Chapter 2

Affine varieties (continued)

2.1 Products

For some problems its not very natural to restrict to irreducible varieties. So we broaden the previous story. Given an affine algebraic set $X \subset \mathbb{A}^n_k$, we can still define the coordinate ring $\mathcal{O}(X)$ as the ring of regular functions. We can again identify it with $\mathcal{O}(X) = k[x_1, \dots x_n]/\mathcal{I}(X)$. Since $\mathcal{I}(X)$ is radical, we get:

Lemma 2.1.1. *$\mathcal{O}(X)$ is reduced (which means that it does not have nonzero nilpotents).*

Proof. Suppose $\bar{r} \in \mathcal{O}(X)$ satisfied $\bar{r}^N = 0$. We can lift $\bar{r}$ to a polynomial $r \in k[x_1, \dots x_n]$ which would have to satisfy $r^N \in \mathcal{I}(X)$. Therefore $r \in \mathcal{I}(X)$, which implies $\bar{r} = 0$. $\square$

One can define a category of algebraic sets as before, and the duality theorem can be extended without much trouble.

Theorem 2.1.2 (Duality version 2)**.** *There is an anti-equivalence between the category of algebraic sets and reduced affine algebras.*

Given two affine algebraic sets $X \subset \mathbb{A}^n_k$ and $Y \subset \mathbb{A}^m_k$. We can form the cartesian product $X \times Y \subset \mathbb{A}^{n+m}_k$. We saw in an exercise that this is algebraic. In fact, we will do that part of it. Let $k[y_1, \dots, y_m]$ denote polynomials on the second $\mathbb{A}^m_k$. We have inclusions

$$k[x_1, \dots x_n] \xrightarrow{\iota_1} k[x_1, \dots, x_n, y_1 \dots, y_m] \xleftarrow{\iota_2} k[y_1, \dots y_m]$$

Expressing $X \times Y = X \times \mathbb{A}^m \cap \mathbb{A}^n \times Y$, shows that $X \times Y = V(\iota_1(\mathcal{I}(X)) + \iota_2(\mathcal{I}(Y))$. Therefore

Lemma 2.1.3. *$X \times Y$ is algebraic with ideal $\mathcal{I}(X) + \mathcal{I}(Y)$ (we suppress the ι's from now on).*

We can jazz this up by observing that

$$k[x_1, \dots, y_m] \cong k[x_1, \dots x_n] \otimes_k k[y_1, \dots, y_m]$$

where the isomorphism sends $f \otimes g$ on the right to the product fg on the left.

Lemma 2.1.4.

$$\mathcal{O}(X \times Y) \cong \mathcal{O}(X) \otimes_k \mathcal{O}(Y)$$

Proof. The left side is isomorphic to

$$k[x_1, \dots, y_m]/(\mathcal{I}(X)) + \mathcal{I}(Y))$$

from above. This is also isomorphic to the tensor product

$$k[x_1, \dots x_n]/\mathcal{I}(X) \otimes_k k[y_1, \dots, y_m]/\mathcal{I}(Y)$$

using the fact that $\otimes$ is right exact. □

We will give a second proof based on universal properties. First, let us consider the product $X \times Y$. The two projections $p : X \times Y \to X$ and $q : X \times Y \to Y$ are both morphisms. We have the following *universal* property.

Lemma 2.1.5. *If Z is any algebraic set equipped with morphisms $p' : Z \to X$ and $q' : Z \to Y$, there is unique morphism r making the diagram*

$$\begin{tikzcd}
Z \arrow[r, "p'"] \arrow[d, "q'"] \arrow[rd, dashed, "r"] & X \\
Y & X \times Y \arrow[l, "q"] \arrow[u, "p"']
\end{tikzcd}$$

commute.

