$

$$(\textit{resp.}\ \ \mathrm{Hom}_{\mathbf{D}^{\geqslant n}}(\tau^{\geqslant n} X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}}(X, \iota'(Y))),$$

i.e., $\tau^{\leqslant n}$ *is right adjoint to* ι, *and* $\tau^{\geqslant n}$ *is left adjoint to* ι'.

Proof. It is sufficient to show that for any $X \in \mathbf{D}$ there exist $Z \in \mathbf{D}^{\leqslant n}$ and $Z' \in \mathbf{D}^{\geqslant m}$ such that

$$\mathrm{Hom}_{\mathbf{D}}(Y, Z) \simeq \mathrm{Hom}_{\mathbf{D}}(Y, X) \qquad (Y \in \mathbf{D}^{\leqslant n}),$$
$$\mathrm{Hom}_{\mathbf{D}}(Z', Y') \simeq \mathrm{Hom}_{\mathbf{D}}(X, Y') \qquad (Y' \in \mathbf{D}^{\geqslant m}).$$

We may assume that $n = 0$ and $m = 1$. Let X_0 and X_1 be as in (T3). We will show that $Z = X_0$ and $Z' = X_1$ satisfy the desired property. We will only show the statement for X_0 (the one for X_1 is proved similarly). Let $Y \in \mathbf{D}^{\leqslant 0}$. Applying the cohomological functor $\mathrm{Hom}_{\mathbf{D}}(Y, \bullet)$ to the distinguished triangle

$$X_0 \longrightarrow X \longrightarrow X_1 \xrightarrow{+1}$$

we obtain an exact sequence

$$\mathrm{Hom}_{\mathbf{D}}(Y, X_1[-1]) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(Y, X_0) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(Y, X) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(Y, X_1).$$

By (T2) we have $\mathrm{Hom}_{\mathbf{D}}(Y, X_1[-1]) = \mathrm{Hom}_{\mathbf{D}}(Y, X_1) = 0$, and hence we obtain

$$\mathrm{Hom}_{\mathbf{D}}(Y, X_0) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}}(Y, X).$$

The proof is complete. □

Note that if a right (resp. left) adjoint functor exists, then it is unique up to isomorphisms. We call the functors

$$\tau^{\leqslant n} : \mathbf{D} \longrightarrow \mathbf{D}^{\leqslant n}, \qquad \tau^{\geqslant n} : \mathbf{D} \longrightarrow \mathbf{D}^{\geqslant n}$$

the truncation functors associated to the t-structure $(\mathbf{D}^{\leqslant 0}, \mathbf{D}^{\geqslant 0})$. We use the convention $\tau^{>n} = \tau^{\geqslant n+1}$ and $\tau^{<n} = \tau^{\leqslant n-1}$.

By definition we have canonical morphisms

$$\tau^{\leqslant n} X \longrightarrow X, \quad X \longrightarrow \tau^{\geqslant n} X \qquad (X \in \mathbf{D}). \tag{8.1.1}$$

By the proof of Proposition 8.1.4 we easily see the following.

Proposition 8.1.5.

(i) *The canonical morphisms* (8.1.1) *is embedded into a distinguished triangle*

$$\tau^{\leqslant n}(X) \longrightarrow X \longrightarrow \tau^{\geqslant n+1}(X) \xrightarrow{+1} \tag{8.1.2}$$

in $\mathbf{D}$.

(ii) *For* $X \in \mathbf{D}$ *let*

$$X_0 \longrightarrow X \longrightarrow X_1 \xrightarrow{+1}$$

be as in (T3). *Then there exist identifications* $X_0 \simeq \tau^{\leqslant 0}X$ *and* $X_1 \simeq \tau^{\geqslant 1}X$ *by which the morphisms* $X_0 \to X$ *and* $X \to X_1$ *are identified with the canonical ones* (8.1.1). *In particular,* X_0 *and* X_1 *are uniquely determined from* $X \in \mathbf{D}$.

Proposition 8.1.6. *The following conditions on* $X \in \mathbf{D}$ *are equivalent:*

(i) *We have* $X \in \mathbf{D}^{\leqslant n}$ (*resp.* $X \in \mathbf{D}^{\geqslant n}$).
(ii) *The canonical morphism* $\tau^{\leqslant n}X \to X$ (*resp.* $X \to \tau^{\geqslant n}X$) *is an isomorphism.*
(iii) *We have* $\tau^{>n}X = 0$ (*resp.* $\tau^{<n}X = 0$).

Proof. The equivalence of (ii) and (iii) is obvious in view of (8.1.2). The implication (ii)$\Longrightarrow$(i) is obvious by $\tau^{\leqslant n}X \in \mathbf{D}^{\leqslant n}$ and $\tau^{\geqslant n}X \in \mathbf{D}^{\geqslant n}$. It remains to show (i)$\Longrightarrow$(ii). Let us show that the canonical morphism $\tau^{\leqslant n}X \to X$ is an isomorphism for $X \in \mathbf{D}^{\leqslant n}$. We may assume that $n = 0$. By applying Proposition 8.1.5 (ii) to the obvious distinguished triangle

$$X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \xrightarrow{+1}$$

we see that the canonical morphism $\tau^{\leqslant 0}X \to X$ is an isomorphism. The remaining assertion is proved similarly. □

Lemma 8.1.7. *Let*

$$X' \longrightarrow X \longrightarrow X'' \xrightarrow{+1}$$

be a distinguished triangle in $\mathbf{D}$. *If* $X', X'' \in \mathbf{D}^{\leqslant 0}$ (*resp.* $\mathbf{D}^{\geqslant 0}$), *then* $X \in \mathbf{D}^{\leqslant 0}$ (*resp.* $\mathbf{D}^{\geqslant 0}$). *In particular, if* $X', X'' \in \mathcal{C}$, *then* $X \in \mathcal{C}$.

Proof. We only prove the assertion for $\mathbf{D}^{\leqslant 0}$. Assume that $X', X'' \in \mathbf{D}^{\leqslant 0}$. By Proposition 8.1.6 it is enough to show $\tau^{>0}(X) = 0$. In the exact sequence

$$\mathrm{Hom}_{\mathbf{D}}(X'', \tau^{>0}(X)) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(X, \tau^{>0}(X)) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(X', \tau^{>0}(X))$$

we have $\mathrm{Hom}_{\mathbf{D}}(X'', \tau^{>0}(X)) = \mathrm{Hom}_{\mathbf{D}}(X', \tau^{>0}(X)) = 0$ by (T2), and hence $\mathrm{Hom}_{\mathbf{D}}(\tau^{>0}(X), \tau^{>0}(X)) = \mathrm{Hom}_{\mathbf{D}}(X, \tau^{>0}(X)) = 0$ by Proposition 8.1.4. This implies $\tau^{>0}(X) = 0$. □

Proposition 8.1.8. *Let* a, b *be two integers.*

(i) *If* $b \geq a$, *then we have* $\tau^{\leqslant b} \circ \tau^{\leqslant a} \simeq \tau^{\leqslant a} \circ \tau^{\leqslant b} \simeq \tau^{\leqslant a}$ *and* $\tau^{\geqslant b} \circ \tau^{\geqslant a} \simeq \tau^{\geqslant a} \circ \tau^{\geqslant b} \simeq \tau^{\geqslant b}$.

(ii) *If* $a > b$, *then* $\tau^{\leqslant b} \circ \tau^{\geqslant a} \simeq \tau^{\geqslant a} \circ \tau^{\leqslant b} \simeq 0$.
(iii) $\tau^{\geqslant a} \circ \tau^{\leqslant b} \simeq \tau^{\leqslant b} \circ \tau^{\geqslant a}$.

Proof. (i) By Proposition 8.1.6 we obtain $\tau^{\leqslant b} \circ \tau^{\leqslant a} \simeq \tau^{\leqslant a}$. We see from Proposition 8.1.4 that for any $X \in \mathbf{D}$ and $Y \in \mathbf{D}^{\leqslant a}$ we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}^{\leqslant a}}(Y, \tau^{\leqslant a}\tau^{\leqslant b}X) &\simeq \mathrm{Hom}_{\mathbf{D}}(Y, \tau^{\leqslant b}X) \simeq \mathrm{Hom}_{\mathbf{D}^{\leqslant b}}(Y, \tau^{\leqslant b}X)\\ &\simeq \mathrm{Hom}_{\mathbf{D}}(Y, X) \simeq \mathrm{Hom}_{\mathbf{D}^{\leqslant a}}(Y, \tau^{\leqslant a}X),\end{aligned}$$

and hence $\tau^{\leqslant a} \circ \tau^{\leqslant b} \simeq \tau^{\leqslant a}$. The remaining assertion can be proved similarly.

(ii) This is an immediate consequence of Proposition 8.1.6.

(iii) By (ii) we may assume $b \geq a$. Let $X \in \mathbf{D}$. We first construct a morphism $\phi : \tau^{\geqslant a}\tau^{\leqslant b}X \longrightarrow \tau^{\leqslant b}\tau^{\geqslant a}X$. By (i) there exists a distinguished triangle

$$\tau^{\leqslant b}\tau^{\geqslant a}X \longrightarrow \tau^{\geqslant a}X \longrightarrow \tau^{>b}X \xrightarrow{+1},$$

from which we conclude $\tau^{\leqslant b}\tau^{\geqslant a}X \in \mathbf{D}^{\geqslant a}$ by Lemma 8.1.7. Hence we obtain a chain of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}}(\tau^{\leqslant b}X, \tau^{\geqslant a}X) &\simeq \mathrm{Hom}_{\mathbf{D}}(\tau^{\leqslant b}X, \tau^{\leqslant b}\tau^{\geqslant a}X)\\ &\simeq \mathrm{Hom}_{\mathbf{D}^{\geqslant a}}(\tau^{\geqslant a}\tau^{\leqslant b}X, \tau^{\leqslant b}\tau^{\geqslant a}X).\end{aligned}$$

Then $\phi \in \mathrm{Hom}_{\mathbf{D}}(\tau^{\geqslant a}\tau^{\leqslant b}X, \tau^{\leqslant b}\tau^{\geqslant a}X)$ is obtained as the image of the composite of natural morphisms $\tau^{\leqslant b}X \longrightarrow X \longrightarrow \tau^{\geqslant a}X$ through these isomorphisms. Let us show that ϕ is an isomorphism. By (i) there exists a distinguished triangle

$$\tau^{<a}X \longrightarrow \tau^{\leqslant b}X \longrightarrow \tau^{\geqslant a}\tau^{\leqslant b}X \xrightarrow{+1},$$

from which we obtain $\tau^{\geqslant a}\tau^{\leqslant b}X \in \mathbf{D}^{\leqslant b}$ by Lemma 8.1.7. On the other hand, applying the *octahedral axiom* to the three distinguished triangles

$$\begin{cases}\tau^{<a}X \xrightarrow{p} \tau^{\leqslant b}X \longrightarrow \tau^{\geqslant a}\tau^{\leqslant b}X \xrightarrow{+1}\\ \tau^{<a}X \xrightarrow{q\circ p} X \longrightarrow \tau^{\geqslant a}X \xrightarrow{+1}\\ \tau^{\leqslant b}X \xrightarrow{q} X \longrightarrow \tau^{>b}X \xrightarrow{+1},\end{cases}$$

we get a new one

$$\tau^{\geqslant a}\tau^{\leqslant b}X \longrightarrow \tau^{\geqslant a}X \longrightarrow \tau^{>b}X \xrightarrow{+1}.$$

Then it follows from $\tau^{\geqslant a}\tau^{\leqslant b}X \in \mathbf{D}^{\leqslant b}$ and Proposition 8.1.5 (ii) that $\tau^{\geqslant a}\tau^{\leqslant b}X \simeq \tau^{\leqslant b}(\tau^{\geqslant a}X)$. □

Theorem 8.1.9.
(i) *The heart* $\mathcal{C} = \mathbf{D}^{\leqslant 0} \cap \mathbf{D}^{\geqslant 0}$ *is an abelian category.*

(ii) *An exact sequence*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in $\mathcal{C}$ gives rise to a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1}$$

in **D**.

Proof. (i) Let $X, Y \in \mathcal{C}$. By applying Lemma 8.1.7 to the distinguished triangle

$$X \longrightarrow X \oplus Y \longrightarrow Y \xrightarrow{+1}$$

in **D** we see that $X \oplus Y \in \mathcal{C}$.

It remains to show that any morphism $f : X \to Y$ in $\mathcal{C}$ admits a kernel and a cokernel and that the canonical morphism $\operatorname{Coim} f \longrightarrow \operatorname{Im} f$ is an isomorphism. Embed f into a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \xrightarrow{+1} .$$

Then we have $Z \in \mathbf{D}^{\leqslant 0} \cap \mathbf{D}^{\geqslant -1}$ by Lemma 8.1.7. We will show that the kernel and the cokernel of f are given by

$$\operatorname{Coker} f \simeq H^0(Z) = \tau^{\geqslant 0} Z,$$
$$\operatorname{Ker} f \simeq H^{-1}(Z) = \tau^{\leqslant 0}(Z[-1]).$$

Consider the exact sequences

$$\operatorname{Hom}_{\mathbf{D}}(X[1], W) \to \operatorname{Hom}_{\mathbf{D}}(Z, W) \to \operatorname{Hom}_{\mathbf{D}}(Y, W) \to \operatorname{Hom}_{\mathbf{D}}(X, W),$$
$$\operatorname{Hom}_{\mathbf{D}}(W, Y[-1]) \to \operatorname{Hom}_{\mathbf{D}}(W, Z[-1]) \to \operatorname{Hom}_{\mathbf{D}}(W, X) \to \operatorname{Hom}_{\mathbf{D}}(W, Y)$$

for $W \in \mathcal{C}$. By (T2) and Proposition 8.1.4 they are rewritten as

$$0 \longrightarrow \operatorname{Hom}_{\mathbf{D}}(\tau^{\geqslant 0} Z, W) \longrightarrow \operatorname{Hom}_{\mathbf{D}}(Y, W) \longrightarrow \operatorname{Hom}_{\mathbf{D}}(X, W),$$
$$0 \longrightarrow \operatorname{Hom}_{\mathbf{D}}(W, \tau^{\leqslant 0}(Z[-1])) \longrightarrow \operatorname{Hom}_{\mathbf{D}}(W, X) \longrightarrow \operatorname{Hom}_{\mathbf{D}}(W, Y).$$

This implies that $\operatorname{Coker} f \simeq \tau^{\geqslant 0} Z$ and $\operatorname{Ker} f \simeq \tau^{\leqslant 0}(Z[-1])$. Let us show that the canonical morphism $\operatorname{Coim} f \longrightarrow \operatorname{Im} f$ is an isomorphism. Let us embed $Y \longrightarrow \operatorname{Coker} f$ into a distinguished triangle

$$I \longrightarrow Y \longrightarrow \operatorname{Coker} f \xrightarrow{+1} .$$

Then $I \in \mathbf{D}^{\geqslant 0}$ by Lemma 8.1.7. Applying the octahedral axiom to the three distinguished triangles

$$\begin{cases} Y \xrightarrow{p} Z \longrightarrow X[1] \xrightarrow{+1} \\ Y \xrightarrow{q \circ p} \mathrm{Coker}\, f \longrightarrow I[1] \xrightarrow{+1} \\ Z \xrightarrow{q} \mathrm{Coker}\, f \longrightarrow \mathrm{Ker}\, f[2] \xrightarrow{+1}, \end{cases}$$

we get new distinguished triangles

$$X[1] \longrightarrow I[1] \longrightarrow \mathrm{Ker}\, f[2] \xrightarrow{+1},$$
$$\mathrm{Ker}\, f \longrightarrow X \longrightarrow I \xrightarrow{+1} .$$

This implies $I \in \mathbf{D}^{\leqslant 0}$ by Lemma 8.1.7 and hence we have $I \in \mathcal{C}$. Then by the argument used in the proof of the existence of a kernel and a cokernel we conclude that

$$\mathrm{Im}\, f = \mathrm{Ker}(Y \to \mathrm{Coker}\, f) \simeq I \simeq \mathrm{Coker}(\mathrm{Ker}\, f \to X) = \mathrm{Coim}\, f.$$

(ii) Embed $X \xrightarrow{f} Y$ into a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow W \xrightarrow{+1} .$$

Then by $\mathrm{Ker}\, f = 0$ and $\mathrm{Coker}\, f \simeq Z$ we obtain $W \simeq Z$ by the proof of (i). □

Definition 8.1.10. We define a functor

$$H^0 : \mathbf{D} \longrightarrow \mathcal{C} = \mathbf{D}^{\leqslant 0} \cap \mathbf{D}^{\geqslant 0}$$

by $H^0(X) = \tau^{\geqslant 0}\tau^{\leqslant 0} X = \tau^{\leqslant 0}\tau^{\geqslant 0} X \in \mathcal{C}$. For $n \in \mathbb{Z}$ we set $H^n(X) = H^0(X[n]) = (\tau^{\geqslant n}\tau^{\leqslant n} X)[n] \in \mathcal{C}$.

Proposition 8.1.11. *The functor* $H^0 : \mathbf{D} \longrightarrow \mathcal{C} = \mathbf{D}^{\leqslant 0} \cap \mathbf{D}^{\geqslant 0}$ *is a* **cohomological functor** *in the sense of Definition* B.3.8.

Proof. We need to show for a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{+1}$$

in $\mathbf{D}$ that

$$H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z)$$

is an exact sequence in $\mathcal{C}$. The proof is divided into several steps.

(a) We prove that

$$0 \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z) \tag{8.1.3}$$

is exact under the condition $X, Y, Z \in \mathbf{D}^{\geqslant 0}$. For $W \in \mathcal{C}$ consider the exact sequence

$$\mathrm{Hom}_{\mathbf{D}}(W, Z[-1]) \to \mathrm{Hom}_{\mathbf{D}}(W, X) \to \mathrm{Hom}_{\mathbf{D}}(W, Y) \to \mathrm{Hom}_{\mathbf{D}}(W, Z).$$

By (T2) we have $\mathrm{Hom}_{\mathbf{D}}(W, Z[-1]) = 0$. Moreover, for $V \in \mathbf{D}^{\geqslant 0}$ we have $\tau^{\leqslant 0}V \simeq \tau^{\leqslant 0}\tau^{\geqslant 0}V = H^0(V)$, and hence $\mathrm{Hom}_{\mathbf{D}}(W, V) \simeq \mathrm{Hom}_{\mathbf{D}}(W, \tau^{\leqslant 0}V) \simeq \mathrm{Hom}_{\mathcal{C}}(W, H^0(V))$ by Proposition 8.1.4. Hence the above exact sequence is rewritten as

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(W, H^0(X)) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(W, H^0(Y)) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(W, H^0(Z)),$$

from which we obtain our assertion.

(b) We prove that (8.1.3) is exact assuming only $Z \in \mathbf{D}^{\geqslant 0}$. Let $W \in \mathbf{D}^{<0}$. Then we have $\mathrm{Hom}_{\mathbf{D}}(W, Z) = \mathrm{Hom}_{\mathbf{D}}(W, Z[-1]) = 0$ and hence $\mathrm{Hom}_{\mathbf{D}}(W, X) \simeq \mathrm{Hom}_{\mathbf{D}}(W, Y)$. By Proposition 8.1.4 this implies that the canonical morphism $\tau^{<0}X \longrightarrow \tau^{<0}Y$ is an isomorphism. Therefore, applying the octahedral axiom to the distinguished triangles

$$\begin{cases} \tau^{<0}X \xrightarrow{p} X \longrightarrow \tau^{\geqslant 0}X \xrightarrow{+1} \\ \tau^{<0}X \xrightarrow{q\circ p} Y \longrightarrow \tau^{\geqslant 0}Y \xrightarrow{+1} \\ X \xrightarrow{q} Y \longrightarrow Z \xrightarrow{+1}, \end{cases}$$

we get a new one

$$\tau^{\geqslant 0}X \longrightarrow \tau^{\geqslant 0}Y \longrightarrow Z \xrightarrow{+1} .$$

Hence our assertion is a consequence of (a).

(c) Similarly to (b) we can prove that

$$H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z) \longrightarrow 0$$

is exact under the condition $X \in \mathbf{D}^{\leqslant 0}$.

(d) Finally, let us consider the general case. Embed the composite of the morphisms $\tau^{\leqslant 0}X \longrightarrow X \longrightarrow Y$ into a distinguished triangle

$$\tau^{\leqslant 0}X \longrightarrow Y \longrightarrow W \xrightarrow{+1} .$$

By applying (c) we have an exact sequence

$$H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(W).$$

Now applying the octahedral axiom to the distinguished triangles

$$\begin{cases} \tau^{\leqslant 0}X \xrightarrow{r} X \longrightarrow \tau^{>0}X \xrightarrow{+1} \\ \tau^{\leqslant 0}X \xrightarrow{s\circ r} Y \longrightarrow W \xrightarrow{+1} \\ X \xrightarrow{s} Y \longrightarrow Z \xrightarrow{+1}, \end{cases}$$

we get a distinguished triangle

$$W \longrightarrow Z \longrightarrow \tau^{>0}X[1] \xrightarrow{+1} .$$

Hence by (b) we have an exact sequence

$$0 \longrightarrow H^0(W) \longrightarrow H^0(Z).$$

This completes the proof. □

For a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{+1}$$

in $\mathbf{D}$ we thus obtain a long exact sequence

$$\cdots \longrightarrow H^{-1}(Z) \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z) \longrightarrow H^1(X) \longrightarrow \cdots$$

in $\mathcal{C}$.

