RIEMANN SURFACE PROBLEM LIST

There are close to 40 problems of various types. Most are elementary, but some are not. Take a look at the list, and find something that appeals to you and that you think you can do. Eventually I’m going to ask you to present either one of them or something of your own devising. The last option might be more appropriate for more advanced people. Whatever you do, you should be able to explain it.

1. Group actions and related problems

(1) The group $GL_2 \mathbb{C}$ acts on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ by fractional linear (or Möbius) transformations

$$z \mapsto \frac{az + b}{cz + d}$$

for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2 \mathbb{C} \) (the case of $z = \infty$ can be handled by continuity, or more formally pretend that $\infty = 1/0$.) $GL_2 \mathbb{C}$ acts on the projective line by

$$\begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \mapsto \begin{pmatrix} az_0 + bz_1 \\ cz_0 + dz_1 \end{pmatrix}$$

Show that these actions are compatible under the identification $\mathbb{C} \cup \{\infty\} \sim \mathbb{CP}^1$ given by $z \mapsto [z, 1]$ and $\infty \mapsto [1, 0]$.

(2) Show that the action of $GL_2(\mathbb{C})$ on $\mathbb{CP}^1$ is 3-transitive which means that given any two triples of distinct points $z_1, z_2, z_3$ and $z'_1, z'_2, z'_3$, there exists $A \in GL_2 \mathbb{C}$ which takes to $Az_i = z'_i$.

(3) Prove that the image of (real) line or circle under a fractional linear transformation is also either a line or a circle. (Hint: it’s probably easier if you treat the case of $z \mapsto az + b$ and $z \mapsto 1/z$ first.)

(4) A line or circle in $\mathbb{H}$ perpendicular to the real axis is called a geodesic. These play the role of “straight lines” in hyperbolic geometry. Prove that an element of $SL_2 \mathbb{R}$ takes a geodesic to a geodesic.

(5) (This is probably hard.) Let $\mathbb{C}(z)$ be the field of rational functions in one variable. Prove that the group of field automorphisms of $\mathbb{C}(z)$ fixing $\mathbb{C}$ is precisely $PGL_2 \mathbb{C} := GL_2 \mathbb{C}/\mathbb{C}^*$.

(6) Verify that the upper half plane $\mathbb{H} = \{z \in \mathbb{C} | \text{Im}(z) > 0\}$ and the unit disk $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ are isomorphic, as a Riemann surfaces, via $z \mapsto \frac{z - i}{z + i}$.

(7) Define the set

$$SU(1, 1) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \left| |a|^2 - |b|^2 = 1 \right. \right\}$$

Verify that $SU(1, 1) \subset GL_2 \mathbb{C}$ is a subgroup.
(8) Show that $SU(1,1)$ acts on $\mathbb{D}$ by fractional linear transformations
\[ z \mapsto \frac{az + b}{bz + \bar{b}} \]

(9) **Further topics**: Tell us more about hyperbolic geometry, modular curves...

2. **Algebraic curves**

(1) Let $y^2 = x^3 + 1$ be an affine curve. Is it nonsingular? What about its projective closure? Find all the singular points of the projective closure if the answer is no.

(2) Let $y^2 + x^3 = 1$ be an affine curve. Is it nonsingular? What about its projective closure? Find all the singular points of the projective closure if the answer is no.

(3) Let $x^2y^2 + x^2 + y^2 = 0$ be an affine curve. Is it nonsingular? What about its projective closure? Find all the singular points of the projective closure if the answer is no.

(4) Let $y^2 = x^6 - 1$ be an affine curve. Is it nonsingular? What about its projective closure? Find all the singular points of the projective closure if the answer is no.

(5) Let $y^2 = x^5 - 1$ be an affine curve. Is it nonsingular? What about its projective closure? Find all the singular points of the projective closure if the answer is no.

(6) Let $y^3 = x^3 - 1$ be an affine curve. Is it nonsingular? What about its projective closure? Find all the singular points of the projective closure if the answer is no.