Proof. The map $r(z) = (p'(z), q'(z))$ is clearly regular, and is the unique solution. □

At first, this may seem to be strange way to think about the product. But one can check that familiar examples of products of sets, groups, topological spaces ... are characterized this way. If we take a familiar category and turn the arrows around, we get the dual notion called a *coproduct*, which less familiar. Let us spell it out. Given objects X, Y in a category, their coproduct (if it exists) is an object $X * Y$ with morphisms $i : X \to X * Y$ and $j : Y \to X * Y$ such that any diagram with solid lines

$$\begin{tikzcd}
Z & X \arrow[l, "i'"'] \arrow[d, "i"] \\
Y \arrow[u, "j'"] \arrow[r, "j"] & X * Y \arrow[lu, dashed, "r"]
\end{tikzcd}$$

and be uniquely completed with the dotted line r. The coproduct of two sets is their disjoint union $X \coprod Y$. We come to the most important example for us.

Theorem 2.1.6. *If A and B are commutative k-algebras, their coproduct is the algebra $A \otimes_k B$ with $i(a) = a \otimes 1$ and $j(b) = 1 \otimes b$.*

Proof. Consider a diagram, as above, with A, B, C in place of X, Y, Z. Then $r(\sum a_i \otimes b_i) = \sum i'(a_i) j'(b_i)$ gives the required homomorphism. $\square$

We can now give a short second proof of lemma 2.1.4:

Second Proof. By duality, products of algebraic sets must correspond to coproducts of their coordinate rings. $\square$

2.2 Algebraic Groups

An algebraic group is to algebraic geometry what a Lie group is to differential geometry. More precisely, an affine[1] algebraic group over k is an affine algebraic set G which is also a group such that the group operations are regular maps. This means that the product

$$m : G \times G \to G,\ m(xy) = xy$$

and inverse

$$i : G \to G,\ i(x) = x^{-1}$$

are morphisms. One reason for not restricting to varieties is because of the following example.

Example 2.2.1. *Any finite group is an algebraic group.*

Example 2.2.2. *As we saw earlier, $G = GL_n(k)$ can be embedded as a closed set in $\mathbb{A}^{n^2+1}$ by sending $A \mapsto (A, 1/\det A)$. This means that $1/\det A$ is a regular function on G Matrix multiplication is clearly regular. So is $A \mapsto A^{-1} = \frac{1}{\det A} adj A$. Therefore G is an algebraic group*

To generate more examples, we observe

Lemma 2.2.3. *If G is an algebraic group and $H \subset G$ is a closed subgroup, then H is an algebraic group on its own.*

Example 2.2.4. *$SL_n(k) \subset GL_n(k)$ is a closed subgroup.*

Example 2.2.5. *The orthogonal group $O_n(k) = \{A \in GL_n(k) \mid A^{-1} = A^T\}$ is closed.*

The next example is tricky. The projective linear group

$$PGL_n(k) = GL_n(k)/k^*$$

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[1] We will suppress the word "affine" now, until we encounter nonaffine examples later such as elliptic curves.

where k^* is identified with the group of nonzero scalar matrices. There is no question it's a group, but it's not obvious that it is an affine algebraic set, let alone an algebraic group. So let us first construct try to the coordinate ring $S = \mathcal{O}(PGL_n(k))$.This can be identified with ring of regular functions on $GL_n(k)$ which is are constant on the orbits of k^*. This means that they are invariant under the action of k^*. We can make this explicit by observing that $R = \mathcal{O}(GL_n(k)) = k[x_{ij}, 1/D]$, where i, j range from $1, \ldots, n$ and $D = \det(x_{ij})$. An element $t \in k^*$ acts by $x_{ij} \mapsto tx_{ij}$. This means that S can be identified with the homogeneous elements of degree 0. A bit of thought shows that S is generated as a k-algebra by ratios m/D where m is a monomial of degree n. Therefore S is finitely generated. Since it is a subring of R, it is also a domain. Therefore S must be the coordinate of an algebraic variety, which, as we'll see in the exercises, is $PGL_n(k)$.This takes care of the first problem. However, we still need to make it into an algebraic group. Given an algebraic group G, $\mathcal{O}(G)$ is an affine algebra, with a co-multiplication