Now let $\mathbf{D}_i$ be triangulated categories endowed with t-structures $(\mathbf{D}_i^{\leqslant 0}, \mathbf{D}_i^{\geqslant 0})$ $(i = 1, 2)$, and let $F : \mathbf{D}_1 \longrightarrow \mathbf{D}_2$ be a functor of triangulated categories. We denote by $\mathcal{C}_i$ the heart of $(\mathbf{D}_i^{\leqslant 0}, \mathbf{D}_i^{\geqslant 0})$.

Definition 8.1.12. We define an additive functor

$${}^pF : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$$

by ${}^pF = H^0 \circ F \circ \varepsilon_1$, where $\varepsilon_1 : \mathcal{C}_1 \to \mathbf{D}_1$ denotes the inclusion functor.

Definition 8.1.13. We say that F is *left t-exact* (resp. *right t-exact*) if $F(\mathbf{D}_1^{\geqslant 0}) \subset \mathbf{D}_2^{\geqslant 0}$ (resp. $F(\mathbf{D}_1^{\leqslant 0}) \subset \mathbf{D}_2^{\leqslant 0}$). We also say that F is *t-exact* if it is both left and right t-exact.

Example 8.1.14. Let $\mathcal{C}_1, \mathcal{C}_2$ be abelian categories and let $G : \mathcal{C}_1 \to \mathcal{C}_2$ be a left exact functor. Assume that $\mathcal{C}_1$ has enough injectives. Then the right derived functor $RG : D^+(\mathcal{C}_1) \longrightarrow D^+(\mathcal{C}_2)$ is left t-exact with respect to the standard t-structures of $D^+(\mathcal{C}_i)$ (see Example 8.1.2).

Proposition 8.1.15. *Let $\mathbf{D}_i$, $(\mathbf{D}_i^{\leqslant 0}, \mathbf{D}_i^{\geqslant 0})$, $(i = 1, 2)$ and $F : \mathbf{D}_1 \longrightarrow \mathbf{D}_2$ be as above. Assume that F is left t-exact.*

(i) *For any $X \in \mathbf{D}_1$ we have $\tau^{\leqslant 0}(F(\tau^{\leqslant 0}X)) \simeq \tau^{\leqslant 0}F(X)$. In particular, for $X \in \mathbf{D}_1^{\geqslant 0}$ there exists an isomorphism ${}^pF(H^0(X)) \simeq H^0(F(X))$ in $\mathcal{C}_2$.*
(ii) *${}^pF : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is a left exact functor between abelian categories.*

Proof. (i) It is sufficient to show that the canonical morphism

$$\operatorname{Hom}_{\mathbf{D}_2^{\leqslant 0}}\big(W,\ \tau^{\leqslant 0}(F(\tau^{\leqslant 0}(X)))\big) \to \operatorname{Hom}_{\mathbf{D}_2^{\leqslant 0}}\big(W,\ \tau^{\leqslant 0}(F(X))\big)$$

is an isomorphism for any $W \in \mathbf{D}_2^{\leqslant 0}$. By Proposition 8.1.4 we have

$$\operatorname{Hom}_{\mathbf{D}_2^{\leqslant 0}}\big(W,\ \tau^{\leqslant 0}(F(\tau^{\leqslant 0}(X)))\big) \simeq \operatorname{Hom}_{\mathbf{D}_2}\big(W,\ F(\tau^{\leqslant 0}(X))\big),$$

$$\operatorname{Hom}_{\mathbf{D}_2^{\leqslant 0}}\big(W,\ \tau^{\leqslant 0}(F(X))\big) \simeq \operatorname{Hom}_{\mathbf{D}_2}\big(W,\ F(X)\big),$$

and hence we have only to show that the canonical morphism

$$\operatorname{Hom}_{\mathbf{D}_2}\big(W,\ F(\tau^{\leqslant 0}(X))\big) \longrightarrow \operatorname{Hom}_{\mathbf{D}_2}\big(W,\ F(X)\big)$$

is an isomorphism for any $W \in \mathbf{D}_2^{\leqslant 0}$. By the distinguished triangle

$$F(\tau^{\leqslant 0}(X)) \longrightarrow F(X) \longrightarrow F(\tau^{\geqslant 1}(X)) \xrightarrow{+1}$$

we obtain an exact sequence

$$\begin{aligned}\operatorname{Hom}_{\mathbf{D}_2}(W, F(\tau^{\geqslant 1}(X))[-1]) &\longrightarrow \operatorname{Hom}_{\mathbf{D}_2}(W, F(\tau^{\leqslant 0}(X)))\\ &\longrightarrow \operatorname{Hom}_{\mathbf{D}_2}(W, F(X))\\ &\longrightarrow \operatorname{Hom}_{\mathbf{D}_2}(W, F(\tau^{\geqslant 1}(X))).\end{aligned}$$

Since F is left t-exact, we have $F(\tau^{\geqslant 1}(X)) \in \mathbf{D}_2^{\geqslant 1}$, and hence

$$\operatorname{Hom}_{\mathbf{D}_2}(W, F(\tau^{\geqslant 1}(X))[n]) = 0 \qquad (n \leq 0)$$

by (T2). Therefore, the assertion follows from the above exact sequence.

(ii) For an exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in $\mathcal{C}_1$ we have a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{+1}$$

in $\mathbf{D}_1$ by Theorem 8.1.9 (ii). Hence we obtain a distinguished triangle

$$F(X) \longrightarrow F(Y) \longrightarrow F(Z) \xrightarrow{+1}$$

in $\mathbf{D}_2$. By considering the cohomology long exact sequence associated to it we obtain an exact sequence

$$H^{-1}(F(Z)) \longrightarrow H^0(F(X)) \longrightarrow H^0(F(Y)) \longrightarrow H^0(F(Z)).$$

It remains to show $H^{-1}(F(Z)) = 0$. Since F is left t-exact, we have $F(Z) \in \mathbf{D}_2^{\geqslant 0}$. Hence we have $\tau^{\leqslant -1}F(Z) = 0$ by Proposition 8.1.6. It follows that we have $H^{-1}(F(Z)) = \tau^{\geqslant -1}\tau^{\leqslant -1}F(Z)[-1] = 0$. □

Lemma 8.1.16. *Let $\mathbf{D}'_i$ be a triangulated category and $\mathbf{D}_i \subset \mathbf{D}'_i$ its full triangulated subcategory with a t-structure $(\mathbf{D}_i^{\leqslant 0}, \mathbf{D}_i^{\geqslant 0})$ $(i = 1, 2)$. Assume that $F : \mathbf{D}'_1 \longrightarrow \mathbf{D}'_2$ and $G : \mathbf{D}'_2 \longrightarrow \mathbf{D}'_1$ are functors of triangulated categories and F is the left adjoint functor of G.*

(i) *If* $F(\mathbf{D}_1) \subset \mathbf{D}_2$ *and* $F(\mathbf{D}_1^{\leqslant 0}) \subset \mathbf{D}_2^{\leqslant d}$ *for* $d \in \mathbb{Z}$*, then for any* $Y \in \mathbf{D}_2^{\geqslant 0}$ *satisfying* $G(Y) \in \mathbf{D}_1$ *we have* $G(Y) \in \mathbf{D}_1^{\geqslant -d}$.
(ii) *If* $G(\mathbf{D}_2) \subset \mathbf{D}_1$ *and* $G(\mathbf{D}_2^{\geqslant 0}) \subset \mathbf{D}_1^{\geqslant -d}$ *for* $d \in \mathbb{Z}$*, then for any* $X \in \mathbf{D}_1^{\leqslant 0}$ *satisfying* $F(X) \in \mathbf{D}_2$ *we have* $F(X) \in \mathbf{D}_2^{\leqslant d}$.

Proof. We prove only the assertion (i). By Proposition 8.1.6 it is enough to show $\tau^{<-d}G(Y) = 0$. According to Proposition 8.1.4, for any $X \in \mathbf{D}_1^{<-d}$ we have an isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}_1^{<-d}}(X,\ \tau^{<-d}G(Y)) &\simeq \mathrm{Hom}_{\mathbf{D}_1}(X,\ G(Y)) \\ &\simeq \mathrm{Hom}_{\mathbf{D}_2}(F(X),\ Y) = 0\end{aligned}$$

(note $F(X) \in \mathbf{D}_2^{<0}$ and $Y \in \mathbf{D}_2^{\geqslant 0}$). Therefore, we have $\tau^{<-d}G(Y) = 0$. □

Corollary 8.1.17. *Let* $\mathbf{D}_i$ *be a triangulated category with a* t*-structure* $(i = 1, 2)$. *Assume that* $F : \mathbf{D}_1 \longrightarrow \mathbf{D}_2$ *and* $G : \mathbf{D}_2 \longrightarrow \mathbf{D}_1$ *are functors of triangulated categories and* F *is the left adjoint functor of* G*. Then* F *is right* t*-exact if and only if* G *is left* t*-exact.*

8.1.2 Perverse sheaves

From now on, let X be a (not necessarily smooth) algebraic variety or an analytic space and denote by $D^b_c(X)$ the full subcategory of $D^b(X) = D^b(\mathrm{Mod}(\mathbb{C}_X))$ consisting of objects $F^\cdot \in D^b(X)$ such that $H^j(F^\cdot)$ is a constructible sheaf on X for any j. For the definition of constructible sheaves and basic properties of $D^b_c(X)$ see Section 4.5. The aim of this subsection is to introduce the *perverse* t*-structure* $({}^pD_c^{\leqslant 0}(X),\ {}^pD_c^{\geqslant 0}(X))$ on $\mathbf{D} = D^b_c(X)$ and define the category of perverse sheaves on X to be its heart ${}^pD_c^{\leqslant 0}(X) \cap {}^pD_c^{\geqslant 0}(X)$. We follow the basic reference [BBD]. We are also indebted to [GM1], [G1], and [KS2, Chapter X].

Remark 8.1.18.
(i) Although we restrict ourselves to the case of complex coefficients, all of the results that we present in Sections 8.1.2 and 8.2 remain valid even after replacing $\mathrm{Mod}(\mathbb{C}_X)$ with $\mathrm{Mod}(\mathbb{Q}_X)$. In particular, we have the notion perverse sheaves and intersection cohomology groups with coefficients in $\mathbb{Q}$. They are essential for the theory of Hodge modules to be explained in Section 8.3.
(ii) The t-structure that we treat here is the one with respect to the "*middle perversity*" in the terminology of [GM1].
(iii) There exists a more general theory of perverse sheaves on subanalytic spaces as explained in [KS2, Chapter X].

Notation 8.1.19. For a locally closed analytic subspace S of X we denote its dimension by d_S. The inclusion map $S \hookrightarrow X$ is usually denoted by i_S.

Recall that we denote by $\mathbf{D}_X : D^b_c(X)^{\mathrm{op}} \xrightarrow{\sim} D^b_c(X)$ the Verdier duality functor.

Definition 8.1.20. We define full subcategories ${}^pD_c^{\leqslant 0}(X)$ and ${}^pD_c^{\geqslant 0}(X)$ of $D_c^b(X)$ as follows. For $F^{\cdot} \in D_c^b(X)$ we have $F^{\cdot} \in {}^pD_c^{\leqslant 0}(X)$ if and only if

(i) $\dim\{\mathrm{supp}\, H^j(F^{\cdot})\} \leq -j$ for any $j \in \mathbb{Z}$,

and $F^{\cdot} \in {}^pD_c^{\geqslant 0}(X)$ if and only if

(ii) $\dim\{\mathrm{supp}\, H^j(\mathbf{D}_X F^{\cdot})\} \leq -j$ for any $j \in \mathbb{Z}$,

We define a full subcategory $\mathrm{Perv}(\mathbb{C}_X)$ of $D_c^b(X)$ by

$$\mathrm{Perv}(\mathbb{C}_X) = {}^pD_c^{\leqslant 0}(X) \cap {}^pD_c^{\geqslant 0}(X).$$

We will show later that the pair $({}^pD_c^{\leqslant 0}(X), {}^pD_c^{\geqslant 0}(X))$ defines a t-structure on $D_c^b(X)$, and hence $\mathrm{Perv}(\mathbb{C}_X)$ will turn out to be an abelian category. Since we have $\mathbf{D}_X\mathbf{D}_X F^{\cdot} \simeq F^{\cdot}$ for any $F^{\cdot} \in D_c^b(X)$, the Verdier duality functor $\mathbf{D}_X(\bullet)$ exchanges ${}^pD_c^{\leqslant 0}(X)$ with ${}^pD_c^{\geqslant 0}(X)$.

Lemma 8.1.21. *Let $F^{\cdot} \in D_c^b(X)$. Then we have*

$$\mathrm{supp}\, H^j(\mathbf{D}_X F^{\cdot}) = \{x \in X \mid \quad H^{-j}(i_{\{x\}}^! F^{\cdot}) \neq 0\}$$

for any $j \in \mathbb{Z}$, where $i_{\{x\}} : \{x\} \hookrightarrow X$ are inclusion maps.

Proof. Since for each $x \in X$ we have

$$i_{\{x\}}^! F^{\cdot} \simeq i_{\{x\}}^! \mathbf{D}_X \mathbf{D}_X F^{\cdot} \simeq \mathbf{D}_{\{x\}} i_{\{x\}}^{-1}(\mathbf{D}_X F^{\cdot}),$$

we obtain an isomorphism $H^{-j}(i_{\{x\}}^! F^{\cdot}) \simeq [H^j(\mathbf{D}_X F^{\cdot})_x]^*$ for any $j \in \mathbb{Z}$. □

Proposition 8.1.22. *Let $F^{\cdot} \in D_c^b(X)$ and $X = \bigsqcup_{\alpha \in A} X_\alpha$ be a complex stratification of X consisting of connected strata such that $i_{X_\alpha}^{-1} F^{\cdot}$ and $i_{X_\alpha}^! F^{\cdot}$ have locally constant cohomology sheaves for any $\alpha \in A$. Then*

(i) *$F^{\cdot} \in {}^pD_c^{\leqslant 0}(X)$ if and only if $H^j(i_{X_\alpha}^{-1} F^{\cdot}) = 0$ for any α and $j > -d_{X_\alpha}$.*
(ii) *$F^{\cdot} \in {}^pD_c^{\geqslant 0}(X)$ if and only if $H^j(i_{X_\alpha}^! F^{\cdot}) = 0$ for any α and $j < -d_{X_\alpha}$.*

Proof. (i) Trivial.

(ii) By Lemma 8.1.21, $F^{\cdot} \in {}^pD_c^{\geqslant 0}(X)$ if and only if

$$\dim\{x \in X \mid \quad H^{-j}(i_{\{x\}}^! F^{\cdot}) \neq 0\} \leq -j$$

for any $j \in \mathbb{Z}$. For $x \in X_\alpha$ decompose the morphism $i_{\{x\}} : \{x\} \hookrightarrow X$ into $\{x\} \xhookrightarrow{j_{\{x\}}} X_\alpha \xhookrightarrow{i_{X_\alpha}} X$. Then we have an isomorphism

$$i_{\{x\}}^! F^{\cdot} \simeq j_{\{x\}}^! i_{X_\alpha}^! F^{\cdot} \simeq j_{\{x\}}^{-1} i_{X_\alpha}^! F^{\cdot}[-2d_{X_\alpha}], \tag{8.1.4}$$

where we used our assumption on $i_{X_\alpha}^! F^{\cdot}$ in the last isomorphism. Hence for any $j \in \mathbb{Z}$ by the connectedness of X_α, $X_\alpha \cap \{x \in X \mid \quad H^{-j}(i_{\{x\}}^! F^{\cdot}) \neq 0\} = X_\alpha \cap \mathrm{supp}\, H^j(\mathbf{D}_X F^{\cdot})$ is X_α or $\emptyset$. Moreover, from (8.1.4) we easily see that the following conditions are equivalent for any $\alpha \in A$:

(a) $H^j(i^!_{X_\alpha} F^{\cdot}) = 0$ for any $j < -d_{X_\alpha}$.
(b) $H^{-j}(i^!_{\{x\}} F^{\cdot}) = 0$ for any $x \in X_\alpha$ and $j > -d_{X_\alpha}$.
(c) $X_\alpha \cap \operatorname{supp} H^j(\mathbf{D}_X F^{\cdot}) = \emptyset$ for any $j > -d_{X_\alpha}$.

The last condition (c) implies that for any stratum X_α such that $X_\alpha \subset \operatorname{supp} H^j(\mathbf{D}_X F^{\cdot})$ we must have $d_{X_\alpha} \leq -j$. This completes the proof. □

Corollary 8.1.23. *Assume that X is a connected complex manifold and all the cohomology sheaves of $F^{\cdot} \in D^b_c(X)$ are locally constant on X. Then*

(i) $F^{\cdot} \in {}^pD^{\leqslant 0}_c(X)$ *if and only if* $H^j(F^{\cdot}) = 0$ *for any* $j > -d_X$.
(ii) $F^{\cdot} \in {}^pD^{\geqslant 0}_c(X)$ *if and only if* $H^j(F^{\cdot}) = 0$ *for any* $j < -d_X$.

Proposition 8.1.24. *Let $F^{\cdot} \in D^b_c(X)$. Then the following four conditions are equivalent:*

(i) $F^{\cdot} \in {}^pD^{\geqslant 0}_c(X)$.
(ii) *For any locally closed analytic subset S of X we have*

$$H^j(i^!_S(F^{\cdot})) = 0 \quad \textit{for any} \quad j < -d_S.$$

(iii) *For any locally closed analytic subset S of X we have*

$$H^j_S(F^{\cdot}) = H^j \mathrm{R}\Gamma_S(F^{\cdot}) = 0 \quad \textit{for any} \quad j < -d_S.$$

(iv) *For any locally closed "smooth" analytic subset S of X we have*

$$H^j(i^!_S(F^{\cdot})) = 0 \quad \textit{for any} \quad j < -d_S.$$

Proof. First, since we have $H^i_S(\bullet) = H^i R i_{S*} i^!_S(\bullet)$, the conditions (ii) and (iii) are equivalent. Let us prove the equivalence of (ii) and (iv). Assume that the condition (iv) is satisfied for $F^{\cdot} \in D^b_c(X)$. We will show that

$$H^j(i^!_Z(F^{\cdot})) = 0 \quad \text{for any} \quad j < -d_Z. \tag{8.1.5}$$

for any locally closed (possibly singular) analytic subset Z of X by induction on $\dim Z$. Denote by Z_{reg} the smooth part of Z and set $Z' = Z \setminus Z_{\mathrm{reg}}$. Then $\dim Z' < \dim Z$ and our hypothesis of induction implies that

$$H^j_{Z'}(F^{\cdot}) = 0 \quad \text{for any} \quad j < -d_{Z'}.$$

In particular, we have

$$H^j_{Z'}(F^{\cdot}) = 0 \quad \text{for any} \quad j < -d_Z.$$

So the assertion (8.1.5) follows from (iv) and the distinguished triangle

$$\mathrm{R}\Gamma_{Z'}(F^{\cdot}) \longrightarrow \mathrm{R}\Gamma_Z(F^{\cdot}) \longrightarrow \mathrm{R}\Gamma_{Z_{\mathrm{reg}}}(F^{\cdot}) \xrightarrow{+1} .$$

Now let us take a complex stratification $X = \sqcup_{\alpha \in A} X_\alpha$ of X consisting of connected strata such that $i_{X_\alpha}^{-1} F^\cdot$ and $i_{X_\alpha}^! F^\cdot$ have locally constant cohomology sheaves for any $\alpha \in A$. Then by Proposition 8.1.22 the condition (i) is equivalent to the one $H^j(i_{X_\alpha}^! F^\cdot) = 0$ for any α and $j < -d_{X_\alpha}$. Therefore, if we take $S = X_\alpha$ in (iv) we see that (iv) implies (i). It remains to prove that (i) implies (iv). Assume that $H^j(i_{X_\alpha}^! F^\cdot) = 0$ for any α and $j < -d_{X_\alpha}$. For any locally closed (smooth) analytic subset S in X, we need to show that

$$H^j(i_S^!(F^\cdot)) = 0 \quad \text{for any} \quad j < -d_S.$$

Set $X_k = \bigsqcup_{\dim X_\alpha \leq k} X_\alpha$ in X ($k = -1, 0, 1, \dots, d_X$). Since

$$X_{-1} = \emptyset \subset X_0 \subset \cdots \subset X_{d_X} = X,$$

it is enough to prove the following assertions $(\mathrm{P})_k$ by induction on k:

$$(\mathrm{P})_k: \quad H^j_{S \cap X_k}(F^\cdot) = 0 \quad \text{for any} \quad j < -d_S. \tag{8.1.6}$$