(7) Let $y^3 = (x^2 + 1)^2(x^3 - 1)$ be an affine curve. Is it nonsingular? What about its projective closure? Find all the singular points of the projective closure if the answer is no.

(8) Let $y^7 = x^2(x - 1)$ be an affine curve. Is it nonsingular? What about its projective closure? Find all the singular points of the projective closure if the answer is no.

(9) Let $y^2 + x^3 + xy^3 = 0$ be an affine curve. Is it nonsingular? What about its projective closure? Find all the singular points of the projective closure if the answer is no.

(10) Given any nonconstant polynomial $f \in \mathbb{C}[x,y]$, prove that the set of zeros $V(f)$ is an infinite proper subset of $\mathbb{C}^2$.

(11) A polynomial $f \in \mathbb{C}[x,y]$ is called irreducible if cannot be factored into a product of nonconstant polynomials. Given an irreducible polynomial $f$, prove that the ring $\mathbb{C}[x,y]/(f)$, called the coordinate ring, is an integral domain, i.e. that it has no zero divisors. (Even if you know commutative algebra, you should be able to explain it in a way that doesn’t assume it.)

(12) Let $f \in \mathbb{C}[x,y]$ be irreducible, and let $X = V(f)$. The field of fractions of $\mathbb{C}[x,y]/(f)$ is called the function field of $X$, and it is denoted by $\mathbb{C}(X)$. Prove that $\mathbb{C}(X)$ has transcendence degree 1 over $\mathbb{C}$, i.e. that it is an algebraic extension of the field of rational function in one variable $\mathbb{C}(x)$.

(13) **Further topics**: Tell us about curves in higher dimensional projective spaces....
3. Valuations

Given a field $K$, a discrete valuation is a function $v : K^* \to \mathbb{Z}$ such that $v(xy) = v(x) + v(y)$ and $v(x + y) \geq \min(v(x), v(y))$. Let us say that $v$ is normalized if it is surjective. If $v$ is nonzero, then $\frac{1}{n}v$ is normalized if $n > 0$ is the degree of a generator of $v(K^*)$. It is convenient to set $v(0) = \infty$.

1. Given a complex number $a$, let $v_a : \mathbb{C}(x) \to \mathbb{Z}$ be defined by $v_a((x - a)^n f(x)/g(x)) = n$ where $f$ and $g$ are coprime to $(x - a)$. Check that this determines a discrete valuation.

2. Let $v_\infty : \mathbb{C}(x)^* \to \mathbb{Z}$ be defined by $v_\infty(f/g) = -v_0(f/g)$. Check that this determines a discrete valuation.

3. Show that any discrete valuation on $\mathbb{C}^*$ is zero.

4. Prove all the normalized discrete valuations on $\mathbb{C}(x)^*$ are the ones given in the first two exercises. (These are in one to one correspondence with points of $\mathbb{CP}^1$.)

5. Prove that the set $R = \{ x \in K \mid v(x) \geq 0 \}$ is a subring of $K$ called the valuation ring associated to $v$. Furthermore, it is principal ideal domain with a unique maximal ideal $m = \{ x \in K \mid v(x) > 0 \}$.

6. Prove the converse to the last problem that a principal ideal domain with a unique maximal ideal the ring $R = \{ x \in K \mid v(x) \geq 0 \}$ for a uniquely determined discrete valuation $v$.

7. Let $f \in \mathbb{C}[x, y]$ be a nonconstant irreducible polynomial, then $R = \mathbb{C}[x, y]/(f)$ is an integral domain (by a previous exercise). Let $K$ denote the field of fractions $R$. Show if $a \in V(f)$ is a nonsingular point, then there exists a discrete valuation $v_a$ on $K$ such that for $g \in \mathbb{C}[x, y]$ $v_a(\bar{g}) > 0$ precisely when $g(a) = 0$, where $\bar{g} \in R$ is the image.