$$m^* : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$$

and a co-iniverse (usually called an antipode)

$$i^* : \mathcal{O}(G) \to \mathcal{O}(G)$$

and co-identity (or augmentation)

$$e^* : \mathcal{O}(G) \to k$$

Let explain what these mean, using g, h for elements of G and Greek letters for functions. If $e \in G$ is the identity, then $e^*(\phi) = \phi(e)$. The second is required to satisfy

$$(i^*\phi)(g) = \phi(g^{-1})$$

Finally suppose that

$$m^*(\phi) = \sum \tau_i \otimes \psi_i$$

Then we must have

$$\phi(gh) = \sum \tau_i(g)\psi_i(h)$$

These maps are required to satisfy various identities dual the axioms of a group. For example the associative law can be expressed as saying that the diagram

$$\begin{tikzcd}
G \times G \times G \arrow[r, "m\times 1"] \arrow[d, "1\times m"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}$$

commutes. So by duality, we have that the diagram

$$\begin{tikzcd}
\mathcal{O}(G) \arrow[r, "m*"] \arrow[d, "m^*"] & \mathcal{O}(G) \otimes \mathcal{O}(G) \arrow[d, "1\otimes m^*"] \\
\mathcal{O}(G) \otimes \mathcal{O}(G) \arrow[r, "m^*\otimes 1"] & \mathcal{O}(G) \otimes \mathcal{O}(G) \otimes \mathcal{O}(G)
\end{tikzcd}$$

should commute. $\mathcal{O}(G)$ together with these maps, satisfying all of these laws, is called a Hopf algebra. As a corollary of the duality theorem, we see that

Proposition 2.2.6. *Isomorphism classes of algebraic groups correspond to isomorphism classes of Hopf algebras.*

When $G = GL_n(k)$, we can see that on $R = \mathcal{O}(G)$

$$e^*(x_{ij}) = \delta_{ij}$$

$$i^*(x_{ij}) = \frac{1}{D}(ij\text{th cofactor of the generic matrix})$$

$$m^*(x_{ij}) = \sum_{\ell} x_{i\ell} \otimes x_{\ell j}$$

These are ring homomorphisms, so the above formulas determine these maps. The map i^* sends a homogeneous element of degree d to a homogeneous element of degree $-d$, and m^* sends a degree element to a degree $2d$ element. It follows that $S = \mathcal{O}(PGL_n(k))$ is preserved under these operations. Therefore it becomes a Hopf algebra. This completes the description of $PGL_n(k)$ as an algebraic group.

2.3 Chevalley's theorem, and algebraic groups

Before saying more about algebraic groups, we need to talk about regular maps in more detail. If $f : X \to Y$ is a regular map of algebraic sets, then we say that preimages of closed sets are closed. By contrast images are more complicated.

Example 2.3.1. *Let $\pi : \mathbb{A}^2 \to \mathbb{A}^1$ be projection onto x (using x, y as coordinates). Then $p(V(xy-1)) = A^1 - \{0\}$ is open, but $p(V(x)) = \{0\}$ is closed.*

In the above examples, images can be either open or closed. But for more general maps, we need a new concept. A set is locally closed if it is an open subset of a closed set. More generally a set is constructible if it is a finite union locally closed sets.

Theorem 2.3.2 (Chevalley)**.** *If $f : X \to Y$ is a regular map between affine algebraic sets, then images of constructible sets are constructible. In particular, $f(X)$ is constructible.*

We will just prove the last statement that $f(X)$ is constructible. Our proof will be long and meandering; it is meant to be instructive, rather than efficient. Let us make a preliminary reduction.