Moreover, by the distinguished triangles

$$\mathrm{R}\Gamma_{S \cap X_{k-1}}(F^\cdot) \longrightarrow \mathrm{R}\Gamma_{S \cap X_k}(F^\cdot) \longrightarrow \mathrm{R}\Gamma_{S \cap (X_k \setminus X_{k-1})}(F^\cdot) \xrightarrow{+1}$$

for $k = 0, 1, \dots, d_X$, we can reduce the problem to the proof of the assertions

$$(\mathrm{Q})_k: \quad H^j_{S \cap (X_k \setminus X_{k-1})}(F^\cdot) = 0 \quad \text{for any} \quad j < -d_S. \tag{8.1.7}$$

Note that $X_k \setminus X_{k-1}$ is the union of k-dimensional strata. Hence we obtain a direct sum decomposition

$$H^j_{S \cap (X_k \setminus X_{k-1})}(F^\cdot) \simeq \bigoplus_{\dim X_\alpha = k} H^j_{S \cap X_\alpha}(F^\cdot)$$

and it remains to show $H^j(i_{S \cap X_\alpha}^! F^\cdot) \simeq 0$ for any $\alpha \in A$ and $j < -d_S$. Decomposing $i_{S \cap X_\alpha}: S \cap X_\alpha \hookrightarrow X$ into $S \cap X_\alpha \xhookrightarrow{j_{X_\alpha}} X_\alpha \xhookrightarrow{i_{X_\alpha}} X$ we obtain an isomorphism $i_{S \cap X_\alpha}^! F^\cdot \simeq j_{X_\alpha}^!(i_{X_\alpha}^!(F^\cdot))$. Therefore, by applying Lemma 8.1.25 below to $Y = X_\alpha$ and $G^\cdot = i_{X_\alpha}^!(F^\cdot) \in D_c^b(Y)$ we obtain

$$H^j j_{X_\alpha}^!(G^\cdot) \simeq H^j i_{S \cap X_\alpha}^!(F^\cdot) \simeq 0 \quad \text{for any} \quad j < -d_{S \cap X_\alpha}.$$

This completes the proof. □

Lemma 8.1.25. *Let Y be a complex manifold and $G^\cdot \in D_c^b(Y)$. Assume that all the cohomology sheaves of $G^\cdot$ are locally constant on Y and for an integer $d \in \mathbb{Z}$ we have $H^j G^\cdot = 0$ for $j < d$. Then for any locally closed analytic subset Z of Y we have*

$$H^j_Z(G^\cdot) = 0 \quad \textit{for any} \quad j < d + 2\,\mathrm{codim}_Y Z.$$

Proof. By induction on the cohomological length of $G^{\cdot}$ we may assume that $G^{\cdot}$ is a local system L on Y. Since the question is local on Y, we may further assume that L is the constant sheaf $\mathbb{C}_Y$. Hence it is sufficient to show

$$H^j_Z(\mathbb{C}_Y) = 0 \quad \text{for any} \quad j < 2\,\mathrm{codim}_Y Z.$$

This well-known result can be proved by induction on the dimension of Z with the help of the distinguished triangle

$$\mathrm{R}\Gamma_{Z\setminus Z_{\mathrm{reg}}}(\mathbb{C}_Y) \longrightarrow \mathrm{R}\Gamma_Z(\mathbb{C}_Y) \longrightarrow \mathrm{R}\Gamma_{Z_{\mathrm{reg}}}(\mathbb{C}_Y) \xrightarrow{+1} . \qquad \square$$

It is easily seen that $F^{\cdot} \in D^b_c(X)$ belongs to ${}^pD^{\leqslant 0}_c(X)$ if and only if the condition

(i)* for any Zariski locally closed irreducible subvariety S of X there exists a Zariski open dense smooth subset S_0 of S such that $H^j(i^{-1}_{S_0}F^{\cdot})$ is a local system for any j and $H^j(i^{-1}_{S_0}F^{\cdot}) = 0$ for any $j > -d_S$

is satisfied (see the proof of Lemma 7.2.9). Also by the proof of Proposition 8.1.24, we easily see that $F^{\cdot} \in D^b_c(X)$ belongs to ${}^pD^{\geqslant 0}_c(X)$ if and only if the condition

(ii)* for any Zariski locally closed irreducible subvariety S of X there exists a Zariski open dense smooth subset S_0 of S such that $H^j(i^!_{S_0}F^{\cdot})$ is a local system for any j and $H^j(i^!_{S_0}F^{\cdot}) = 0$ for any $j < -d_S$

is satisfied.

Proposition 8.1.26. *Let $F^{\cdot} \in {}^pD^{\leqslant 0}_c(X)$ and $G^{\cdot} \in {}^pD^{\geqslant 0}_c(X)$.*

(i) *We have*

$$H^j(R\mathcal{H}om_{\mathbb{C}_X}(F^{\cdot}, G^{\cdot})) = 0$$

for any $j < 0$.

(ii) *The correspondence*

$$\{\textit{open subsets of } X\} \ni U \longmapsto \mathrm{Hom}_{D^b(U)}(F^{\cdot}|_U,\ G^{\cdot}|_U)$$

defines a sheaf on X.

Proof. (i) Set $S = \bigcup_{j<0} \mathrm{supp}(H^j(R\mathcal{H}om_{\mathbb{C}_X}(F^{\cdot}, G^{\cdot}))) \subset X$. Assume that $S \neq \emptyset$. Let $i_S : S \to X$ be the embedding. For $j < 0$ we have

$$\mathrm{supp}(H^j R\mathcal{H}om_{\mathbb{C}_X}(F^{\cdot}, G^{\cdot})) \subset S,$$

and hence

$$\begin{aligned} H^j R\mathcal{H}om_{\mathbb{C}_X}(F^{\cdot}, G^{\cdot}) &\simeq H^j(\mathrm{R}\Gamma_S R\mathcal{H}om_{\mathbb{C}_X}(F^{\cdot}, G^{\cdot})) \\ &\simeq H^j(i_{S*}i^!_S R\mathcal{H}om_{\mathbb{C}_X}(F^{\cdot}, G^{\cdot})) \\ &\simeq i_{S*}H^j(R\mathcal{H}om_{\mathbb{C}_S}(i^{-1}_S F^{\cdot}, i^!_S G^{\cdot})). \end{aligned}$$

Our assumption $F^\cdot \in {}^pD_c^{\leqslant 0}(X)$ implies that

$$\dim \operatorname{supp}\{H^k(i_S^{-1}F^\cdot)\} \leq -k$$

for any $k \in \mathbb{Z}$, and the dimension of

$$Z := \bigcup_{k>-d_S} \operatorname{supp}\{H^k(i_S^{-1}F^\cdot)\} \subset S$$

is less than d_S. Therefore, we obtain $S_0 = S \setminus Z \neq \emptyset$ and $H^j i_{S_0}^{-1} F^\cdot = 0$ for any $j > -d_S$. On the other hand, we have $H^j i_S^! G^\cdot = 0$ for any $j < -d_S$. Hence we obtain $H^j R\mathcal{H}om_{\mathbb{C}_S}(i_S^{-1}F^\cdot, i_S^! G^\cdot)|_{S_0} = 0$ for any $j < 0$. But this contradicts our definition $S = \bigcup_{j<0} \operatorname{supp}\{H^j R\mathcal{H}om_{\mathbb{C}_X}(F^\cdot, G^\cdot)\}$.

(ii) By (i) we have

$$\begin{aligned}\operatorname{Hom}_{D^b(U)}(F^\cdot|_U, G^\cdot|_U) &= H^0\bigl(U, R\mathcal{H}om_{\mathbb{C}_X}(F^\cdot, G^\cdot)\bigr)\\ &= \Gamma\bigl(U, H^0(R\mathcal{H}om_{\mathbb{C}_X}(F^\cdot, G^\cdot))\bigr).\end{aligned}$$

Hence the correspondence $U \longmapsto \operatorname{Hom}_{D^b(U)}(F^\cdot|_U, G^\cdot|_U)$ gives a sheaf isomorphic to $H^0(R\mathcal{H}om_{\mathbb{C}_X}(F^\cdot, G^\cdot))$. □

Now we are ready to prove the following.

Theorem 8.1.27. *The pair* $({}^pD_c^{\leqslant 0}(X), {}^pD_c^{\geqslant 0}(X))$ *defines a* t*-structure on* $D_c^b(X)$.

Proof. Among the conditions of t-structures in Definition 8.1.1, (T1) is trivially satisfied and (T2) follows from Proposition 8.1.26 above. Let us show (T3). For $F^\cdot \in D_c^b(X)$, take a stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ of X such that $i_{X_\alpha}^{-1}F^\cdot$ and $i_{X_\alpha}^! F^\cdot$ have locally constant cohomology sheaves for any $\alpha \in A$. Set $X_k = \bigsqcup_{\dim X_\alpha \leq k} X_\alpha \subset X$ ($k = -1, 0, 1, 2, \dots$) and consider the following assertions:

$(\mathrm{S})_k$ $\begin{cases} \text{There exists } F_0^\cdot \in {}^pD_c^{\leqslant 0}(X \setminus X_k),\ F_1^\cdot \in {}^pD_c^{\geqslant 1}(X \setminus X_k), \text{ and a distinguished triangle} \\ \qquad F_0^\cdot \longrightarrow F^\cdot|_{X\setminus X_k} \longrightarrow F_1^\cdot \xrightarrow{+1} \\ \text{in } D_c^b(X \setminus X_k) \text{ such that } F_0^\cdot|_{X_\alpha} \text{ and } F_1^\cdot|_{X_\alpha} \text{ have locally constant cohomology sheaves for any } \alpha \in A \text{ satisfying } X_\alpha \subset X \setminus X_k. \end{cases}$

Note that what we want to prove is $(\mathrm{S})_{-1}$. We will show $(\mathrm{S})_k$'s by descending induction on $k \in \mathbb{Z}$. It is trivial for $k \gg 0$. Assume that $(\mathrm{S})_k$ holds. Let us prove $(\mathrm{S})_{k-1}$. Take a distinguished triangle

$$F_0^\cdot \longrightarrow F^\cdot|_{X\setminus X_k} \longrightarrow F_1^\cdot \xrightarrow{+1} \tag{8.1.8}$$

in $D_c^b(X \setminus X_k)$ as in $(\mathrm{S})_k$. Let $j : X \setminus X_k \hookrightarrow X \setminus X_{k-1}$ be the open embedding and $i : X_k \setminus X_{k-1} \hookrightarrow X \setminus X_{k-1}$ be the closed embedding. Since $j_!$ is left adjoint to $j^!$, the morphism $F_0^{\cdot} \longrightarrow j^!(F^{\cdot}|_{X\setminus X_{k-1}}) \simeq F^{\cdot}|_{X\setminus X_k}$ gives rise to a morphism $j_!F_0^{\cdot} \longrightarrow F^{\cdot}|_{X\setminus X_{k-1}}$. Let us embed this morphism into a distinguished triangle

$$j_!F_0^{\cdot} \longrightarrow F^{\cdot}|_{X\setminus X_{k-1}} \longrightarrow G^{\cdot} \xrightarrow{+1} . \tag{8.1.9}$$

We also embed the composite of the morphisms $\tau^{\leqslant -k}i_!i^!G^{\cdot} \longrightarrow i_!i^!G^{\cdot} \longrightarrow G^{\cdot}$ into a distinguished triangle

$$\tau^{\leqslant -k}i_!i^!G^{\cdot} \longrightarrow G^{\cdot} \longrightarrow \tilde{F}_1^{\cdot} \xrightarrow{+1} . \tag{8.1.10}$$

We finally embed the composite of $F^{\cdot}|_{X\setminus X_{k-1}} \longrightarrow G^{\cdot} \longrightarrow \tilde{F}_1^{\cdot}$ into a distinguished triangle

$$\tilde{F}_0^{\cdot} \longrightarrow F^{\cdot}|_{X\setminus X_{k-1}} \longrightarrow \tilde{F}_1^{\cdot} \xrightarrow{+1} . \tag{8.1.11}$$

By our construction $\tilde{F}_0^{\cdot}|_{X_\alpha}$ and $\tilde{F}_1^{\cdot}|_{X_\alpha}$ have locally constant cohomology sheaves for any $\alpha \in A$ satisfying $X_\alpha \subset X \setminus X_{k-1}$. It remains to show $\tilde{F}_0^{\cdot} \in {}^pD_c^{\leqslant 0}(X \setminus X_{k-1})$ and $\tilde{F}_1^{\cdot} \in {}^pD_c^{\geqslant 1}(X \setminus X_{k-1})$. Applying the functor $j^{-1}(\bullet)$ to (8.1.10) and (8.1.9), we get an isomorphism $j^{-1}\tilde{F}_1^{\cdot} \simeq j^{-1}G^{\cdot}$ and a distinguished triangle

$$F_0^{\cdot} \longrightarrow F^{\cdot}|_{X\setminus X_k} \longrightarrow j^{-1}\tilde{F}_1^{\cdot} \xrightarrow{+1} .$$

Hence we have $j^{-1}\tilde{F}_1^{\cdot} \simeq F_1^{\cdot}$, and $j^{-1}\tilde{F}_0^{\cdot} \simeq F_0^{\cdot}$ by (8.1.8) and (8.1.11). Therefore, we have only to show that

(i) $H^j(i^{-1}\tilde{F}_0^{\cdot}) = 0$ for ${}^\forall j > -k$
(ii) $H^j(i^!\tilde{F}_1^{\cdot}) = 0$ for ${}^\forall j < -k+1$

in view of Proposition 8.1.22 (note that $X_k \setminus X_{k-1}$ is a union of k-dimensional strata). By applying the octahedral axiom to the three distinguished triangles

$$\begin{cases} j_!F_0^{\cdot} \longrightarrow F^{\cdot}|_{X\setminus X_{k-1}} \xrightarrow{f} G^{\cdot} \xrightarrow{+1} \\ \tilde{F}_0^{\cdot} \longrightarrow F^{\cdot}|_{X\setminus X_{k-1}} \xrightarrow{g\circ f} \tilde{F}_1^{\cdot} \xrightarrow{+1} \\ \tau^{\leqslant -k}i_!i^!G^{\cdot} \longrightarrow G^{\cdot} \xrightarrow{g} \tilde{F}_1^{\cdot} \xrightarrow{+1} \end{cases}$$

we obtain a distinguished triangle

$$j_!F_0^{\cdot} \longrightarrow \tilde{F}_0^{\cdot} \longrightarrow \tau^{\leqslant -k}i_!i^!G^{\cdot} \xrightarrow{+1} .$$

Hence we have $i^{-1}\tilde{F}_0^{\cdot} \simeq i^{-1}\tau^{\leqslant -k}i_!i^!G^{\cdot} \simeq i^{-1}i_!\tau^{\leqslant -k}i^!G^{\cdot} \simeq \tau^{\leqslant -k}i^!G^{\cdot}$. The assertion (i) is proved. By applying the functor $i^!$ to (8.1.10) we obtain a distinguished triangle

$$i^!\tau^{\leqslant -k}i_!i^!G^{\cdot} \longrightarrow i^!G^{\cdot} \longrightarrow i^!\tilde{F}_1^{\cdot} \xrightarrow{+1} .$$

We have $i^!\tau^{\leqslant -k}i_!i^!G \simeq i^!i_!\tau^{\leqslant -k}i^!G \simeq \tau^{\leqslant -k}i^!G$, and hence we obtain $i^!\tilde{F}_1^{\cdot} \simeq \tau^{\geqslant -k+1}(i^!G^{\cdot})$ by this distinguished triangle. The assertion (ii) is also proved. □

Definition 8.1.28. The t-structure $({}^pD_c^{\leqslant 0}(X), {}^pD_c^{\geqslant 0}(X))$ of the triangulated category $D_c^b(X)$ is called the perverse t-structure. An object of its heart $\mathrm{Perv}(\mathbb{C}_X) = {}^pD_c^{\leqslant 0}(X) \cap {}^pD_c^{\geqslant 0}(X)$ is called a *perverse sheaf* on X. We denote by

$$ {}^p\tau^{\leqslant 0} : D_c^b(X) \longrightarrow {}^pD_c^{\leqslant 0}(X), \qquad {}^p\tau^{\geqslant 0} : D_c^b(X) \longrightarrow {}^pD_c^{\geqslant 0}(X) $$

the truncation functors with respect to the perverse t-structure. For $n \in \mathbb{Z}$ we define a functor

$$ {}^pH^n : D_c^b(X) \longrightarrow \mathrm{Perv}(\mathbb{C}_X) $$

by ${}^pH^n(F^\cdot) = {}^p\tau^{\leqslant 0}\,{}^p\tau^{\geqslant 0}(F^\cdot[n])$. For $F^\cdot \in D_c^b(X)$ its image ${}^pH^n(F^\cdot)$ in $\mathrm{Perv}(\mathbb{C}_X)$ is called the nth *perverse cohomology* (or the nth perverse part) of $F^\cdot$.

Note that for any perverse sheaf $F^\cdot \in \mathrm{Perv}(\mathbb{C}_X)$ on X we have $H^i(F^\cdot) = 0$ for $i \notin [-d_X, 0]$. By Proposition 8.1.11 a distinguished triangle

$$ F^\cdot \longrightarrow G^\cdot \longrightarrow H^\cdot \xrightarrow{+1} $$

in $D_c^b(X)$ gives rise to a long exact sequence

$$ \cdots \to {}^pH^{n-1}(H^\cdot) \to {}^pH^n(F^\cdot) \to {}^pH^n(G^\cdot) \to {}^pH^n(H^\cdot) \to {}^pH^{n+1}(F^\cdot) \to \cdots $$

in the abelian category $\mathrm{Perv}(\mathbb{C}_X)$.

By definition, being a perverse sheaf is a local property in the following sense. Let $X = \bigcup_{i\in I} U_i$ be an open covering of X. Then $F^\cdot \in D_c^b(X)$ is a perverse sheaf if and only if $F^\cdot|_{U_i}$ is so for any $i \in I$.

Remark 8.1.29. It can be shown that the correspondence

$$ \{\text{open subsets of } X\} \ni U \longmapsto \mathrm{Perv}(\mathbb{C}_U) $$

defines a *stack*, i.e., a kind of sheaf with values in categories. More precisely, let $X = \bigcup_{i\in I} U_i$ be an open covering and assume that we are given a family $F_i^\cdot \in \mathrm{Perv}(\mathbb{C}_{U_i})$ equipped with isomorphism $F_i|_{U_i\cap U_j} \simeq F_j|_{U_i\cap U_j}$ satisfying obvious compatibility conditions. Then we can glue it uniquely to get $F^\cdot \in \mathrm{Perv}(\mathbb{C}_X)$. This is the reason why we call a complex of sheaves $F^\cdot \in \mathrm{Perv}(\mathbb{C}_X)$ a perverse "sheaf."

Definition 8.1.30. For a perverse sheaf $F^\cdot \in \mathrm{Perv}(\mathbb{C}_X)$, we define the *support* $\mathrm{supp}\, F^\cdot$ of $F^\cdot$ to be the complement of the largest open subset $U \hookrightarrow X$ such that $F^\cdot|_U = 0$.

Proposition 8.1.31. *Assume that X is a smooth algebraic variety or a complex manifold. Then for any local system L on X^{an} we have* $L[d_X] \in \mathrm{Perv}(\mathbb{C}_X)$.

Proof. Assume that X is a complex manifold. By $\omega_X{}^\cdot \simeq \mathbb{C}_X[2d_X]$ we have

$$ \mathbf{D}_X(L[d_X]) = R\mathcal{H}om_{\mathbb{C}_X}(L[d_X], \mathbb{C}_X[2d_X]) \simeq L^*[d_X], $$

where L^* denotes the dual local system $\mathcal{H}om_{\mathbb{C}_X}(L, \mathbb{C}_X)$. Hence the assertion is clear. The proof for the case where X is a smooth algebraic variety is the same. $\square$

Remark 8.1.32. More generally, it is known that if X is a pure-dimensional algebraic variety (resp. analytic space) which is locally a complete intersection, then we have $L[d_X] \in \mathrm{Perv}(\mathbb{C}_X)$ for any local system L on X^{an} (see for instance [Di, Theorem 5.1.20]).

The following proposition is obvious in view of the definition of ${}^pD_c^{\leqslant 0}(X)$ and ${}^pD_c^{\geqslant 0}(X)$.

Proposition 8.1.33. *The Verdier duality functor* $\mathbf{D}_X : D_c^b(X) \to D_c^b(X)^{\mathrm{op}}$ *is* t*-exact and induces an exact functor*

$$\mathbf{D}_X : \mathrm{Perv}(\mathbb{C}_X) \longrightarrow \mathrm{Perv}(\mathbb{C}_X)^{\mathrm{op}}.$$

Proposition 8.1.34. *Let* $i : Z \hookrightarrow X$ *be an embedding of a closed subvariety* Z *of* X*. Then the functor* i_* *sends* $\mathrm{Perv}(\mathbb{C}_Z)$ *to* $\mathrm{Perv}(\mathbb{C}_X)$.