8. Let $R$ be as above. Let $\bar{R}$ denote the intersection of all discrete valuation rings containing $R$. Prove that $\bar{R}$ is integrally closed which means that if $r$ in the field of fractions satisfies a monic polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0 \in R[z]$, then $r \in \bar{R}$.

9. Continuing the notation from above, show that $\bar{R}$ is the smallest integrally closed ring containing $R$. This is called the integral closure or normalization of $R$.

10. Further topics: Tell us about Dedekind domains, the class group...

4. Branched covers

1. Use the Riemann-Hurwitz formula to calculate the genus of the projective closure of $x^3 + y^3 = 1$. (Hint: Consider the projection to the $x$-axis.)

2. Use the Riemann-Hurwitz formula to calculate the genus of the projective closure of $x^4 + y^4 = 1$. (Hint: Consider the projection to the $x$-axis.)

3. Let a finite group $G$ of holomorphic automorphisms of a compact Riemann surface $X$. It is known that the set theoretic quotient (i.e. orbit space) $X/G$ admits the structure of a compact Riemann surface such that the canonical map $X \to X/G$ is holomorphic. Give a formula for the genus of $X/G$, assuming that $G$ acts without fixed points.
4 Riemann Surface Problem List

(4) Generalize the formula from the last exercise to the case where $G$ acts with fixed points. Test the formula for

$$G = \left\{ \begin{pmatrix} \exp(2\pi ik/n) & 0 \\ 0 & 1 \end{pmatrix} \mid k = 0, \ldots, n - 1 \right\} \subset GL_2(\mathbb{C})$$

acting on the projective line.

(5) Same instructions as the last one, but with the action of $G = \{\pm 1\}$ acting on the projective closure of $x^4 + y^4 = 1$ by $-1 : (x, y) \mapsto (-x, -y)$ on the affine part.

(6) This exercise assumes you know about the fundamental group. Given a distinct points $\{p_0, \ldots, p_n\} \subset \mathbb{CP}^1$, the fundamental group $\pi_1(\mathbb{CP}^1 - \{p_0, \ldots, p_n\})$ is generated by loops $g_i$ around the $p_i$ with a single relation $g_0g_1\ldots g_n = 1$. Thus it is generated freely by $g_1, \ldots, g_n$. By constructing a surjective homomorphism $\pi_1(\mathbb{CP}^1 - \{p_1, \ldots, p_n\}) \to S_n$ and some covering space theory, show that the symmetric group $S_n$ can made to act a compact Riemann surface. Calculate its genus.

(7) This exercise assumes that you know about homology and group representations. Suppose that a finit group $G$ acts without fixed points on a compact Riemann surface $X$. Calculate the character of the representation of $G$ on $H_1(X, \mathbb{C})$. (Hint: Use the Lefschetz trace formula.)

5. Function Theory

Fix $\tau \in \mathbb{H}$, the Jacobi theta function is given by the Fourier series

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i nz + \pi in^2 \tau} = \sum_{n=-\infty}^{\infty} e^{\pi n^2 \tau} e^{2\pi i nz}$$

(1) Show that the above series converges uniformly on compact sets. It follows that $\theta(z)$ is an entire function. You can assume that $\tau = i$ if it makes it easier.

(2) Show, at least formally, that $\theta(z + 1) = \theta(z)$ and $\theta(z + \tau) = e^{-\pi i \tau - 2\pi i z} \theta(z)$.

(3) Prove that the second logarithmic derivative is $f(z) = \frac{d^2}{dz^2} \log \theta(z)$ is nonconstant meromorphic function satisfying $f(z + 1) = f(z)$ and $f(z + \tau) = f(z)$. In other words, it is elliptic with respect to the lattice $\mathbb{Z} + \mathbb{Z} \tau$.

(4) Further topics: This barely scratched the surface about elliptic functions, theta functions, modular functions...