Lemma 2.3.3. *If the last part of theorem 2.3.2 is true for varieties, then it is true for algebraic sets.*

Proof. If $X = \bigcup X_i$ is a decomposition into irreducible components, then it suffices to prove that $f(X_i)$ is constructible. So we may as well assume that X is irreducible. Now suppose that $Y = \bigcup Y_i$ is a decomposition into irreducible components. Let $f_i = f|_{f^{-1}Y_i}$. Then $f(X) = \bigcup \operatorname{im} f_i$. So we may as well assume that Y is irreducible. □

So from now on until we say otherwise, assume that X and Y are irreducible, so that $A = \mathcal{O}(X)$ and $B = \mathcal{O}(Y)$ are domains. Then f corresponds to a homomorphism $\phi = f^* : B \to A$. Let us catalogue the various kinds of maps we will encounter.

A closed immersion is an inclusion $X \subset Y$ where X is closed. In terms of rings, we have the canonical map $B \to A = B/I$, where $I = \mathcal{I}(X)$. The converse is also clear. Therefore

Lemma 2.3.4. *The map f is a closed immersion if and only if $B \to A$ is surjective.*

A basic open immersion is an inclusion $X = D(g) \subset Y$. We saw earlier that $D(g)$ is affine. In fact $D(g)$ can be viewed as the closed subset $V(gt-1) \subset Y \times \mathbb{A}^1$ where t is the coordinate of the second factor. Therefore $A = B/(gt-1) \cong B[1/g]$.

Lemma 2.3.5. *The map f is a basic open immersion if and only if ϕ is an inclusion of the form $B \subset B[1/g]$.*

A map $f : X \to Y$ is dominant if $f(X)$ is dense.

Lemma 2.3.6. *f is dominant if and only if $B \to A$ is injective.*

Proof. If f is dominant, the nonzero regular function g on Y has nonzero restriction to $f(X)$, and therefore $f^*g \neq 0$.

In the other direction, if ϕ is not injective, it factors through $A/\ker\phi$. This implies that f factors through a closed immersion, so can't be dominant. □

In the next couple of items, we will work backwards from the algebra.

Lemma 2.3.7. *The map ϕ is the inclusion $B \subset A = B[x_1, \ldots, x_n]$ if and only if $f : X = \mathbb{A}^n_k \times Y \to Y$ is the projection.*

Proof. The key point is that if $X = \mathbb{A}^n \times Y$ then $A = \mathcal{O}(\mathbb{A}^n) \otimes_k B = k[x_1, \ldots x_n] \otimes_k B \cong B[x_1, \ldots, x_n]$. □

Recall that an inclusion $B \subset A$ is an integral extension if A is finite as a B-module.

Definition 2.3.8. *$f : X \to Y$ is called finite, if $B \subset A$ is an integral extension.*

A terminology is explained by the following

Lemma 2.3.9. *A finite map is surjective and set theoretically finite to one.*

Proof. Remember that points of X (or Y) correspond to maximal ideals of A (or B). If $m \in \operatorname{Max} B$, then the preimage $f^{-1}(m)$ can be identified with the set of maximal ideals of A containing m. This is nonempty by the so called "going up theorem". We can also idenity $f^{-1}(m) = \operatorname{Max} C$ where $C = (B/m \otimes_B A)$. This a finite dimensional k algebra, so it has only finitely many maximal ideals. $\square$

Here is the key proposition.

Proposition 2.3.10. *$f(X)$ contains an open subset of $\overline{f(X)}$.*

Proof. By factoring f as $X \to \overline{f(X)} \subseteq Y$, we can reduce to the case where f is dominant. So we have an inclusion $B \subseteq A$. Let K be the fraction field of B. Since A is finitely generated over k, $K \otimes_B A$ is necessarily finitely generated as a K-algebra. We can express the generators as a_i/b for some $b \in B$. By the Noether normalization, we can write $K \otimes_A B$ as an integral extension of a polynomial ring $K[x_1, \ldots, x_n]$. It follows that $A[1/b] = B[1/b] \otimes_B A$ is an integral extension of $B[1/b][x_1, \ldots, x_n]$. This implies that over $D(b)$, the restriction of f is the composite of finite map and a projection, so it is surjective. Therefore $D(b) \subseteq f(X)$. $\square$

To finish the proof, we use dimensions. Recall that $\dim X$ is the transcendence degree of the field of fractions of A. It is also equal to the Krull dimension of A. This implies that any (closed) subvariety of X has strictly small dimension.