Proof. It is easily seen that i_* sends ${}^pD_c^{\leqslant 0}(Z)$ to ${}^pD_c^{\leqslant 0}(X)$. Since Z is closed, we have $i_* = i_! = \mathbf{D}_X \circ i_* \circ \mathbf{D}_Z$. Hence i_* sends ${}^pD_c^{\geqslant 0}(Z)$ to ${}^pD_c^{\geqslant 0}(X)$. □

By Propositions 8.1.31 and 8.1.34 we obtain the following.

Example 8.1.35.

(i) Let X be a (possibly singular) analytic space and $Y \subset X$ a smooth complex manifold contained in X as a closed subset. Then for any local system L on Y, the complex $i_{Y*}(L[d_Y]) \in D_c^b(X)$ is a perverse sheaf on X.

(ii) Let $X = \mathbb{C}$ and $U = \mathbb{C} \setminus \{0\} \overset{j}{\hookrightarrow} X$. Then for any local system L on U, $Rj_*(L[1])$, $Rj_!(L[1]) = j_!(L[1]) \in D_c^b(X)$ ($j_!$ is an exact functor) are perverse sheaves on $X = \mathbb{C}$.

Definition 8.1.36. Let X, Y be algebraic varieties (or analytic spaces). For a functor $F : D_c^b(X) \longrightarrow D_c^b(Y)$ of triangulated categories we define a functor ${}^pF : \mathrm{Perv}(\mathbb{C}_X) \longrightarrow \mathrm{Perv}(\mathbb{C}_Y)$ to be the composite of the functors

$$\mathrm{Perv}(\mathbb{C}_X) \hookrightarrow D_c^b(X) \xrightarrow{F} D_c^b(Y) \xrightarrow{{}^pH^0} \mathrm{Perv}(\mathbb{C}_Y).$$

Let $f : X \longrightarrow Y$ be a morphism of algebraic varieties or analytic spaces. Then we have functors

$${}^pf^{-1}, {}^pf^! : \mathrm{Perv}(\mathbb{C}_Y) \longrightarrow \mathrm{Perv}(\mathbb{C}_X).$$

Assume that $f : X \longrightarrow Y$ is a proper morphism. Then we also have functors

$${}^pRf_*, {}^pRf_! : \mathrm{Perv}(\mathbb{C}_X) \longrightarrow \mathrm{Perv}(\mathbb{C}_Y).$$

Notation 8.1.37. We sometimes denote the functors ${}^pRf_*$, ${}^pRf_!$ by ${}^pf_*$, ${}^pf_!$, respectively, to simplify our notation.

Lemma 8.1.38.

(i) *For an object $F^\cdot$ of $D_c^b(X)$ we have $F^\cdot = 0$ if and only if ${}^pH^j(F^\cdot) = 0$ for any $j \in \mathbb{Z}$.*

(ii) *A morphism $f : F^\cdot \longrightarrow G^\cdot$ in $D_c^b(X)$ is an isomorphism if and only if the morphism ${}^pH^j(f) : {}^pH^j(F^\cdot) \to {}^pH^j(G^\cdot)$ is an isomorphism for any $j \in \mathbb{Z}$.*

Proof. (i) Assume that ${}^pH^j(F^\cdot) = 0$ for any $j \in \mathbb{Z}$. Since $F^\cdot$ is represented by a bounded complex of sheaves, there exist integers $a \leq b$ such that $F^\cdot \in {}^pD_c^{\leqslant b}(X) \cap {}^pD_c^{\geqslant a}(X)$. In the distinguished triangle

$$ {}^p\tau^{\leqslant b-1}(F^\cdot) \longrightarrow F^\cdot \longrightarrow {}^p\tau^{\geqslant b}(F^\cdot) \xrightarrow{+1} $$

we have ${}^p\tau^{\geqslant b}(F^\cdot) \simeq {}^p\tau^{\geqslant b}\,{}^p\tau^{\leqslant b}(F^\cdot) \simeq {}^pH^b(F^\cdot)[-b] = 0$, and hence we have $F^\cdot \simeq {}^p\tau^{\leqslant b-1}(F^\cdot) \in {}^pD_c^{\leqslant b-1}(X) \cap {}^pD_c^{\geqslant a}(X)$. By repeating this procedure we finally obtain $F^\cdot \in {}^pD_c^{\leqslant a-1}(X) \cap {}^pD_c^{\geqslant a}(X)$, and hence $F^\cdot = 0$.

(ii) Assume that the morphism ${}^pH^j(f) : {}^pH^j(F^\cdot) \to {}^pH^j(G^\cdot)$ is an isomorphism for any $j \in \mathbb{Z}$. Embed the morphism $f : F^\cdot \longrightarrow G^\cdot$ into a distinguished triangle

$$ F^\cdot \xrightarrow{f} G^\cdot \longrightarrow H^\cdot \xrightarrow{+1} . $$

By considering the long exact sequence of perverse cohomologies associated to it we obtain ${}^pH^j(H^\cdot) \simeq 0$ for ${}^\forall j \in \mathbb{Z}$. Hence we have $H^\cdot = 0$ by (i). It follows that $F^\cdot \longrightarrow G^\cdot$ is an isomorphism. □

The following result is an obvious consequence of Proposition 8.1.5 (ii).

Lemma 8.1.39. *Let*

$$ F^\cdot \xrightarrow{f} G^\cdot \xrightarrow{g} H^\cdot \xrightarrow{+1} $$

be a distinguished triangle in $D_c^b(X)$ and assume $F^\cdot \in {}^pD_c^{\leqslant 0}(X)$ and $H^\cdot \in {}^pD_c^{\geqslant 1}(X)$. Then we have $F^\cdot \simeq {}^p\tau^{\leqslant 0}(G^\cdot)$ and $H^\cdot \simeq {}^p\tau^{\geqslant 1}(G^\cdot)$.

Proposition 8.1.40. *Let $f : Y \longrightarrow X$ be a morphism of algebraic varieties or analytic spaces such that* $\dim f^{-1}(x) \leq d$ *for any $x \in X$.*

(i) *For any $F^\cdot \in {}^pD_c^{\leqslant 0}(X)$ we have $f^{-1}(F^\cdot) \in {}^pD_c^{\leqslant d}(Y)$.*

(ii) *For any $F^\cdot \in {}^pD_c^{\geqslant 0}(X)$ we have $f^!(F^\cdot) \in {}^pD_c^{\geqslant -d}(Y)$.*

Proof. (i) For $F^\cdot \in {}^pD_c^{\leqslant 0}(X)$ we have

$$ \begin{aligned} \dim\big(\operatorname{supp} H^j(f^{-1}F^\cdot[d])\big) &= \dim\big(f^{-1}(\operatorname{supp} H^{j+d}(F^\cdot))\big) \\ &\leq \dim\big(\operatorname{supp} H^{j+d}(F^\cdot)\big) + d \leq -j - d + d = -j, \end{aligned} $$

and hence $f^{-1}F^\cdot[d] \in {}^pD_c^{\leqslant 0}(Y)$, Therefore, we have $f^{-1}F^\cdot \in {}^pD_c^{\leqslant d}(Y)$.

(ii) This follows from (i) in view of $f^! = \mathbf{D}_Y \circ f^{-1} \circ \mathbf{D}_X$ and Proposition 8.1.33. □

Corollary 8.1.41. *Let Z be a locally closed subvariety of X and let $i : Z \to X$ be the embedding.*

(i) *The functor $i^{-1} : D_c^b(X) \longrightarrow D_c^b(Z)$ is right t-exact with respect to the perverse t-structures.*
(ii) *The functor $i^! : D_c^b(X) \longrightarrow D_c^b(Z)$ is left t-exact with respect to the perverse t-structures.*

The following propositions are immediate consequences of Proposition 8.1.40 and Corollary 8.1.41 in view of Lemma 8.1.16.

Proposition 8.1.42. *Let $f : X \to Y$ be as in Proposition* 8.1.40.

(i) *For any $G^\cdot \in {}^pD_c^{\geqslant 0}(Y)$ such that $Rf_*(G^\cdot) \in D_c^b(X)$ we have $Rf_*(G^\cdot) \in {}^pD_c^{\geqslant -d}(X)$.*
(ii) *For any $G^\cdot \in {}^pD_c^{\leqslant 0}(Y)$ such that $Rf_!(G^\cdot) \in D_c^b(X)$ we have $Rf_!(G^\cdot) \in {}^pD_c^{\leqslant d}(X)$.*

Proposition 8.1.43. *Let $i : Z \to X$ be as in Corollary* 8.1.41.

(i) *For any $G^\cdot \in {}^pD_c^{\geqslant 0}(Z)$ such that $Ri_*(G^\cdot) \in D_c^b(X)$ we have $Ri_*(G^\cdot) \in {}^pD_c^{\geqslant 0}(X)$.*
(ii) *For any $G^\cdot \in {}^pD_c^{\leqslant 0}(Z)$ such that $i_!(G^\cdot) \in D_c^b(X)$ we have $i_!(G^\cdot) \in {}^pD_c^{\leqslant 0}(X)$.*

Corollary 8.1.44.

(i) *Let $j : U \hookrightarrow X$ be an inclusion of an open subset U of X. Then $j^{-1} = j^!$ is t-exact with respect to the perverse t-structures and induces an exact functor*

$$ {}^pj^{-1} = {}^pj^! : \mathrm{Perv}(\mathbb{C}_X) \longrightarrow \mathrm{Perv}(\mathbb{C}_U). $$

(ii) *Let $i : Z \hookrightarrow X$ be an inclusion of a closed subvariety Z. Then $i_* = i_!$ is t-exact with respect to the perverse t-structures and induces an exact functor*

$$ {}^pi_* = {}^pi_! : \mathrm{Perv}(\mathbb{C}_Z) \longrightarrow \mathrm{Perv}(\mathbb{C}_X). $$

Moreover, if we denote by $\mathrm{Perv}_Z(\mathbb{C}_X)$ *the category of perverse sheaves on X whose supports are contained in Z, then the functor ${}^pi^{-1} = {}^pi^! :$* $\mathrm{Perv}_Z(\mathbb{C}_X) \longrightarrow \mathrm{Perv}(\mathbb{C}_Z)$ *is well defined and induces an equivalence*

$$ \mathrm{Perv}_Z(\mathbb{C}_X) \underset{{}^pi_*={}^pi_!}{\overset{{}^pi^{-1}={}^pi^!}{\rightleftarrows}} \mathrm{Perv}(\mathbb{C}_Z) $$

of categories. The functor ${}^pi^{-1}$ is the quasi-inverse of ${}^pi_*$.

Lemma 8.1.45. *For an exact sequence* $0 \longrightarrow F^\cdot \longrightarrow G^\cdot \longrightarrow H^\cdot \longrightarrow 0$ *of perverse sheaves on X, we have* $\mathrm{supp}\, G^\cdot = \mathrm{supp}\, F^\cdot \cup \mathrm{supp}\, H^\cdot$.

Proof. We first show $\operatorname{supp} F^{\cdot} \subset \operatorname{supp} G^{\cdot}$. Set $U = X \setminus \operatorname{supp} G^{\cdot}$. It is enough to show $F^{\cdot}|_U = 0$. By Corollary 8.1.44 (i) we have an exact sequence

$$0 \longrightarrow F^{\cdot}|_U \xrightarrow{\phi} G^{\cdot}|_U$$

in the abelian category $\operatorname{Perv}(\mathbb{C}_U)$. Hence we obtain $F^{\cdot}|_U = 0$ from $G^{\cdot}|_U = 0$. The inclusion $\operatorname{supp} H^{\cdot} \subset \operatorname{supp} G^{\cdot}$ can be proved similarly. Let us show $\operatorname{supp} G^{\cdot} \subset \operatorname{supp} F^{\cdot} \cup \operatorname{supp} H^{\cdot}$. It is sufficient to show that if we have $F^{\cdot}|_U = H^{\cdot}|_U = 0$ for an open subset U of X, then $G^{\cdot}|_U = 0$. This follows easily from Theorem 8.1.9 (ii). □

Remark 8.1.46. The formal properties of perverse sheaves on X that we listed above carry rich information on the singularities of the base space X. Indeed, using perverse sheaves, we can easily recover and even extend various classical results in singularity theory. For example, see [Di], [Mas1], [Mas2], [NT], [Schu], [Tk2]. Note also that Kashiwara [Kas19] recently introduced an interesting t-structure on the category $D^b_{rh}(D_X)$ whose heart corresponds to the category of constructible sheaves on X (here X is a smooth algebraic variety) through the Riemann–Hilbert correspondence.

8.2 Intersection cohomology theory

8.2.1 Introduction

Let X be an irreducible projective algebraic variety (or an irreducible compact analytic space) of dimension d_X. If X is non-singular, then we have the Poincaré duality $H^i(X, \mathbb{C}_X) \simeq \left[H^{2d_X-i}(X, \mathbb{C}_X)\right]^*$ for any $0 \leq i \leq 2d_X$. However, for a general (singular) variety X, we cannot expect such a nice symmetry in its usual cohomology groups. The intersection cohomology theory of Goresky–MacPherson [GM1] is a new theory which enables us to overcome this problem. The basic idea in their theory is to replace the constant sheaf $\mathbb{C}_X$ with a new complex $\mathrm{IC}_X{}^{\cdot}[-d_X] \in D^b_c(X)$ of sheaves on X and introduce the *intersection cohomology groups*

$$IH^i(X) = H^i\left(X, \mathrm{IC}_X{}^{\cdot}[-d_X]\right) \quad (0 \leq i \leq 2d_X)$$

by taking the hypercohomology groups of $\mathrm{IC}_X{}^{\cdot}$. Then we obtain a generalized Poincaré duality

$$IH^i(X) = \left[IH^{2d_X-i}(X)\right]^* \quad (0 \leq i \leq 2d_X)$$

for any projective variety X. Moreover, it turns out that intersection cohomology groups admit the Hodge decomposition. Indeed, Morihiko Saito constructed his theory of Hodge modules to obtain this remarkable generalization of the Hodge–Kodaira theory to singular varieties (see Section 8.3). To define the *intersection cohomology complex* $\mathrm{IC}_X{}^{\cdot} \in D^b_c(X)$ of X, first we take a constant perverse sheaf $\mathbb{C}_U[d_X]$ on a Zariski open dense subset U of the smooth part X_{reg} of X. Then $\mathrm{IC}_X{}^{\cdot}$ is a "minimal" extension of $\mathbb{C}_U[d_X] \in \operatorname{Perv}(\mathbb{C}_U)$ to a perverse sheaf on the whole X. Let us briefly explain the construction of $\mathrm{IC}_X{}^{\cdot}$. To begin with, take a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X and set $X_k = \bigsqcup_{\dim X_\alpha \leq k} X_\alpha \subset X$ $(k = -1, 0, 1, 2, \dots)$. Then we get a filtration

$$X = X_{d_X} \supset X_{d_X - 1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

of X by closed analytic subsets X_k of X such that $X_k \setminus X_{k-1}$ is a smooth k-dimensional complex manifold for any $k \in \mathbb{Z}$. Set $U_k = X \setminus X_{k-1}$ and $j_k : U_k \hookrightarrow U_{k-1}$. Then we have

$$\emptyset = U_{d_X+1} \xhookrightarrow{j_{d_X+1}} U_{d_X} \xhookrightarrow{j_{d_X}} U_{d_X-1} \hookrightarrow \cdots \xhookrightarrow{j_2} U_1 \xhookrightarrow{j_1} U_0 = X$$

and the perverse sheaf $\mathrm{IC}_X{}^{\cdot}$ is, in this case, isomorphic to the complex

$$\left(\tau^{\leqslant -1} Rj_{1*}\right) \circ \left(\tau^{\leqslant -2} Rj_{2*}\right) \circ \cdots \circ \left(\tau^{\leqslant -d_X} Rj_{d_X *}\right)(\mathbb{C}_U[d_X])$$

for $U = U_{d_X} \subset X$. We can prove that the Verdier dual of $\mathrm{IC}_X{}^{\cdot}$ is isomorphic to $\mathrm{IC}_X{}^{\cdot}$ itself. Namely, we have $\mathbf{D}_X(\mathrm{IC}_X{}^{\cdot}) \simeq \mathrm{IC}_X{}^{\cdot}$. This self-duality of $\mathrm{IC}_X{}^{\cdot}$ is the main reason why the intersection cohomology groups of X satisfy the generalized Poincaré duality. In order to see that this construction of $\mathrm{IC}_X{}^{\cdot}$ is canonical, it is, in fact, necessary to check that it does not depend on the choice of a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$. For this purpose, in [GM1] Goresky and MacPherson introduced a "maximal" filtration

$$X \supset \bar{X}_{d_X} \supset \bar{X}_{d_X - 1} \supset \cdots \supset \bar{X}_0 \supset \bar{X}_{-1} = \emptyset$$

of X (that is, any Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X is finer than the stratification $X = \bigsqcup_{k \in \mathbb{Z}} (\bar{X}_k \setminus \bar{X}_{k-1})$). But here, we define the intersection cohomology complex $\mathrm{IC}_X{}^{\cdot}$ by using the perverse t-structures and prove that it is isomorphic to the complex

$$\left(\tau^{\leqslant -1} Rj_{1*}\right) \circ \cdots \circ \left(\tau^{\leqslant -d_X} Rj_{d_X *}\right)(\mathbb{C}_U[d_X])$$

whenever we fix a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X. More generally, for any pair (X, U) of an irreducible complex analytic space X and its Zariski open dense subset U $(j : U \hookrightarrow X)$, we can introduce a functor

$${}^p j_{!*} : \mathrm{Perv}(\mathbb{C}_U) \longrightarrow \mathrm{Perv}(\mathbb{C}_X)$$

such that ${}^p j_{!*}(\mathbb{C}_U[d_X]) \simeq \mathrm{IC}_X{}^{\cdot}$ in the above case. Such an extension of a perverse sheaf on U to the one on X will be called a *minimal extension* or a Deligne–Goresky–MacPherson extension (D-G-M extension for short). In addition to the two fundamental papers [BBD] and [GM1] on this subject, we are also indebted to [Bor2], [CG], [Di], [G1], [Ki], [Na2], [Schu].

8.2.2 Minimal extensions of perverse sheaves

Let X be an irreducible algebraic variety or an irreducible analytic space and U a Zariski open dense subset of X. In what follows, we set $Z = X \setminus U$ and denote by $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$ the embeddings.

We say that a stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X is *compatible* with $F^\cdot \in D_c^b(U)$ if $U = \bigsqcup_{\alpha \in B} X_\alpha$ for some $B \subset A$, and both $F^\cdot|_{X_\alpha}$, $\mathbf{D}_U F^\cdot|_{X_\alpha}$ have locally constant cohomology sheaves for any $\alpha \in B$. Such a stratification always exists if X is an algebraic variety; however, it does not always exist in the case where X is an analytic space.

Example 8.2.1. Regard $X = \mathbb{C}$ and $U = \mathbb{C} \setminus \{0\}$ as an analytic space and its Zariski open subset, respectively. Set $Y_n = \{1/n\}$ for each positive integer n. We further set $V = U \setminus (\bigcup_{n=1}^{\infty} Y_n)$. Then the stratification $U = V \sqcup (\bigsqcup_{n=1}^{\infty} Y_n)$ of U (and any of its refinement) cannot be extended to that of X. In particular, if F is a constructible sheaf on U whose support is exactly $U \setminus V$, then there exists no stratification of X compatible with F.

Assume that a stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X is compatible with $F^\cdot \in D_c^b(U)$. By replacing it with its refinement we may assume that the stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ satisfies the Whitney condition (see Definition E.3.7). In this situation the cohomology sheaves of $Rj_* F^\cdot|_{X_\alpha}$ and $j_! F^\cdot|_{X_\alpha}$ are locally constant for any $\alpha \in A$. In particular, we have $Rj_* F^\cdot, j_! F^\cdot \in D_c^b(X)$.

Now let $F^\cdot$ be a perverse sheaf on U and assume that there exists a Whitney stratification of X compatible with $F^\cdot$. We shall consider the problem of extending $F^\cdot$ to a perverse sheaf on X.

By taking the 0th perverse cohomology ${}^pH^0$ of the canonical morphism $j_! F^\cdot \longrightarrow Rj_* F^\cdot$ we get a morphism ${}^pj_! F^\cdot \longrightarrow {}^pj_* F^\cdot$ (see Notation 8.1.37) in $\mathrm{Perv}(\mathbb{C}_X)$.

Definition 8.2.2. We denote by ${}^pj_{!*} F^\cdot$ the image of the canonical morphism

$$ {}^pj_! F^\cdot \longrightarrow {}^pj_* F^\cdot $$

in $\mathrm{Perv}(\mathbb{C}_X)$, and call it the *minimal extension* of $F^\cdot \in \mathrm{Perv}(\mathbb{C}_U)$.

In other words, the morphism ${}^pj_! F^\cdot \longrightarrow {}^pj_* F^\cdot$ factorizes as ${}^pj_! F^\cdot \twoheadrightarrow {}^pj_{!*} F^\cdot \hookrightarrow {}^pj_* F^\cdot$ ($\twoheadrightarrow$ is an epimorphism and $\hookrightarrow$ is a monomorphism) in $\mathrm{Perv}(\mathbb{C}_X)$. Moreover, by definition, in the algebraic case for any morphism $F^\cdot \to G^\cdot$ in $\mathrm{Perv}(\mathbb{C}_U)$ we obtain a canonical morphism ${}^pj_{!*} F^\cdot \to {}^pj_{!*} G^\cdot$ in $\mathrm{Perv}(\mathbb{C}_X)$.

Proposition 8.2.3. *For $F^\cdot \in \mathrm{Perv}(\mathbb{C}_U)$, we have $\mathbf{D}_X({}^pj_{!*} F^\cdot) \simeq {}^pj_{!*}(\mathbf{D}_U F^\cdot)$.*

Proof. By applying $\mathbf{D}_X$ to the sequence

$$ {}^pj_! F^\cdot \twoheadrightarrow {}^pj_{!*} F^\cdot \hookrightarrow {}^pj_* F^\cdot $$

of morphisms in $\mathrm{Perv}(\mathbb{C}_X)$ we obtain a sequence

$$ \mathbf{D}_X({}^pj_* F^\cdot) \twoheadrightarrow \mathbf{D}_X({}^pj_{!*} F^\cdot) \hookrightarrow \mathbf{D}_X({}^pj_! F^\cdot) $$

of morphisms in $\mathrm{Perv}(\mathbb{C}_X)$ by Proposition 8.1.33. Furthermore, it follows also from Proposition 8.1.33 that

$$\begin{cases} \mathbf{D}_X({}^p j_* F^\cdot) \simeq {}^p H^0 \mathbf{D}_X(Rj_* F^\cdot) \simeq {}^p j_!(\mathbf{D}_U F^\cdot) \\ \mathbf{D}_X({}^p j_! F^\cdot) \simeq {}^p H^0 \mathbf{D}_X(Rj_! F^\cdot) \simeq {}^p j_*(\mathbf{D}_U F^\cdot). \end{cases}$$

Therefore, we obtain a sequence

$${}^p j_!(\mathbf{D}_U F^\cdot) \twoheadrightarrow \mathbf{D}_X({}^p j_{!*} F^\cdot) \hookrightarrow {}^p j_*(\mathbf{D}_U F^\cdot)$$

of morphisms in $\mathrm{Perv}(\mathbb{C}_X)$, which shows that we should have $\mathbf{D}_X({}^p j_{!*} F^\cdot) \simeq {}^p j_{!*}(\mathbf{D}_U F^\cdot)$. □

Lemma 8.2.4. *Let U' be a Zariski open subset of X containing U such that we have $U' = \sqcup_{\alpha \in B'} X_\alpha$ for some $B' \subset A$. We denote by $j_1 : U \to U'$ and $j_2 : U' \to X$ the embeddings.*

(i) *We have* ${}^p j_* F^\cdot \simeq {}^p j_{2*} {}^p j_{1*} F^\cdot$ *and* ${}^p j_! F^\cdot \simeq {}^p j_{2!} {}^p j_{1!} F^\cdot$.
(ii) ${}^p j_{!*} F^\cdot \simeq {}^p j_{2!*} {}^p j_{1!*} F^\cdot$.

Proof. (i) Since the functors Rj_{1*} and Rj_{2*} are left t-exact, we have

$${}^p j_* F^\cdot = {}^p H^0(Rj_{2*} Rj_{1*} F^\cdot) \simeq {}^p H^0\big(Rj_{2*} {}^p H^0(Rj_{1*} F^\cdot)\big) = {}^p j_{2*} {}^p j_{1*} F^\cdot$$

by Proposition 8.1.15 (i). The proof of the assertion ${}^p j_! F^\cdot \simeq {}^p j_{2!} {}^p j_{1!} F^\cdot$ is similar.

(ii) Recall that ${}^p j_{1!*} F^\cdot$ is a subobject of ${}^p j_{1*} F^\cdot$ in $\mathrm{Perv}(\mathbb{C}_{U'})$ such that the morphism ${}^p j_{1!} F^\cdot \longrightarrow {}^p j_{1*} F^\cdot$ factorizes as

$${}^p j_{1!} F^\cdot \twoheadrightarrow {}^p j_{1!*} F^\cdot \hookrightarrow {}^p j_{1*} F^\cdot.$$

By using the right t-exactness of ${}^p j_{2!}$ (Propositions 8.1.43 (ii) and 8.1.15 (ii)) and the left t-exactness of ${}^p j_{2*}$ (Proposition 8.1.43 (i) and Proposition 8.1.15 (ii)) we obtain

$$\begin{aligned} {}^p j_! F^\cdot = {}^p j_{2!} \circ {}^p j_{1!} F^\cdot &\twoheadrightarrow {}^p j_{2!} \circ {}^p j_{1!*} F^\cdot \twoheadrightarrow {}^p j_{2!*} \circ {}^p j_{1!*} F^\cdot \\ &\hookrightarrow {}^p j_{2*} \circ {}^p j_{1!*} F^\cdot \hookrightarrow {}^p j_{2*} \circ {}^p j_{1*} F^\cdot = {}^p j_* F^\cdot. \end{aligned}$$

It follows that we have ${}^p j_{2!*} \circ {}^p j_{1!*} F^\cdot \simeq {}^p j_{!*} F^\cdot$. □

Proposition 8.2.5. *The minimal extension $G^\cdot = {}^p j_{!*} F^\cdot$ of $F^\cdot \in$* Perv$(\mathbb{C}_U)$ *is characterized as the unique perverse sheaf on X satisfying the conditions*

(i) $G^\cdot|_U \simeq F^\cdot$,
(ii) $i^{-1} G^\cdot \in {}^p D_c^{\leqslant -1}(Z)$,
(iii) $i^! G^\cdot \in {}^p D_c^{\geqslant 1}(Z)$.

Proof. We first show that the minimal extension $G^\cdot = {}^p j_{!*} F^\cdot$ satisfies the conditions (i), (ii), (iii). Since the functor $j^{-1} = j^!$ is t-exact by Corollary 8.1.44 (i), we have

$$\begin{aligned} {}^p j_{!*} F^\cdot|_U &= j^{-1}\, \mathrm{Im}\big[{}^p j_! F^\cdot \longrightarrow {}^p j_* F^\cdot\big] \\ &= \mathrm{Im}\big[j^{-1} {}^p j_! F^\cdot \longrightarrow j^{-1} {}^p j_* F^\cdot\big] \\ &= \mathrm{Im}\big[{}^p H^0(j^{-1} Rj_! F^\cdot) \longrightarrow {}^p H^0(j^{-1} Rj_* F^\cdot)\big] \end{aligned}$$

$$= \mathrm{Im}\big[F^{\cdot} \longrightarrow F^{\cdot}\big]$$
$$= F^{\cdot}.$$

Hence the condition (i) is satisfied. By the distinguished triangle

$$j_! j^{-1} G^{\cdot} \longrightarrow G^{\cdot} \longrightarrow i_* i^{-1} G^{\cdot} \xrightarrow{+1}$$

in $D^b_c(X)$ we obtain an exact sequence

$${}^pH^0(j_! j^{-1} G^{\cdot}) \longrightarrow {}^pH^0(G^{\cdot}) \longrightarrow {}^pH^0(i_* i^{-1} G^{\cdot}) \longrightarrow {}^pH^1(j_! j^{-1} G^{\cdot}).$$

By the definition of $G^{\cdot}$ we have ${}^pH^0(G^{\cdot}) = {}^pj_{!*}F^{\cdot}$. By (i) we have $j^{-1}G^{\cdot} = F^{\cdot}$, and hence ${}^pH^0(j_! j^{-1}G^{\cdot}) = {}^pj_! F^{\cdot}$. Moreover, we have ${}^pH^1(j_! j^{-1}G^{\cdot}) = 0$ by Proposition 8.1.43 (ii). Finally, the canonical morphism ${}^pj_! F^{\cdot} \longrightarrow {}^pj_{!*}F^{\cdot}$ is an epimorphism by the definition of ${}^pj_{!*}$. Therefore, we obtain ${}^pH^0(i_* i^{-1}G^{\cdot}) = 0$ by the above exact sequence. Since i_* is t-exact, we have ${}^pH^0(i^{-1}G^{\cdot}) = 0$. Since i^{-1} is right t-exact, we have $i^{-1}G^{\cdot} \in {}^pD_c^{\leq 0}(Z)$. It follows that $i^{-1}G^{\cdot} \in {}^pD_c^{\leqslant -1}(Z)$. Hence the condition (ii) is satisfied. The condition (iii) can be checked similarly to (ii) by using the distinguished triangle

$$i_* i^! G^{\cdot} \longrightarrow G^{\cdot} \longrightarrow Rj_* j^{-1} G^{\cdot} \xrightarrow{+1} .$$

Let us show that $G^{\cdot} \in \mathrm{Perv}(\mathbb{C}_X)$ satisfying the conditions (i), (ii), (iii) is canonically isomorphic to ${}^pj_{!*}F^{\cdot}$. Since $j^{-1}(= j^!)$ is left adjoint to Rj_* and right adjoint to $j_!$, we obtain canonical morphisms $j_! F^{\cdot} \longrightarrow G^{\cdot} \longrightarrow Rj_* F^{\cdot}$ in $D^b_c(X)$. Hence we obtain canonical morphisms ${}^pj_! F^{\cdot} \longrightarrow G^{\cdot} \longrightarrow {}^pj_* F^{\cdot}$ in $\mathrm{Perv}(\mathbb{C}_X)$. It is sufficient to show that ${}^pj_! F^{\cdot} \longrightarrow G^{\cdot}$ is an epimorphism and $G^{\cdot} \longrightarrow {}^pj_* F^{\cdot}$ is a monomorphism in $\mathrm{Perv}(\mathbb{C}_X)$. We only show that ${}^pj_! F^{\cdot} \longrightarrow G^{\cdot}$ is an epimorphism (the proof of the remaining assertion is similar). Since the cokernel of ${}^pj_! F^{\cdot} \longrightarrow G^{\cdot}$ is supported by Z, there exists an exact sequence

$${}^pj_! F^{\cdot} \longrightarrow G^{\cdot} \longrightarrow i_* H^{\cdot} \longrightarrow 0$$

for some $H^{\cdot} \in \mathrm{Perv}(\mathbb{C}_Z)$ (Corollary 8.1.44 (ii)). Since i^{-1} is right t-exact, we have an exact sequence ${}^pi^{-1}G^{\cdot} \longrightarrow {}^pi^{-1} i_* H^{\cdot} \longrightarrow 0$. By Corollary 8.1.44 (ii) we have ${}^pi^{-1} i_* H^{\cdot} = i^{-1} i_* H^{\cdot} = H^{\cdot}$. Moreover, by our assumption (ii) we have ${}^pi^{-1}G^{\cdot} = 0$. It follows that $H^{\cdot} = 0$, and hence ${}^pj_! F^{\cdot} \longrightarrow G^{\cdot}$ is an epimorphism. □

Corollary 8.2.6. *Assume that X is smooth. Then for any local system $L \in \mathrm{Loc}(X)$ on X we have $L[d_X] \simeq {}^pj_{!*}(L|_U[d_X])$.*

Proof. By Proposition 8.2.5 it is sufficient to show $i^{-1}L[d_X] \in {}^pD_c^{\leqslant -1}(Z)$ and $i^! L[d_X] \in {}^pD_c^{\geqslant 1}(Z)$. By $d_Z < d_X$ we easily see that $i^{-1}L[d_X] \in {}^pD_c^{\leqslant -1}(Z)$. Furthermore, we have

$$\begin{aligned} i^! L[d_X] &\simeq i^! \mathbf{D}_X \mathbf{D}_X (L[d_X]) \\ &\simeq \mathbf{D}_X i^{-1}(L^*[d_X]) \in {}^pD_c^{\geqslant 1}(X), \end{aligned}$$

where L^* is the dual local system of L. □

Proposition 8.2.7. *Let* $F^{\cdot} \in \mathrm{Perv}(\mathbb{C}_U)$ *be as above. Then*

(i) ${}^p j_* F^{\cdot}$ *has no non-trivial subobject whose support is contained in* Z.
(ii) ${}^p j_! F^{\cdot}$ *has no non-trivial quotient object whose support is contained in* Z.

Proof. (i) Let $G^{\cdot} \subset {}^p j_* F^{\cdot}$ be a subobject of ${}^p j_* F^{\cdot}$ such that $\mathrm{supp}\, G^{\cdot} \subset Z$. Then by Corollary 8.1.41 $i^! G^{\cdot} \simeq i^{-1} G^{\cdot}$ is a perverse sheaf on Z and we obtain ${}^p i^! G^{\cdot} \simeq i^! G^{\cdot}$. Since we have $G^{\cdot} \simeq i_* i^! G^{\cdot}$, it suffices to show that ${}^p i^! G^{\cdot} \simeq 0$. Now let us apply the left t-exact functor ${}^p i^!$ to the exact sequence $0 \to G^{\cdot} \to {}^p j_* F^{\cdot}$. Then we obtain an exact sequence $0 \to {}^p i^! G^{\cdot} \to {}^p i^! {}^p j_* F^{\cdot}$. By Proposition 8.1.15 (i) and $i^! R j_* F^{\cdot} \simeq 0$ we obtain ${}^p i^! {}^p j_* F^{\cdot} \simeq {}^p H^0(i^! R j_* F^{\cdot}) \simeq 0$. Hence we get ${}^p i^! G^{\cdot} \simeq 0$. The proof of (ii) is similar. □

Corollary 8.2.8. *The minimal extension* ${}^p j_{!*} F^{\cdot}$ *has neither non-trivial subobject nor non-trivial quotient object whose support is contained in* Z.

Proof. By the definition of the minimal extension ${}^p j_{!*} F^{\cdot}$ the result follows immediately from Proposition 8.2.7. □

In the algebraic case we also have the following.

Corollary 8.2.9.
(i) *Let* $0 \to F^{\cdot} \to G^{\cdot}$ *be an exact sequence in* $\mathrm{Perv}(\mathbb{C}_U)$. *Then the associated sequence* $0 \to {}^p j_{!*} F^{\cdot} \to {}^p j_{!*} G^{\cdot}$ *in* $\mathrm{Perv}(\mathbb{C}_X)$ *is also exact.*
(ii) *Let* $F^{\cdot} \to G^{\cdot} \to 0$ *be an exact sequence in* $\mathrm{Perv}(\mathbb{C}_U)$. *Then the associated sequence* ${}^p j_{!*} F^{\cdot} \to {}^p j_{!*} G^{\cdot} \to 0$ *in* $\mathrm{Perv}(\mathbb{C}_X)$ *is also exact.*

Proof. (i) Since the kernel $K^{\cdot}$ of the morphism ${}^p j_{!*} F^{\cdot} \to {}^p j_{!*} G^{\cdot}$ is a subobject of ${}^p j_{!*} F^{\cdot}$ whose support is contained in Z, we obtain $K^{\cdot} \simeq 0$ by Corollary 8.2.8. The proof of (ii) is similar. □

Corollary 8.2.10. *Assume that* $F^{\cdot}$ *is a simple object in* $\mathrm{Perv}(\mathbb{C}_U)$. *Then the minimal extension* ${}^p j_{!*} F^{\cdot}$ *is also a simple object in* $\mathrm{Perv}(\mathbb{C}_X)$.

Proof. Let $G^{\cdot} \subset {}^p j_{!*} F^{\cdot}$ be a subobject of ${}^p j_{!*} F^{\cdot}$ in $\mathrm{Perv}(\mathbb{C}_X)$ and consider the exact sequence $0 \to G^{\cdot} \to {}^p j_{!*} F^{\cdot} \to H^{\cdot} \to 0$ associated to it. If we apply the t-exact functor $j^! = j^{-1}$ to it, then we obtain an exact sequence $0 \to j^{-1} G^{\cdot} \to F^{\cdot} \to j^{-1} H^{\cdot} \to 0$. Since $F^{\cdot}$ is simple, $j^{-1} G^{\cdot}$ or $j^{-1} H^{\cdot}$ is zero. In other words, $G^{\cdot}$ or $H^{\cdot}$ is supported by Z. It follows from Corollary 8.2.8 that $G^{\cdot}$ or $H^{\cdot}$ is zero. □

Now we focus our attention on the case where U is smooth and $F^{\cdot} = L[d_X]$ for a local system $L \in \mathrm{Loc}(U)$ on U. Note that we assume that X is irreducible as before. We can take a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X such that U is a union of strata in it. In view of Lemma 8.2.4 and Corollary 8.2.6 we may assume that U is the unique open stratum in considering the minimal extension of $L[d_X]$. In other words we fix a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X and consider the minimal extension ${}^p j_{!*}(L[d_X])$ of $L \in \mathrm{Loc}(U)$, where U is the open stratum and $j : U \to X$ is the embedding.

Set $X_k = \bigsqcup_{\dim X_\alpha \leq k} X_\alpha$ for each $k \in \mathbb{Z}$. Then we get a filtration

$$X = X_{d_X} \supset X_{d_X - 1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

of X by (closed) analytic subsets. Set $U_k = X \setminus X_{k-1} = \bigsqcup_{\dim X_\alpha \geq k} X_\alpha$ and consider the sequence

$$U = U_{d_X} \xhookrightarrow{j_{d_X}} U_{d_X - 1} \xhookrightarrow{j_{d_X - 1}} \cdots \xhookrightarrow{j_2} U_1 \xhookrightarrow{j_1} U_0 = X$$

of inclusions of open subsets in X.

Proposition 8.2.11. *In the situation as above, we have an isomorphism*

$${}^p j_{!*}(L[d_X]) \simeq \left(\tau^{\leqslant -1} R j_{1*}\right) \circ \cdots \circ \left(\tau^{\leqslant -d_X} R j_{d_X *}\right)(L[d_X]).$$

Proof. In view of Lemma 8.2.4 it is sufficient to show that for any perverse sheaf $F^\cdot$ on U_k whose restriction to each stratum $X_\alpha \subset U_k$ has locally constant cohomology sheaves we have

$${}^p j_{k!*} F^\cdot \simeq \tau^{\leqslant -k} R j_{k*}(F^\cdot).$$

We will show that the conditions (i), (ii), (iii) of Proposition 8.2.5 is satisfied for $G^\cdot = \tau^{\leqslant -k} R j_{k*}(F^\cdot) \in D_c^b(U_{k-1})$. Since U_k consists of strata with dimension $\geq k$, we have $H^r(F^\cdot) = 0$ for $r > -k$ by Proposition 8.1.22. It follows that $\left[\tau^{\leqslant -k} R j_{k*}(F^\cdot)\right]\big|_{U_k} \simeq F^\cdot$, and hence the condition (i) is satisfied. Set $Z = U_{k-1} \setminus U_k = \bigsqcup_{\dim X_\alpha = k-1} X_\alpha$ and let $i : Z \hookrightarrow U_k$ be the embedding. Then $i^{-1} G^\cdot$ has locally constant cohomology sheaves on each $(k-1)$-dimensional stratum $X_\alpha \subset Z$ and we have $H^r(i^{-1} G^\cdot) = 0$ for $r > -k$. It follows that $i^{-1} G^\cdot \in {}^p D_c^{\leqslant -1}(Z)$ by Proposition 8.1.22 (i), and hence the condition (ii) is satisfied. Consider the distinguished triangle

$$G^\cdot = \tau^{\leqslant -k} R j_{k*} F^\cdot \longrightarrow R j_{k*} F^\cdot \longrightarrow \tau^{\geqslant -k+1} R j_{k*} F^\cdot \xrightarrow{+1}.$$

Applying the functor $i^!$ to it, we get $i^! G^\cdot \simeq i^!(\tau^{\geqslant -k+1} R j_{k*} F^\cdot)[-1]$ because $i^! R j_{k*} F^\cdot \simeq 0$. Hence we have $H^r(i^! G^\cdot) = 0$ for $r \leq -k+1$. Since $i^! G^\cdot$ has locally constant cohomology sheaves on each $(k-1)$-dimensional stratum $X_\alpha \subset Z$, we have $i^! G^\cdot \in {}^p D_c^{\geqslant 1}(Z)$ by Proposition 8.1.22 (ii). The condition (iii) is also satisfied. □

Corollary 8.2.12. *There exists a canonical morphism* $(j_* L)[d_X] \to {}^p j_{!*}(L[d_X])$ *in* $D_c^b(X)$.

Proof. The result follows from the isomorphism

$$\tau^{\leqslant -d_X}\, {}^p j_{!*}(L[d_X]) \simeq (j_{1*} \circ j_{2*} \cdots \circ j_{d_X *})(L)[d_X] \simeq (j_* L)[d_X]$$

obtained by Proposition 8.2.11. □

Definition 8.2.13. For an irreducible algebraic variety (resp. an irreducible analytic space) X we define its *intersection cohomology complex* $\mathrm{IC}_X{}^{\cdot} \in \mathrm{Perv}(\mathbb{C}_X)$ by

$$\mathrm{IC}_X{}^{\cdot} = {}^p j_{!*}(\mathbb{C}_{X_{\mathrm{reg}}^{\mathrm{an}}}[d_X]) \qquad (\text{resp. } \mathrm{IC}_X{}^{\cdot} = {}^p j_{!*}(\mathbb{C}_{X_{\mathrm{reg}}}[d_X]),$$

where X_{reg} denotes the regular part of X and $j : X_{\mathrm{reg}} \hookrightarrow X$ is the embedding.