Proof of last part lof theorem 2.3.2. We can assume, by induction, that the result holds when X is replaced by aset whose irreducible components have dimension less than $\dim X$. By the last proposition, $f(X) = U \cup Z$, where $U \subset \overline{f(X)}$ is open and $Z = (f(X) - U)$. The set U is locally closed in Y. By induction $Z = f|_{f^{-1}Z}$ is constructible. Therefore $f(X)$ is constructible. $\square$

Let us return to the study of algebraic groups. A homomorphism of algebraic groups is a homomorphism which is also regular.

Theorem 2.3.11. *Let G be an algebraic group*

- *(a) If $U, V \subseteq G$ are dense open sets, $U \cdot V = G$.*
- *(b) The closure of a subgroup is a subgroup*
- *(c) A constructible subgroup of G is a closed subgroup.*
- *(d) If $f : G_1 \to G_2$ is a homomorphism, then both $\ker f$ and $\operatorname{im} f$ are closed subgroups of G_1 and G_2 respectively.*

Proof. Suppose $g \in G$. Then gV^{-1} is dense open because $x \mapsto gx^{-1}$ is a homeomorphism. Therefore this has a nonempty intersection with U. Let $h \in U \cap gV^{-1}$. This means that $h = g\ell^{-1}$ for some $\ell \in V$. Therefore $g = h\ell \in UV$. This proves (a).

(b) Suppose that $H \subseteq G$ is a subgroup. Then since inversion is a homeomorphism which takes H itself, it takes $\overline{H}$ to itself. If $h \in H$, then left or right multiplication by h also a homeomorphism preserving H, $h\overline{H}, \overline{H}h \subset \overline{H}$. This implies that if $h \in \overline{H}$, then $hH \subset \overline{H}$ and $Hh \subset \overline{H}$. The same reasoning shows that $h\overline{H} \subset \overline{H}$ and $\overline{H}h \subset \overline{H}$. So $\overline{H}$ is a subgroup.

(c) Let H be a constructible subgroup. Then $\overline{H}$ is a subgroup by (b). We also know that H contains a subset U open and dense in $\overline{H}$. By (a), $U \cdot U = \overline{H}$. This forces $H = \overline{H}$.

(d) Finally if f is a homomorphism, it's continuous. Therefore $\ker f$ is closed. By Chevalley, $\operatorname{im} f$ is a constructible subgroup of G_2. So it's closed by (c).

□

2.4 Exercises

1. Prove that the product of two irreducible affine algebraic sets is irreducible.
2. Let $X \coprod Y$ denote the disjoint union of two affine algebraic sets. Give two proofs that $\mathcal{O}(X \coprod Y) = \mathcal{O}(X) \times \mathcal{O}(Y)$, an direct elementary proof, and a proof using universal properties.
3. Let $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ where the blocks $n \times n$. Prove that $Sp_{2n}(k) = \{A \in GL_{2n}(k) \mid A^T J A = J\}$ is a closed subgroup.
4. Given an algebraic group G, prove that the center $Z(G) = \{g \in G \mid \forall h \in G, hg = gh\}$ is a closed subgroup.
5. Give an example of a regular map of varieties, where the image is neither open nor closed.
6. Given an algebraic group G and an affine (for now) algebraic set X, we say that G acts X is there is a regular map

   $$\alpha : G \times X \to X$$

   satisfying the usual conditions for an action $\alpha(e, x) = x$ and $\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$. Usually, we just write $gx = \alpha(g, x)$. Given $x \in X$, show that its orbit $\{gx \mid g \in G\}$ is constructible
7. Show that the converse to lemma 2.3.9 fails.