By Proposition 8.2.3 we have the following.

Theorem 8.2.14. *We have* $\mathbf{D}_X(\mathrm{IC}_X{}^{\cdot}) = \mathrm{IC}_X{}^{\cdot}$.

Proposition 8.2.15. *There exist canonical morphisms*

$$\mathbb{C}_X \longrightarrow \mathrm{IC}_X{}^{\cdot}[-d_X] \longrightarrow \omega_X{}^{\cdot}[-2d_X]$$

in $D_c^b(X)$.

Proof. By Corollary 8.2.12 there exists a natural morphism $\mathbb{C}_X \longrightarrow \mathrm{IC}_X{}^{\cdot}[-d_X]$. By taking the Verdier dual we obtain a morphism $\mathrm{IC}_X{}^{\cdot}[d_X] \longrightarrow \omega_X{}^{\cdot}$. □

Definition 8.2.16. Let X be an irreducible analytic space. For $i \in \mathbb{Z}$ we set

$$\begin{cases} IH^i(X) := H^i(R\Gamma(X, \mathrm{IC}_X{}^{\cdot}[-d_X])), \\ IH_c^i(X) := H^i(R\Gamma_c(X, \mathrm{IC}_X{}^{\cdot}[-d_X])). \end{cases}$$

We call $IH^i(X)$ (resp. $IH_c^i(X)$) the ith *intersection cohomology group* (resp. the ith intersection cohomology group with compact supports) of X.

The following theorem is one of the most important results in intersection cohomology theory.

Theorem 8.2.17. *Let X be an irreducible analytic space of dimension d. Then we have the* **generalized Poincaré duality**

$$IH^i(X) \simeq \left[IH_c^{2d-i}(X)\right]^*$$

for any $0 \leq i \leq 2d$.

Proof. Let $a_X : X \longrightarrow\!\!\!\!\rightarrow \{\mathrm{pt}\}$ be the unique morphism from X to the variety $\{\mathrm{pt}\}$ consisting of a single point. Then we have an isomorphism

$$R\mathcal{H}om_{\mathbb{C}}(Ra_{X!}\mathrm{IC}_X{}^{\cdot}, \mathbb{C}) \simeq Ra_{X*}R\mathcal{H}om_{\mathbb{C}_X}(\mathrm{IC}_X{}^{\cdot}, \omega_X{}^{\cdot})$$

in $D^b(\{\mathrm{pt}\}) \simeq D^b(\mathrm{Mod}(\mathbb{C}))$ by the Poincaré–Verdier duality theorem. Since $R\mathcal{H}om_{\mathbb{C}_X}(\mathrm{IC}_X{}^{\cdot}, \omega_X{}^{\cdot}) = \mathbf{D}_X(\mathrm{IC}_X{}^{\cdot}) \simeq \mathrm{IC}_X{}^{\cdot}$ by Theorem 8.2.14, we get an isomorphism

$$\left[R\Gamma_c(X, \mathrm{IC}_X{}^{\cdot})\right]^* \simeq R\Gamma(X, \mathrm{IC}_X{}^{\cdot})$$

By taking the $(i-d)$th cohomology groups of both sides, we obtain the desired isomorphism. □

In what follows, let us set $H^i(X) = H^i(X, \mathbb{C}_X)$ and $H_c^i(X) = H_c^i(X, \mathbb{C}_X)$. By the Verdier duality theorem we have an isomorphism $H^{-i}(X, \omega_X^{\cdot}) \simeq [H_c^i(X)]^*$ for any $i \in \mathbb{Z}$. This hypercohomology group $H^{-i}(X, \omega_X^{\cdot})$ is called the ith *Borel–Moore homology group* of X and denoted by $H_i^{BM}(X)$. If X is complete (or compact), then $H_i^{BM}(X)$ is isomorphic to the usual homology group $H_i(X) = H_i(X, \mathbb{C})$ of X. By Proposition 8.2.15 we obtain the following.

Proposition 8.2.18. *There exist canonical morphisms*

$$H^i(X) \longrightarrow IH^i(X) \longrightarrow H_{2d_X-i}^{BM}(X).$$

for any $i \in \mathbb{Z}$.

The morphism $H^i(X) \longrightarrow H_{2d_X-i}^{BM}(X)$ can be obtained more directly as follows. Recall that the top-dimensional Borel–Moore homology group $H_{2d_X}^{BM}(X) = [H_c^{2d_X}(X)]^* \simeq \mathbb{C}$ of X contains a canonical generator $[X]$ called the *fundamental class* of X (see, for example, Fulton [F, Section 19.1]). Then by the cup product

$$H^i(X) \times H_c^{2d_X-i}(X) \longrightarrow H_c^{2d_X}(X)$$

and the morphism $H_c^{2d_X}(X) \longrightarrow \mathbb{C}$ obtained by the fundamental class $[X] \in [H_c^{2d_X}(X)]^*$ we obtain a bilinear map

$$H^i(X) \times H_c^{2d_X-i}(X) \longrightarrow \mathbb{C}.$$

This gives a morphism $H^i(X) \longrightarrow H_{2d_X-i}^{BM}(X) = [H_c^{2d_X-i}(X)]^*$.

Proposition 8.2.19. *Let X be a projective variety with isolated singular points. Then we have*

$$IH^i(X) = \begin{cases} H^i(X_{\mathrm{reg}}) & 0 \leq i < d_X \\ \mathrm{Im}[H^{d_X}(X) \longrightarrow H^{d_X}(X_{\mathrm{reg}})] & i = d_X \\ H^i(X) & d_X < i \leq 2d_X. \end{cases}$$

In particular we have $H^i(X_{\mathrm{reg}}) \simeq H^{2d_X-i}(X)$ for any $0 \leq i < d_X$.

Proof. Let $p_1, p_2, \ldots, p_k$ be the singular points of X and $j : X_{\mathrm{reg}} \hookrightarrow X$ the embedding. Then $X = X_{\mathrm{reg}} \sqcup (\sqcup_{i=1}^k \{p_i\})$ is a Whitney stratification of X and we have $\mathrm{IC}_X{}^{\cdot}[-d_X] \simeq \tau^{\leqslant d_X-1}(Rj_*\mathbb{C}_{X_{\mathrm{reg}}})$. Hence we obtain a distinguished triangle

$$\mathrm{IC}_X{}^{\cdot}[-d_X] \longrightarrow Rj_*\mathbb{C}_{X_{\mathrm{reg}}} \longrightarrow \tau^{\geqslant d_X}(Rj_*\mathbb{C}_{X_{\mathrm{reg}}}) \xrightarrow{+1} .$$

Applying the functor $\mathrm{R}\Gamma(X, \bullet)$, we easily see that

$$IH^i(X) = H^i(X, \mathrm{IC}_X{}^{\cdot}[-d_X]) \simeq H^i(X_{\mathrm{reg}})$$

for $0 \leq i < d_X$ and the morphism

$$IH^{d_X}(X) = H^{d_X}(X, \mathrm{IC}_X{}^{\boldsymbol{\cdot}}[-d_X]) \hookrightarrow H^{d_X}(X_{\mathrm{reg}})$$

is injective. Now let us embed the canonical morphism $\mathbb{C}_X \longrightarrow \mathrm{IC}_X{}^{\boldsymbol{\cdot}}[-d_X]$ (Proposition 8.2.15) into a distinguished triangle

$$\mathbb{C}_X \longrightarrow \mathrm{IC}_X{}^{\boldsymbol{\cdot}}[-d_X] \longrightarrow F^{\boldsymbol{\cdot}} \xrightarrow{+1} .$$

Then $F^{\boldsymbol{\cdot}}$ is supported by the zero-dimensional subset $\sqcup_{i=1}^k \{p_i\}$ of X and $H^i(F^{\boldsymbol{\cdot}}) = 0$ for any $i \geq d_X$. Therefore, applying $\mathrm{R}\Gamma(X, \bullet)$ to this distinguished triangle, we obtain

$$H^i(X) \simeq IH^i(X) = H^i(X, \mathrm{IC}_X{}^{\boldsymbol{\cdot}}[-d_X])$$

for $d_X < i \leq 2d_X$ and the morphism

$$H^{d_X}(X) \twoheadrightarrow IH^{d_X}(X) = H^{d_X}(X, \mathrm{IC}_X{}^{\boldsymbol{\cdot}}[-d_X])$$

is surjective. This completes the proof. □

For some classes of varieties with mild singularities, intersection cohomology groups are isomorphic to the usual cohomology groups. For example, let us recall the following classical notion.

Definition 8.2.20. Let X be an algebraic variety or an analytic space. We say that X is *rationally smooth* (or a rational homology "manifold") if for any point $x \in X$ we have

$$H^i_{\{x\}}(X, \mathbb{C}_X) = \begin{cases} \mathbb{C} & i = 2d_X \\ 0 & \text{otherwise.} \end{cases}$$

By definition, rationally smooth varieties are pure-dimensional. Smooth varieties are obviously rationally smooth. Typical examples of rationally smooth varieties with singularities are complex surfaces with Kleinian singularities and moduli spaces of algebraic curves. More generally V-manifolds are rationally smooth. It is known that the usual cohomology groups of a rationally smooth variety satisfy Poincaré duality (see Corollary 8.2.22 below) and the hard Lefschetz theorem.

Proposition 8.2.21. *Let X be a rationally smooth irreducible analytic space. Then the canonical morphisms $\mathbb{C}_X \to \omega_X{}^{\boldsymbol{\cdot}}[-2d_X]$ and $\mathbb{C}_X \to \mathrm{IC}_X{}^{\boldsymbol{\cdot}}[-d_X]$ are isomorphisms.*

Proof. Let $i_{\{x\}} : \{x\} \hookrightarrow X$ be the embedding. Then we have

$$i_{\{x\}}^{-1}\omega_X{}^{\boldsymbol{\cdot}} = i_{\{x\}}^{-1}\mathbf{D}_X(\mathbb{C}_X) \simeq \mathbf{D}_{\{x\}} i_{\{x\}}^!(\mathbb{C}_X) \simeq \mathrm{RHom}_{\mathbb{C}}(\mathrm{R}\Gamma_{\{x\}}(X, \mathbb{C}_X), \mathbb{C})$$

and hence an isomorphism

$$H^{j-2d_X}[i_{\{x\}}^{-1}\omega_X{}^{\boldsymbol{\cdot}}] \simeq [H^{2d_X-j}_{\{x\}}(X, \mathbb{C}_X)]^*$$

for any $j \in \mathbb{Z}$. Then the isomorphism $\mathbb{C}_X \simeq \omega_X{}^{\cdot}[-2d_X]$ follows from these isomorphisms and the rationally smoothness of X. Now let us set $F^{\cdot} := \mathbb{C}_X[d_X] \simeq \omega_X{}^{\cdot}[-d_X] \in D^b_c(X)$. Then the complex $F^{\cdot}$ satisfies the condition $\mathbf{D}_X(F^{\cdot}) \simeq F^{\cdot}$. Therefore, by Proposition 8.2.5 we can easily show that $F^{\cdot}$ is isomorphic to the intersection cohomology complex $\mathrm{IC}_X{}^{\cdot}$. □

Corollary 8.2.22. *Let X be a rationally smooth irreducible analytic space. Then we have an isomorphism $H^i(X) \simeq IH^i(X)$ for any $i \in \mathbb{Z}$.*

Definition 8.2.23. Let X be an irreducible algebraic variety or an irreducible analytic space. Let U be a Zariski open dense subset of X_{reg} and $j : U \to X$ the embedding. For a local system $L \in \mathrm{Loc}(U)$ on U we set

$$\mathrm{IC}_X(L)^{\cdot} = {}^p j_{!*}(L[d_X]) \in \mathrm{Perv}(\mathbb{C}_X)$$

and call it a *twisted intersection cohomology complex* of X. We sometimes denote $\mathrm{IC}_X(L)^{\cdot}[-d_X]$ by ${}^{\pi}L^{\cdot}$.

Let X be an algebraic variety or an analytic space. Consider an irreducible closed subvariety Y of X and a simple object L of $\mathrm{Loc}(Y_0)$ (i.e., an irreducible local system on Y_0), where Y_0 is a smooth Zariski open dense subset of Y. Then the minimal extension $\mathrm{IC}_Y(L)^{\cdot}$ of the locally constant perverse sheaf $L[d_Y]$ to Y can be naturally considered as a perverse sheaf on X by Corollary 8.1.44 (ii). By Corollary 8.2.10 and Lemma 8.2.24 below this perverse sheaf on X is a simple object in $\mathrm{Perv}(\mathbb{C}_X)$. Moreover, it is well known that any simple object in $\mathrm{Perv}(\mathbb{C}_X)$ can be obtained in this way (see [BBD]).

Lemma 8.2.24. *Let L be an irreducible local system on a smooth irreducible variety X. Then the locally constant perverse sheaf $L[d_X]$ on X is a simple object in* $\mathrm{Perv}(\mathbb{C}_X)$.

Proof. Let $0 \to F_1{}^{\cdot} \to L[d_X] \to F_2{}^{\cdot} \to 0$ be an exact sequence in $\mathrm{Perv}(\mathbb{C}_X)$. Choose a Zariski open dense subset U of X on which $F_1{}^{\cdot}$ and $F_2{}^{\cdot}$ have locally constant cohomology sheaves and set $j : U \hookrightarrow X$. Then by Lemma 8.1.23 there exist local systems M_1, M_2 on U such that $F_1{}^{\cdot}|_U \simeq M_1[d_X]$, $F_2{}^{\cdot}|_U \simeq M_2[d_X]$. Hence we obtain an exact sequence $0 \to M_1 \to L|_U \to M_2 \to 0$ of local systems on U. Since M_1 can be extended to the local system $j_*M_1 \subset j_*(L|_U) \simeq L$ of the same rank on X, it follows from the irreducibility of L that M_1 or M_2 is zero. Namely, $F_1{}^{\cdot}$ or $F_2{}^{\cdot}$ is supported by $Z = X \setminus U$. Since $L[d_X] \simeq {}^p j_{!*}(L|_U[d_X])$ by Corollary 8.2.6, $F_1{}^{\cdot}$ or $F_2{}^{\cdot}$ should be zero by Corollary 8.2.8. □

Remark 8.2.25. Assume that X is a complex manifold. For a $\mathbb{C}^{\times}$-invariant Lagrangian analytic subset Λ of the cotangent bundle T^*X denote by $\mathrm{Perv}_{\Lambda}(\mathbb{C}_X)$ the subcategory of $\mathrm{Perv}(\mathbb{C}_X)$ consisting of objects whose micro-supports are contained in Λ. From the results in some simple cases it is generally expected that for any $\Lambda \subset T^*X$ as above there exists a finitely presented algebra R such that the category $\mathrm{Perv}_{\Lambda}(\mathbb{C}_X)$ is equivalent to that of finite-dimensional representations of R. In some special (but important) cases this algebraic (or quiver) description of the category $\mathrm{Perv}_\Lambda(\mathbb{C}_X)$ was established by [GGM]. Recently this problem has been completely solved for any smooth projective variety X and any $\Lambda \subset T^*X$ by S. Gelfand–MacPherson–Vilonen [GMV].

Now we state the decomposition theorem due to Beilinson–Bernstein–Deligne–Gabber (see [BBD]) without proofs.

Theorem 8.2.26 (Decomposition theorem). *For a proper morphism $f : X \longrightarrow Y$ of algebraic varieties, we have*

$$Rf_*\big[\mathrm{IC}_X{}^{\boldsymbol{\cdot}}\big] \simeq \bigoplus_k^{\textit{finite}} i_{k*}\mathrm{IC}_{Y_k}(L_k)^{\boldsymbol{\cdot}}[l_k]. \tag{8.2.1}$$

Here for each k, Y_k is an irreducible closed subvariety of Y, $i_k : Y_k \hookrightarrow Y$ is the embedding, $L_k \in \mathrm{Loc}(Y'_k)$ for some smooth Zariski open subset Y'_k of Y_k, and l_k is an integer.

The proof relies on a deep theory of weights for étale perverse sheaves in positive characteristics. This result can be extended to analytic situation via the theory of Hodge modules (see Section 8.3 below).

Corollary 8.2.27. *Let X be a projective variety and $\pi : \widetilde{X} \to X$ a resolution of singularities of X. Then $IH^i(X)$ is a direct summand of $H^i(\widetilde{X})$ for any $i \in \mathbb{Z}$.*

Proof. By the decomposition theorem we have

$$R\pi_*(\mathbb{C}_{\widetilde{X}}[d_X]) \simeq \mathrm{IC}_X{}^{\boldsymbol{\cdot}} \oplus F^{\boldsymbol{\cdot}}$$

for some $F^{\boldsymbol{\cdot}} \in D^b_c(X)$, from which the result follows. □

Remark 8.2.28. The decomposition theorem has various important applications in geometric representation theory of reductive algebraic groups. For details we refer the reader to Lusztig [L2] and Chriss–Ginzburg [CG].

In general the direct image of a perverse sheaf is not necessarily a perverse sheaf. We will give a sufficient condition on a morphism $f : X \to Y$ so that $Rf_*(\mathrm{IC}_X{}^{\boldsymbol{\cdot}})$ is perverse.

Definition 8.2.29. Let $f : X \to Y$ be a dominant morphism of irreducible algebraic varieties. We say that f is *small* (resp. *semismall*) if the condition $\mathrm{codim}_Y\{y \in Y \mid \dim f^{-1}(y) \geq k\} > 2k$ (resp. $\mathrm{codim}_Y\{y \in Y \mid \dim f^{-1}(y) \geq k\} \geq 2k$) is satisfied for any $k \geq 1$.

Note that if $f : X \to Y$ is semismall then there exists a smooth open dense subset $U \subset Y$ such that $f|_{f^{-1}(U)}: f^{-1}(U) \to U$ is a finite morphism. In particular we have $d_X = d_Y$.

Proposition 8.2.30. *Let $f : X \to Y$ be a dominant proper morphism of irreducible algebraic varieties. Assume that X is rationally smooth.*

(i) *Assume that f is semismall. Then the direct image $Rf_*(\mathbb{C}_{X^{\mathrm{an}}}[d_X])$ of the constant perverse sheaf $\mathbb{C}_{X^{\mathrm{an}}}[d_X] \simeq \mathrm{IC}_X{}^{\cdot}$ is a perverse sheaf on Y.*

(ii) *Assume that f is small. Then we have an isomorphism $Rf_*(\mathbb{C}_{X^{\mathrm{an}}}[d_X]) \simeq \mathrm{IC}_Y(L)^{\cdot}$ for some $L \in \mathrm{Loc}(U)$, where U is a smooth open subset of Y.*

Proof. (i) By $\mathbf{D}_Y Rf_* \simeq Rf_* \mathbf{D}_X$ (f is proper) it suffices to check the condition $Rf_*(\mathbb{C}_{X^{\mathrm{an}}}[d_X]) \in {}^pD_c^{\leqslant 0}(X)$. This follows easily from $R^j f_*(\mathbb{C}_{X^{\mathrm{an}}})_y = H^j(f^{-1}(y), \mathbb{C})$ and the fact that $H^j(f^{-1}(y), \mathbb{C}) = 0$ for $j > 2 \dim f^{-1}(y)$.

(ii) By our assumption there exists an open subset $U \subset Y$ such that $f|_{f^{-1}(U)}: f^{-1}(U) \to U$ is a finite morphism. If necessary, we can shrink U so that U is smooth and $f_*\mathbb{C}_{X^{\mathrm{an}}}|_{U^{\mathrm{an}}} \in \mathrm{Loc}(U)$. We denote this local system by L. Set $Z = Y \setminus U$ and let $i : Z \hookrightarrow Y$ be the embedding. By Proposition 8.2.5 we have only to show that $i^{-1}Rf_*(\mathbb{C}_{X^{\mathrm{an}}}[d_X]) \in {}^pD_c^{\leqslant -1}(Z)$ and $i^! Rf_*(\mathbb{C}_{X^{\mathrm{an}}}[d_X]) \in {}^pD_c^{\geqslant 1}(Z)$. Again by $\mathbf{D}_Y Rf_* \simeq Rf_* \mathbf{D}_X$ it is enough to check only the condition $i^{-1}Rf_*(\mathbb{C}_{X^{\mathrm{an}}}[d_X]) \in {}^pD_c^{\leqslant -1}(Z)$. This can be shown by the argument used in the proof of (i). □

Recall that the normalization $\pi : \widetilde{X} \to X$ of a projective variety X is a finite map which induces an isomorphism $\pi|_{\pi^{-1}X_{\mathrm{reg}}} : \pi^{-1}X_{\mathrm{reg}} \xrightarrow{\sim} X_{\mathrm{reg}}$.

Corollary 8.2.31. *Let X be a projective variety and $\pi : \widetilde{X} \to X$ its normalization. Then we have an isomorphism*

$$R\pi_*\big[\mathrm{IC}_{\widetilde{X}}{}^{\cdot}\big] \simeq \mathrm{IC}_X{}^{\cdot}.$$

In particular, there exists an isomorphism $IH^i(\widetilde{X}) \simeq IH^i(X)$ for any $i \in \mathbb{Z}$.

Since the normalization $\pi : \widetilde{C} \to C$ of an algebraic curve C is smooth, we obtain an isomorphism $IH^i(C) \simeq H^i(\widetilde{C})$ for any $i \in \mathbb{Z}$. However, this is not always true in higher-dimensional cases.

Example 8.2.32. Let C be an irreducible plane curve defined by $C = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{C}) | x_0^3 + x_1^3 = x_0 x_1 x_2\}$. Since C has an isolated singular point, we obtain

$$IH^i(C) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i = 1 \\ \mathbb{C} & i = 2 \end{cases}$$

by Proposition 8.2.19. In this case the normalization $\widetilde{C}$ of C is isomorphic to $\mathbb{P}^1(\mathbb{C})$ and hence we observe that $IH^i(C) \simeq H^i(\widetilde{C})$ for any $i \in \mathbb{Z}$.

Remark 8.2.33. Proposition 8.2.30 has important consequences in representation theory. For example, let G be a semisimple algebraic group over $\mathbb{C}$ and $B \subset G$ a Borel subgroup (see Chapter 9 for the definitions). For the flag variety $X = G/B$ of G let us set $\widetilde{G} = \{(g, x) \in G \times X \mid gx = x\} \subset G \times X$. Then $\widetilde{G}$ is a smooth complex manifold because the second projection $\widetilde{G} \to X$ is a fiber bundle on X. Furthermore, it turns out that the first projection $f : \widetilde{G} \to G$ is small (see Lusztig [L1]). Therefore, via the Riemann–Hilbert correspondence, Proposition 8.2.30 implies that

$$H^j \int_f \mathcal{O}_{\widetilde{G}} = 0 \quad \text{for} \quad j \neq 0.$$

The remaining non-zero term $H^0 \int_f \mathcal{O}_{\widetilde{G}}$ is a regular holonomic system on G and coincides with the one satisfied by the characters of representations (invariant distributions) of a real form of G (see [HK1]). Harish-Chandra obtained many important results in representation theory through the detailed study of this system of equations. We also note that Proposition 8.2.30 and Theorem 8.2.36 below play crucial roles in the recent progress of the geometric Langlands program (see, for example, [MV]).

We end this section by presenting a beautiful application of the decomposition theorem due to Borho–MacPherson [BM] on the explicit description of the direct images of constant perverse sheaves. Let $f : X \to Y$ be a dominant projective morphism of irreducible algebraic varieties. Then by a well-known result in analytic geometry (see, for example, Thom [Th, p. 276]), there exists a complex stratification $Y = \sqcup_{\alpha \in A} Y_\alpha$ of Y by connected strata Y_α's such that $f|_{f^{-1}Y_\alpha} : f^{-1}Y_\alpha \to Y_\alpha$ is a topological fiber bundle with the fiber $F_\alpha := f^{-1}(y_\alpha)$ ($y_\alpha \in Y_\alpha$). Let us assume, moreover, that f is semismall. Then we have $\operatorname{codim}_Y Y_\alpha \geq 2 \dim F_\alpha$ for any $\alpha \in A$. Note that for any $i > 2 \dim F_\alpha$ we have $H^i(F_\alpha) = 0$. In particular $[H^i Rf_*(\mathbb{C}_X)]|_{Y_\alpha} = 0$ for any $i > \operatorname{codim}_Y Y_\alpha$. Set $c_\alpha = \operatorname{codim}_Y Y_\alpha$ and denote by L_α the local system $[H^{c_\alpha} Rf_*(\mathbb{C}_X)]|_{Y_\alpha}$ on Y_α.

Definition 8.2.34. We say that a stratum Y_α is *relevant* if the condition $c_\alpha = 2 \dim F_\alpha$ holds.

We easily see that f is small if and only if the only relevant stratum is the open dense one. Moreover, a stratum Y_α is relevant if and only if $L_\alpha \neq 0$. For a relevant stratum Y_α the top-dimensional cohomology group $H^{c_\alpha}(F_\alpha) \simeq (L_\alpha)_{y_\alpha}$ of the fiber F_α has a basis corresponding to the d_{F_α}-dimensional irreducible components of F_α. The fundamental group of Y_α acts on the $\mathbb{C}$-vector space $H^{c_\alpha}(F_\alpha) \simeq (L_\alpha)_{y_\alpha}$ by permutations of these irreducible components. This action completely determines the local system L_α. For each $\alpha \in A$ let

$$L_\alpha = \bigoplus_\phi (L_\phi)^{\oplus m_\phi}$$

be the irreducible decomposition of the local system L_α, where ϕ ranges through the set of all irreducible representations of the fundamental group of Y_α and $m_\phi \geq 0$ is the multiplicity of the irreducible local system L_ϕ corresponding to ϕ.

Definition 8.2.35. We say that a pair (Y_α, ϕ) of a stratum Y_α and an irreducible representation of the fundamental group of Y_α is *relevant* if the stratum Y_α is relevant and $m_\phi \neq 0$.

Theorem 8.2.36 (Borho–MacPherson [BM]). *In the situation as above, assume, moreover, that X is rationally smooth. Then the direct image $Rf_*(\mathbb{C}_X[d_X])$ of the constant perverse sheaf $\mathbb{C}_X[d_X] \simeq \mathrm{IC}_X{}^{\cdot}$ is explicitly given by*

$$Rf_*(\mathbb{C}_X[d_X]) \simeq \bigoplus_{(Y_\alpha,\phi)} [i_{\alpha *}\mathrm{IC}_{Z_\alpha}(L_\phi)^{\cdot}]^{\oplus m_\phi},$$

where (Y_α, ϕ) ranges through the set of all relevant pairs, Z_α is the closure of Y_α and $i_\alpha : Z_\alpha \hookrightarrow Y$ is the embedding.

Proof. By Proposition 8.2.30 the direct image $Rf_*(\mathbb{C}_X[d_X])$ is a perverse sheaf on Y. By the decomposition theorem we can prove recursively that on each strata Y_β it is written more explicitly as

$$Rf_*(\mathbb{C}_X[d_X]) \simeq \bigoplus_{(Y_\alpha,\phi)} [i_{\alpha *}\mathrm{IC}_{Z_\alpha}(L_\phi)^{\cdot}]^{\oplus n_\phi},$$

where (Y_α, ϕ) ranges through the set of all pairs of Y_α and irreducible representations ϕ of the fundamental group of Y_α, and n_ϕ are some non-negative integers. Indeed, let Y_{α_0} be the unique open dense stratum in Y. Then on Y_{α_0} the direct image $Rf_*(\mathbb{C}_X[d_X])$ is obviously written as

$$Rf_*(\mathbb{C}_X[d_X]) \simeq \bigoplus_{(Y_{\alpha_0},\phi)} (L_\phi[d_X])^{\oplus m_\phi},$$

where (Y_{α_0}, ϕ) ranges through the set of relevant pairs. Namely, for any pair (Y_α, ϕ) such that $\alpha = \alpha_0$ we have $n_\phi = m_\phi$. Let Y_{α_1} be a stratum such that $\mathrm{codim}_Y Y_{\alpha_1} = 1$. Since $H^i Rf_*(\mathbb{C}_X[d_X]) = 0$ on Y_{α_1} for $i \neq -d_X$ by the semismallness of f, for any pair (Y_α, ϕ) such that $\alpha = \alpha_1$ we must have $n_\phi = 0$. Therefore, on $Y_{\alpha_0} \sqcup Y_{\alpha_1}$ we obtain an isomorphism

$$Rf_*(\mathbb{C}_X[d_X]) \simeq \bigoplus_{(Y_{\alpha_0},\phi)} [i_{\alpha_0 *}\mathrm{IC}_{Z_{\alpha_0}}(L_\phi)^{\cdot}]^{\oplus m_\phi},$$

where (Y_{α_0}, ϕ) ranges through the set of relevant pairs. By repeating this argument, we can finally prove the theorem. □

Remark 8.2.37. By Theorem 8.2.36 we can geometrically construct and study representations of Weyl groups of semisimple algebraic groups. This is the so-called theory of Springer representations. For this very important application of Theorem 8.2.36, see Borho–MacPherson [BM]. As another subject where Theorem 8.2.36 is applied effectively we also point out the work by Göttsche [Go] computing the Betti numbers of Hilbert schemes of points of algebraic surfaces. Inspired by this result Nakajima [Na1] and Grojnowski [Gr] found a beautiful symmetry in the cohomology groups of these Hilbert schemes (see [Na2] for the details).

8.3 Hodge modules

8.3.1 Motivation

Let k an the algebraic closure of the prime field $\mathbb{F}_p$ of characteristic p. Starting from a given separated scheme X of finite type over $\mathbb{Z}$, we can construct by base changes, the schemes $X_{\mathbb{C}} = X \otimes_{\mathbb{Z}} \mathbb{C}$ over $\mathbb{C}$ and $X_k = X \otimes_{\mathbb{Z}} k$ over k. The scheme X_k is regarded as the counterpart of $X_{\mathbb{C}}$ in positive characteristic. We have cohomology groups $H^*(X_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Q})$, $H_c^*(X_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Q})$ for the underlying analytic space $X_{\mathbb{C}}^{\mathrm{an}}$ of $X_{\mathbb{C}}$; the corresponding notions in positive characteristic are the so-called l-adic étale cohomology groups $H^*(X_k, \bar{\mathbb{Q}}_l)$, $H_c^*(X_k, \bar{\mathbb{Q}}_l)$ (l is a prime number different from p). More precisely, under some suitable conditions on X over $\mathbb{Z}$ we have the isomorphisms

$$H^*(X_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l \simeq H^*(X_k, \bar{\mathbb{Q}}_l), \quad H_c^*(X_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l \simeq H_c^*(X_k, \bar{\mathbb{Q}}_l),$$

which show that the étale topology for X_k corresponds to the classical topology for $X_{\mathbb{C}}$. This correspondence can be extended to the level of local systems and perverse sheaves on $X_{\mathbb{C}}^{\mathrm{an}}$ and X_k.

For X_k we have the Frobenius morphism, which is an operation peculiar to the case of positive characteristic. This allows us, compared with the case of $X_{\mathbb{C}}$, to develop a more detailed theory on X_k by considering étale local systems and étale perverse sheaves endowed with the action of the Frobenius morphisms (Weil sheaves). That is the theory of weights for étale sheaves, which played an important role in the proof of the Weil conjecture [De3], [BBD].

However, Grothendieck's philosophy predicted the existence of the theory of weights for objects over $\mathbb{C}$. In the case of local systems, the theory of the variation of Hodge structures had been known as a realization of such a theory over $\mathbb{C}$ [De2]. In the case of general perverse sheaves, a theory of weight which is based on the Riemann–Hilbert correspondence was constructed, and gave the final answer to this problem. This is the theory of Hodge modules due to Morihiko Saito [Sa1], [Sa3].

In this section, we present a brief survey on the theory of Hodge modules.

8.3.2 Hodge structures and their variations

In this subsection we discuss standard notions on Hodge structures. For more precise explanations, refer to [De2], [GS].

Let X be a smooth projective algebraic variety. The complexifications $H_{\mathbb{C}} = H^n(X^{\mathrm{an}}, \mathbb{C})$ of its rational cohomology groups $H = H^n(X^{\mathrm{an}}, \mathbb{Q})$ ($n \in \mathbb{Z}$) can be naturally identified with the de Rham cohomology groups. Moreover, in such cases, a certain family $\{ H^{p,q} \mid p, q \in \mathbb{N},\ p + q = n \}$ of subspaces of $H_{\mathbb{C}}$ is defined by the theory of harmonic forms so that we have the Hodge decomposition

$$H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}, \quad \bar{H}^{p,q} = H^{q,p}.$$

Here we denote the complex conjugation map of $H_{\mathbb{C}}$ with respect to H by $h \mapsto \bar{h}$ ($h \in H_{\mathbb{C}}$). Let us set $F^p(H_{\mathbb{C}}) := \bigoplus_{i \geq p} H^{i,n-i}$. Then F defines a decreasing filtration of $H_{\mathbb{C}}$ and the equality $H_{\mathbb{C}} = F^p \oplus \bar{F}^{n-p+1}$ holds for every p. We call it the Hodge filtration of $H_{\mathbb{C}}$. Note that we can reconstruct the subspaces $H^{p,q}$ from the Hodge filtration by using $H^{p,q} = F^p \cap \bar{F}^q$.

Therefore, it would be natural to define the notion of Hodge structures in the following way. Let H be a finite-dimensional vector space over $\mathbb{Q}$ and $H_{\mathbb{C}}$ its complexification. Denote by $h \mapsto \bar{h}$ ($h \in H_{\mathbb{C}}$) the complex conjugation map of $H_{\mathbb{C}}$ and consider a finite decreasing filtration $F = \{ F^p(H_{\mathbb{C}}) \}_{p \in \mathbb{Z}}$ by subspaces in $H_{\mathbb{C}}$. That is, we assume that $F^p(H_{\mathbb{C}})$'s are subspaces of $H_{\mathbb{C}}$ satisfying $F^p(H_{\mathbb{C}}) \supset F^{p+1}(H_{\mathbb{C}})$ $({}^{\forall} p)$ and $F^p(H_{\mathbb{C}}) = \{0\}, \ F^{-p}(H_{\mathbb{C}}) = H_{\mathbb{C}}$ $(p \gg 0)$. For an integer n, we say that the pair (H, F) is a *Hodge structure* of *weight* n if the condition

$$H_{\mathbb{C}} = F^p \oplus \bar{F}^{n-p+1}$$

holds for any p. The filtration F is called the *Hodge filtration*. In this case, if we set $H^{p,q} = F^p \cap \bar{F}^q$, then we obtain the Hodge decomposition

$$H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}, \qquad \bar{H}^{p,q} = H^{q,p}.$$

We can naturally define the morphisms between Hodge structures as follows. Given two Hodge structures (H, F), (H', F) of the same weight n, a linear map $f : H \to H'$ is called a *morphism of Hodge structures* if it satisfies the condition

$$f(F^p(H_{\mathbb{C}})) \subset F^p(H'_{\mathbb{C}})$$

for any p. Here we used the same symbol f for the complexified map $H_{\mathbb{C}} \to H'_{\mathbb{C}}$ of f. Thus we have defined the category $SH(n)$ of the Hodge structures of weight n. The morphisms in $SH(n)$ are *strict* with respect to the Hodge filtration F. Namely, for $f \in \mathrm{Hom}_{SH(n)}\big((H, F), (H', F)\big)$ we always have

$$f(F^p(H_{\mathbb{C}})) = f(H_{\mathbb{C}}) \cap F^p(H'_{\mathbb{C}})$$

for any $p \in \mathbb{Z}$, from which we see that $SH(n)$ is an abelian category.

Next let us explain the *polarizations of Hodge structures*. We say a Hodge structure $(H, F) \in SH(n)$ is *polarizable* if there exists a bilinear form S on $H_{\mathbb{C}}$ satisfying the following properties:

(i) If n is even, S is symmetric. If n is odd, S is anti-symmetric.
(ii) If $p + p' \neq n$, we have $S(H^{p,n-p}, H^{p',n-p'}) = 0$.
(iii) For any $v \in H^{p,n-p}$ such that $v \neq 0$ we have $(\sqrt{-1})^{n-2p} S(v, \bar{v}) > 0$.

Denote by $SH(n)^p$ the full subcategory of $SH(n)$ consisting of polarizable Hodge structures of weight n. Then it turns out that $SH(n)^p$ is an abelian category, and any object from it can be expressed as a direct sum of irreducible objects.

As we have explained above, for a smooth projective algebraic variety X, a natural Hodge structure of weight n can be defined on $H^n(X^{\mathrm{an}}, \mathbb{Q})$; however, the situation is much more complicated for non-projective varieties with singularities. In such cases, the cohomology group $H^n(X^{\mathrm{an}}, \mathbb{Q})$ is a sort of mixture of the Hodge structures of various weights. This is the theory of mixed Hodge structures due to Deligne.

Let us give the definition of mixed Hodge structures. Let H be a finite-dimensional vector space over $\mathbb{Q}$, F a finite decreasing filtration of $H_{\mathbb{C}}$ and $W = \{W_n(H)\}_{n\in\mathbb{Z}}$ a finite increasing filtration of H. By complexifying W, we get an increasing filtration of $H_{\mathbb{C}}$, which we also denote by W. Then the complexification of $\mathrm{gr}_n^W(H) = W_n(H)/W_{n-1}(H)$ is identified with $W_n(H_{\mathbb{C}})/W_{n-1}(H_{\mathbb{C}})$ and its decreasing filtration is defined by

$$\tilde{F}^p(W_n(H_{\mathbb{C}})/W_{n-1}(H_{\mathbb{C}})) = (W_n(H_{\mathbb{C}}) \cap F^p(H_{\mathbb{C}}) + W_{n-1}(H_{\mathbb{C}}))/W_{n-1}(H_{\mathbb{C}}).$$

We say that a triplet (H, F, W) is a *mixed Hodge structure* if for any n the filtered vector space $\mathrm{gr}_n^W(H, F) := (\mathrm{gr}_n^W(H), \tilde{F})$ is a Hodge structure of weight n. We can naturally define the morphisms between mixed Hodge structures, and hence the category SHM of mixed Hodge structures is defined. Let SHM^p be the full subcategory of SHM consisting of objects $(H, F, W) \in SHM$ such that $\mathrm{gr}_n^W(H, F) \in SH(n)^p$ for any n. They are abelian categories.

Next we explain the notion of the variations of Hodge structures, which naturally appears in the study of deformation (moduli) theory of algebraic varieties.

Let $f : Y \to X$ be a smooth projective morphism between two smooth algebraic varieties. Then the nth higher direct image sheaf $R^n f_*^{\mathrm{an}}(\mathbb{Q}_{Y^{\mathrm{an}}})$ is a local system on X^{an} whose stalk at $x \in X$ is isomorphic to $H^n(f^{-1}(x)^{\mathrm{an}}, \mathbb{Q})$. Since the fiber $f^{-1}(x)$ is a smooth projective variety, there exists a Hodge structure of weight n on $H^n(f^{-1}(x)^{\mathrm{an}}, \mathbb{Q})$. Namely, the sheaf $R^n f_*^{\mathrm{an}}(\mathbb{Q}_{Y^{\mathrm{an}}})$ is a local system whose stalks are Hodge structures of weight n. Extracting properties of this local system, we come to the definition of the variations of Hodge structures as follows.

Let X be a smooth algebraic variety and H a $\mathbb{Q}$-local system on X^{an}. Then by a theorem of Deligne (Theorem 5.3.8), there is a unique (up to isomorphisms) regular integrable connection $\mathcal{M}$ on X such that $DR(\mathcal{M}) = \mathbb{C} \otimes_{\mathbb{Q}} H\,[\dim X]$. Let us denote this regular integrable connection $\mathcal{M}$ by $\mathcal{M}(H)$. Assume that $F = \{F^p(\mathcal{M}(H))\}_{p\in\mathbb{Z}}$ is a finite decreasing filtration of $\mathcal{M}(H)$ by $\mathcal{O}_X$-submodules such that F^p/F^{p+1} is a locally free $\mathcal{O}_X$-module for any p. Namely, F corresponds to a filtration of the associated complex vector bundle by its subbundles. Since the complexification $(H_x)_{\mathbb{C}}$ of the stalk H_x of H at $x \in X$ coincides with $\mathbb{C} \otimes_{\mathcal{O}_{X,x}} \mathcal{M}(H)_x$, a decreasing filtration $F(x)$ of $(H_x)_{\mathbb{C}}$ is naturally defined by F. Now we say that the pair (H, F) is a *variation of Hodge structures* of weight n if it satisfies the conditions

(i) For any $x \in X$, $(H_x, F(x)) \in SH(n)$.
(ii) For any $p \in \mathbb{Z}$, we have $\Theta_X \cdot F^p(\mathcal{M}(H)) \subset F^{p-1}(\mathcal{M}(H))$, where Θ_X stands for the sheaf of holomorphic vector fields on X.

The last condition is called *Griffiths transversality*. Also, we say that the variation of Hodge structures (H, F) is *polarizable* if there exists a morphism

$$S : H \otimes_{\mathbb{Q}_{X^{\mathrm{an}}}} H \longrightarrow \mathbb{Q}_{X^{\mathrm{an}}}$$

of local systems which defines a polarization of $(H_x, F(x))$ at each point $x \in X$. We denote by $VSH(X, n)$ the category of the variations of Hodge structures on X of weight n. Its full subcategory consisting of polarizable objects is denoted by $VSH(X, n)^p$. The categories $VSHM(X)$, $VSHM(X)^p$ of the variations of mixed Hodge structures on X can be defined in the same way as SHM, SHM^p. All these categories $VSH(X, n)$, $VSH(X, n)^p$, $VSHM(X)$ and $VSHM(X)^p$ are abelian categories.

8.3.3 Hodge modules

The theory of variations of Hodge structures in the previous subsection may be regarded as the theory of weights for local systems. Our problem here is to extend it to the theory of weights for general perverse sheaves. The $\mathbb{Q}$-local systems H appearing in the variations of Hodge structures should be replaced with perverse sheaves K (over $\mathbb{Q}$). In this situation, a substitute for the regular integrable connection $\mathcal{M}(H)$ is a regular holonomic system $\mathcal{M}$ such that

$$DR(\mathcal{M}) = \mathbb{C} \otimes_{\mathbb{Q}} K.$$

Now, what is the Hodge filtration in this generalized setting? Instead of the decreasing Hodge filtration $F = \{F^p\}$ of $\mathcal{M}$, set $F_p = F^{-p}$ and let us now consider the increasing filtration $\{F_p\}$. Then Griffiths transversality can be rephrased as $\Theta_X \cdot F_p \subset F_{p+1}$. Therefore, for a general regular holonomic system $\mathcal{M}$, we can consider good filtrations in the sense of Section 2.1 as a substitute of Hodge filtrations.

Now, for a smooth algebraic variety X denote by $MF_{rh}(D_X)$ the category of the pairs $(\mathcal{M}, F)$ of $\mathcal{M} \in \mathrm{Mod}_{rh}(D_X)$ and a good filtration F of M. We also denote by $MF_{rh}(D_X, \mathbb{Q})$ the category of the triplets $(\mathcal{M}, F, K)$ consisting of $(\mathcal{M}, F) \in MF_{rh}(D_X)$ and a perverse sheaf $K \in \mathrm{Perv}(\mathbb{Q}_X)$ over $\mathbb{Q}$ such that $DR(\mathcal{M}) = \mathbb{C} \otimes_{\mathbb{Q}} K$. Also $MF_{rh}W(D_X, \mathbb{Q})$ stands for the category of quadruplets $(\mathcal{M}, F, K, W)$ consisting of $(\mathcal{M}, F, K) \in MF_{rh}(D_X, \mathbb{Q})$ and its finite increasing filtration $W = \{W_n\}$ in the category $MF_{rh}(D_X, \mathbb{Q})$. Although these categories are not abelian, they are additive categories. Under these definitions, we can show that $VSH(X, n)$ (resp. $VSHM(X)$) is a full subcategory of $MF_{rh}(D_X, \mathbb{Q})$ (resp. $MF_{rh}W(D_X, \mathbb{Q})$). Indeed, by sending $(H, F) \in VSH(X, n)$ (resp. $(H, F, W) \in VSHM(X)$) to $(\mathcal{M}(H), F, H[\dim X]) \in MF_{rh}(D_X, \mathbb{Q})$ (resp. $(\mathcal{M}(H), F, H[\dim X], W) \in MF_{rh}W(D_X, \mathbb{Q})$) we get the inclusions

$$\phi_X^n : VSH(X, n) \longrightarrow MF_{rh}(D_X, \mathbb{Q}),$$
$$\phi_X : VSHM(X) \longrightarrow MF_{rh}W(D_X, \mathbb{Q})$$

of categories. So our problem is now summarized in the following three parts:

(1) Define a full abelian subcategory of $MF_{rh}(D_X, \mathbb{Q})$ (resp. $MF_{rh}W(D_X, \mathbb{Q})$) consisting of objects of weight n (resp. objects of mixed weights).
(2) Define the various operations, such as direct image, inverse image, duality functor for (the derived categories of) the abelian categories defined in (1).

(3) Show that the notions defined in (1), (2) satisfy the properties that deserves the name of the theory of weights. That is, prove basic theorems similar to those in the theory of weights in positive characteristic.

Morihiko Saito tackled these problems for several years and gave a decisive answer. In [Sa1], Saito defined the full abelian subcategory $MH(X, n)$ of $MF_{rh}(D_X, \mathbb{Q})$ consisting of the Hodge modules of weight n. He also gave the definition of the full abelian subcategory $MH(X, n)^p$ of $MH(X, n)$ consisting of polarizable objects and proved its stability through the direct images associated to projective morphisms. Next, in [Sa3], he defined the category $MHM(X)$ of mixed Hodge modules as a subcategory of $MF_{rh}W(D_X, \mathbb{Q})$ and settled the above problems (2), (3). Since the definition of *Hodge modules* requires many steps, we do not give their definition here and explain only their properties.

We first present some basic properties of the categories $MH(X, n)$ and $MH(X, n)^p$ following [Sa1]:

(p1) $MH(X, n)$, $MH(X, n)^p$ are full subcategories of $MF_{rh}(D_X, \mathbb{Q})$, and we have $MH\ (X, n)^p \subset MH(X, n)$.

(p2) (locality) Consider an open covering $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ of X. For $\mathcal{V} \in MF_{rh}(D_X, \mathbb{Q})$ we have $\mathcal{V} \in MH(X, n)$ (resp. $MH(X, n)^p$) if and only if $\mathcal{V}|_{U_\lambda} \in MH(U_\lambda, n)$ (resp. $\mathcal{V}|_{U_\lambda} \in MH(U_\lambda, n)^p$).

(p3) All morphisms in $MH(X, n)$ and $MH(X, n)^p$ are strict with respect to the filtrations F.

We see from (p3) that $MH(X, n)$ and $MH(X, n)^p$ are abelian categories. For $\mathcal{V} = (\mathcal{M}, F, K) \in MH(X, n)$ the support $\mathrm{supp}(\mathcal{M})$ of $\mathcal{M}$ is called the support of $\mathcal{V}$ and we denote it by $\mathrm{supp}(\mathcal{V})$. This is a closed subvariety of X. Now let Z be an irreducible closed subvariety of X. We say $\mathcal{V} = (\mathcal{M}, F, K) \in MH(X, n)$ has the *strict support* Z if the support of $\mathcal{V}$ is Z and there is neither subobject nor quotient object of $\mathcal{V}$ whose support is a non-empty proper subvariety of Z. Let us denote by $MH_Z(X, n)$ the full subcategory of $MH(X, n)$ consisting of objects having the strict support Z. We also set $MH_Z(X, n)^p = MH_Z(X, n) \cap MH(X, n)^p$. Then we have

(p4) $MH(X, n) = \bigoplus_Z MH_Z(X, n)$, $MH(X, n)^p = \bigoplus_Z MH_Z(X, n)^p$, where Z ranges over the family of irreducible closed subvarieties of X.

If one wants to find a category satisfying only the conditions (p1)–(p4) one can set $MH(X, n) = \{0\}$ for all X; however, the category $MH(X, n)$ is really not trivial, as we will explain below.

For $\mathcal{V} = (\mathcal{M}, F, K) \in MF_{rh}(D_X, \mathbb{Q})$ and an integer m we define $\mathcal{V}(m) \in MF_{rh}(D_X, \mathbb{Q})$ by

$$\mathcal{V}(m) = (\mathcal{M} \otimes_{\mathbb{Q}} \mathbb{Q}(m),\ F[m],\ K \otimes_{\mathbb{Q}} \mathbb{Q}(m))$$

(the *Tate twist* of $\mathcal{V}$), where $\mathbb{Q}(m) = (2\pi\sqrt{-1})^m \mathbb{Q} \subset \mathbb{C}$ and $F[m]_p = F_{p-m}$. The readers might feel it strange to write $\otimes_{\mathbb{Q}} \mathbb{Q}(m)$; however, this notation is natural from the viewpoint of Hodge theory.

(p5) $\phi_X^n(VSH(X, n)^p) \subset MH_X(X, n + \dim X)^p$.
(p6) If $\mathcal{V} \in MH(X, n)$ (resp. $MH(X, n)^p$), then $\mathcal{V}(m) \in MH(X, n - 2m)$ (resp. $MH(X, n - 2m)^p$).

From (p5), (p6) and the stability through projective direct images to be stated below, we see that $MH(X, n)$ contains many non-trivial objects (see also (m13) below).

Let $f : X \to Y$ be a projective morphism between two smooth algebraic varieties. Then the derived direct image $f_\bigstar(\mathcal{M}, F)$ of $(\mathcal{M}, F) \in MF_{rh}(D_X)$ is defined as a complex of filtered modules (more precisely it is an object of a certain derived category). As a complex of D-modules it is the ordinary direct image $\int_f \mathcal{M}$.

(p7) If $(\mathcal{M}, F, K) \in MH(X, n)^p$, then the complex $f_\bigstar(\mathcal{M}, F)$ is strict with respect to the filtrations and we have

$$(H^j f_\bigstar(\mathcal{M}, F), {}^pH^j f_*(K)) \in MH(Y, n + j)^p.$$

for any $j \in \mathbb{Z}$ (${}^pH^j f_*(K)$ is the jth perverse part of the direct image complex $f_*(K)$, see Section 8.1).

Let us explain how the filtered complex $f_\bigstar(\mathcal{M}, F)$ can be defined. First consider the case where f is a closed embedding. In this case, for $j \neq 0$ we have $H^j \int_f \mathcal{M} = 0$ and $H^0 \int_f \mathcal{M} = f_*(D_{Y\leftarrow X} \otimes_{D_X} \mathcal{M})$. Using the filtration on $D_{Y\leftarrow X}$ induced from the one on D_Y, let us define a filtration on $\int_f \mathcal{M}$ by

$$F_p\Big(\int_f \mathcal{M}\Big) := f_*\Big(\sum_q F_q(D_{Y\leftarrow X}) \otimes F_{p-q+\dim X-\dim Y}(\mathcal{M})\Big).$$

Then this is the filtered complex $f_\bigstar(\mathcal{M}, F)$ of sheaves. Next consider the case where f is a projection $X = Y \times Z \to Y$ (Z is a smooth projective variety of dimension m). Now $D_{Y\leftarrow X} \otimes^L_{D_X} \mathcal{M}$ can be expressed as the relative de Rham complex

$$DR_{X/Y}(\mathcal{M}) = [\Omega^0_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M} \to \Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M} \to \cdots \to \Omega^m_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}]$$

(here $\Omega^m_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}$ is in degree 0). Hence a filtration of $DR_{X/Y}(\mathcal{M})$ as a complex of $f^{-1}D_Y$-modules is defined by

$$\begin{aligned} F_p(DR_{X/Y}(\mathcal{M})) = \big[\Omega^0_{X/Y} \otimes_{\mathcal{O}_X} F_p(\mathcal{M}) \to \Omega^1_{X/Y} \otimes_{\mathcal{O}_X} F_{p+1}(\mathcal{M}) \\ \to \cdots \to \Omega^m \otimes_{\mathcal{O}_X} F_{p+m}(\mathcal{M})\big], \end{aligned}$$

and its sheaf-theoretical direct image is $f_\bigstar(\mathcal{M}, F)$. Note that the complex $f_\bigstar(\mathcal{M}, F)$ is strict if and only if the morphism

$$H^j\big(Rf_*(F_p(DR_{X/Y}(\mathcal{M})))\big) \longrightarrow H^j\big(Rf_*(DR_{X/Y}(\mathcal{M}))\big) = H^j \int_f \mathcal{M}$$

is injective for any j and p. If this is the case, $H^j \int_f \mathcal{M}$ becomes a filtered module by the filtration

$$F_p\Big(H^j \int_f \mathcal{M}\Big) = H^j\big(Rf_*(F_p(DR_{X/Y}(\mathcal{M})))\big).$$

We next present some basic properties of mixed Hodge modules following [Sa3]:

(m1) $MHM(X)$ is a full subcategory of $MF_{rh}W(D_X, \mathbb{Q})$, and subobjects and quotient objects (in the category $MF_{rh}W(D_X, \mathbb{Q})$) of an object of $MHM(X)$ are again in $MHM(X)$. That is, the category $MHM(X)$ is stable under the operation of taking subquotients in $MF_{rh}W(D_X, \mathbb{Q})$.

(m2) If $\mathcal{V} = (\mathcal{M}, F, K, W) \in MHM(X)$, then $\mathrm{gr}_n^W \mathcal{V} = \mathrm{gr}_n^W(\mathcal{M}, F, K) \in MH(X, n)^p$. Furthermore, if $\mathcal{V} \in MF_{rh}W(D_X, \mathbb{Q})$ satisfies $\mathrm{gr}_k^W \mathcal{V} = 0$ $(k \neq n)$, $\mathrm{gr}_n^W \mathcal{V} \in MH(X, n)^p$, then we have $\mathcal{V} \in MHM(X)$.

(m3) (locality) Let $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ be an open covering of X. Then for $\mathcal{V} \in MF_{rh}W(D_X, \mathbb{Q})$ we have $\mathcal{V} \in MHM(X)$ if and only if $\mathcal{V}|_{U_\lambda} \in MHM(U_\lambda)$ for any $\lambda \in \Lambda$.

(m4) All morphisms in $MHM(X)$ are strict with respect to the filtrations F, W.

(m5) If $\mathcal{V} = (\mathcal{M}, F, K, W) \in MHM(X)$, then $\mathcal{V}(m) = (\mathcal{M} \otimes_{\mathbb{Q}} \mathbb{Q}(m), F[m], K \otimes_{\mathbb{Q}} \mathbb{Q}(m), W[-2m]) \in MHM(X)$.

It follows from (m4) that $MHM(X)$ is an abelian category. Also by (m2) the object

$$\mathbb{Q}_X^H [\dim X] := (\mathcal{O}_X, F, \mathbb{Q}_{X^{\mathrm{an}}}[\dim X], W)$$
$$(\mathrm{gr}_p^F = 0 \ (p \neq 0), \ \mathrm{gr}_n^W = 0 \ (n \neq \dim X))$$

in $MF_{rh}W(D_X, \mathbb{Q})$ belongs to $MHM(X)$. Let us define a functor

$$\mathrm{rat} : MHM(X) \longrightarrow \mathrm{Perv}(\mathbb{Q}_X)$$

by assigning $K \in \mathrm{Perv}(\mathbb{Q}_X)$ to $(\mathcal{M}, F, K, W) \in MHM(X)$. Then the functor rat induces also a functor

$$\mathrm{rat} : D^bMHM(X) \longrightarrow D^b(\mathrm{Perv}(\mathbb{Q}_X)) \simeq D_c^b(\mathbb{Q}_X)$$

of triangulated categories (i.e., it sends distinguished triangles in $D^bMHM(X)$ to those in $D_c^b(\mathbb{Q}_X)$), where we used the isomorphism $D^b(\mathrm{Perv}(\mathbb{Q}_X)) \simeq D_c^b(\mathbb{Q}_X)$ proved by [BBD] and [Bei]. In [Sa3] the functors

$$\mathbb{D} : MHM(X) \longrightarrow MHM(X)^{\mathrm{op}},$$
$$f_\bigstar, \ f_! : D^bMHM(X) \longrightarrow D^bMHM(Y),$$
$$f^\bigstar, \ f^! : D^bMHM(Y) \longrightarrow D^bMHM(X)$$

were defined in the derived categories of mixed Hodge modules and he proved various desired properties ($f : X \to Y$ is a morphism of algebraic varieties). Namely, we have

(m6) $\mathbb{D} \circ \mathbb{D} = \mathrm{Id}, \ \mathbb{D} \circ f_\bigstar = f_! \circ \mathbb{D}, \ \mathbb{D} \circ f^\bigstar = f^! \circ \mathbb{D}.$

(m7) $\mathbf{D} \circ \mathrm{rat} = \mathrm{rat} \circ \mathbb{D}$, $f_* \circ \mathrm{rat} = \mathrm{rat} \circ f_{\bigstar}$, $f_! \circ \mathrm{rat} = \mathrm{rat} \circ f_!$, $f^* \circ \mathrm{rat} = \mathrm{rat} \circ f^{\bigstar}$, $f^! \circ \mathrm{rat} = \mathrm{rat} \circ f^!$.

(m8) If f is a projective morphism, then $f_{\bigstar} = f_!$.

Let $\mathcal{V} \in D^b MHM(X)$. We say that $\mathcal{V}$ has *mixed weights* $\leq n$ (resp. $\geq n$) if $\mathrm{gr}_i^W H^j \mathcal{V} = 0$ $(i > j + n)$ (resp. $\mathrm{gr}_i^W H^j \mathcal{V} = 0$ $(i < j + n)$). Also $\mathcal{V}$ is said to have a *pure weight* n if $\mathrm{gr}_i^W H^j \mathcal{V} = 0$ $(i \neq j + n)$ holds. Now we have the following results, which justify the name "theory of weights":

(m9) If $\mathcal{V}$ has mixed weights $\leq n$ (resp. $\geq n$), then $\mathbb{D}\mathcal{V}$ has mixed weights $\geq -n$ (resp. $\leq -n$).

(m10) If $\mathcal{V}$ has mixed weights $\leq n$, then $f_!\mathcal{V}$, $f^{\bigstar}\mathcal{V}$ have mixed weights $\leq n$.

(m11) If $\mathcal{V}$ has mixed weights $\geq n$, then $f_{\bigstar}\mathcal{V}$, $f^!\mathcal{V}$ have mixed weights $\geq n$.

Next we will explain the relation with the variations of mixed Hodge structures.

(m12) Let $\mathcal{H} \in VSHM(X)^p$. Then $\phi_X(\mathcal{H}) \in MF_{rh}W(D_X, \mathbb{Q})$ is an object of $MHM(X)$ if and only if $\mathcal{H}$ is *admissible* in the sense of Kashiwara [Kas13].

This implies in particular that $MHM(\mathrm{pt}) = SHM^p$ (the case when X is a one-point variety $\{\mathrm{pt}\}$).

Finally, let us describe the objects in $MH_Z(X, n)^p$ by using direct images (Z is an irreducible subvariety of X). Let U be a smooth open subset of Z and assume that the inclusion map $j : U \hookrightarrow X$ is an affine morphism. For $\mathcal{H} \in VSH(U, n)^p$, $j_!\phi_U^n\mathcal{H}$ and $j_*\phi_U^n\mathcal{H}$ are objects in $MHM(X)$ having weights $\leq n$, $\geq n$, respectively. Therefore, if we set

$$j_{!*}\phi_U^n\mathcal{H} = \mathrm{Im}\big[j_!\phi_U^n\mathcal{H} \longrightarrow j_*\phi_U^n\mathcal{H}\big],$$

then $j_{!*}\phi_U^n\mathcal{H}$ is an object in $MHM(X)$ having the pure weight $n + \dim Z$ (this functor $j_{!*}$ is a Hodge-theoretical version of the functor of minimal extensions defined in Section 8.2). Hence $\mathrm{gr}_{n+\dim Z}^W j_{!*}\phi_U^n\mathcal{H}$ is an object in $MH(X, n + \dim Z)^p$. Furthermore, it turns out that this object belongs to $MH_Z(X, n)^p$. Let us denote by $VSH(Z, n)_{\mathrm{gen}}^p$ the category of polarizable variations of Hodge structures of weight n defined on some smooth Zariski open subsets of Z. Then by the above correspondence, we obtain an equivalence

(m13) $VSH(Z, n)_{\mathrm{gen}}^p \xrightarrow{\sim} MH_Z(X, n + \dim Z)^p$.

of categories. Now consider the trivial variation of Hodge structures

$$(\mathcal{O}_{Z_{\mathrm{reg}}}, F, \mathbb{Q}_{Z_{\mathrm{reg}}^{\mathrm{an}}}) \in VSH(Z_{\mathrm{reg}}, 0)^p$$

defined over the regular part Z_{reg} of Z ($\mathrm{gr}_p^F = 0$ $(p \neq 0)$). Let us denote by IC_Z^H the corresponding object in $MH_Z(X, \dim Z)^p$ obtained by (m13). If we define the filtration W on it by the condition $\mathrm{gr}_n^W = 0$ $(n \neq \dim Z)$, then we see that $\mathrm{IC}_Z^H \in MHM(X)$. The underlying D-module (resp. perverse sheaf) of IC_Z^H is the minimal extension $L(Z_{\mathrm{reg}}, \mathcal{O}_{Z_{\mathrm{reg}}})$ (resp. the intersection cohomology complex $\mathrm{IC}(\mathbb{Q}_{Z_{\mathrm{reg}}^{\mathrm{an}}})$).

Also in the case of $Z = X$, we have $\mathrm{IC}_X^H = \mathbb{Q}_X^H[\dim X]$. Let $a_X : X \to \mathrm{pt}$ be the unique map and for $\mathcal{V} \in MHM(X)$ set

$$H^k(X, \mathcal{V}) = H^k\big((a_X)_*\mathcal{V}\big), \quad H_c^k(X, \mathcal{V}) = H^k\big((a_X)_!\mathcal{V}\big).$$

Then these are objects in $MHM(\mathrm{pt}) = SHM^p$. When X is a projective variety, by (m10) and (m11) we get in particular

$$H^k(X, \mathrm{IC}_Z^H) = H_c^k(X, \mathrm{IC}_Z^H) \in SH(k + \dim Z)^p.$$

This result shows that the global intersection cohomology groups of Z have Hodge structures with pure weights.