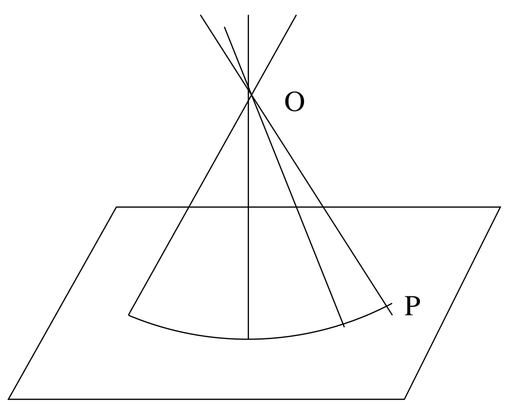
Chapter 3

Quasiprojective Varieties

3.1 Projective space

In Euclidean plane geometry, we need separate the cases of pairs of lines which meet and parallel lines which don't. Geometry becomes a lot simpler if any two lines met possibly "at infinity". There are various ways of arranging this, the most convenient method is to embed the $\mathbb{A}^2$ into 3 dimensional space as depicted. To each point $P \in \mathbb{A}^2$, we can associate the line OP. The lines parallel to the plane correspond to the points at infinity.



Let $k = \bar{k}$ as before. Projective space of dimension n, $\mathbb{P}^n_k$ is the set of lines through 0 in $\mathbb{A}^{n+1}$. Equivalently, it is the set of equivalence classes $\mathbb{P}^n_k = \mathbb{A}^{n+1} - \{0\}/\sim$, where $v \sim tv$, $t \in k^*$. If choose coordinates $x_0, \ldots, x_n$ on the affine space – called homogeneous coordinates, in this context – a point of $\mathbb{P}^n$ is represented by an equivalence class $[x_0, \ldots, x_n]$. We can embed $\mathbb{A}^n_k \subset \mathbb{P}^n_k$ by sending $(y_1, \ldots, y_n) \mapsto [1, y_1, \ldots, y_n]$. So the points in the complement, defined by $x_0 = 0$, are the points "at infinity".

We want to do algebraic geometry on projective space. Given $X \subset \mathbb{P}^n$, define the *cone* over X to be $Cone(X) = \pi^{-1}X \cup \{0\} \subseteq \mathbb{A}^{n+1}$. A subset of $\mathbb{A}^{n+1}$ of this form is called a cone. We define $X \subseteq \mathbb{P}^n$ to be *algebraic* iff $Cone(X)$ is algebraic in $\mathbb{A}^{n+1}$.

Lemma 3.1.1. *The collection of algebraic subsets are the closed sets for a Noetherian topology on $\mathbb{P}^n$ also called the Zariski topology. $\mathbb{A}^n \subset \mathbb{P}^n$ is an open subset.*

Let's make the notion of algebraic set more explicit. We will use variables $x_0, \dots x_n$. Thus X is algebraic iff $Cone(X) = V(S)$ for some set of polynomials in $k[x_0, \dots x_n]$. Let's characterize the corresponding ideals. Given a polynomial f, we can write it as a sum of homogeneous polynomials $f = f_0 + f_1 + \dots$. The f_i will be called the homogeneous components of f.

Lemma 3.1.2. *$I \subset k[x_0, \dots x_n]$ is generated by homogeneous polynomials iff I contains all the homogeneous components of its members.*

An $I \subset k[x_0, \dots x_n]$ is called *homogeneous* if it satisfies the above conditions

Lemma 3.1.3. *If I is homogeneous then $V(I)$ is a cone. If X is a cone, then $\mathcal{I}(X)$ is homogeneous .*

Proof. We will only prove the second statement. Suppose that X is a cone. Suppose that $f \in \mathcal{I}(X)$, and let f_n be its homogenous components. Then for $a \in X$,

$$\sum t^n f_n(a) = f(ta) = 0$$

which implies $f_n \in \mathcal{I}(X)$. □

We let $V_{\mathbb{P}}(I)$ denote the image of $V(I) - \{0\}$ in $\mathbb{P}^n$. Once again, we will revert to assuming k is algebraically closed. Then as a corollary of the weak Nullstellensatz, we obtain

Theorem 3.1.4. *If I is homogeneous , then $V_{\mathbb{P}}(I) = \emptyset$ iff $(x_0, \dots x_n) \subseteq \sqrt{I}$.*

The ideal $(x_0, \dots x_n)$ is often an exception to various statements and constructions in projective geometry. It referred to as the irrelevant ideal. Given a closed subset $X \subset \mathbb{A}^n$, we let $\overline{X} \subset \mathbb{P}^n$ denote its closure. Let us describe this algebraically. Given a polynomial $f \in k[x_1, \dots x_n]$, it homogenization (with respect to x_0) is

$$f^H = x_0^{\deg f} f\left(\frac{x_1}{x_0}, \dots \frac{x_n}{x_0}\right)$$

The inverse operation is $f^D = f(1, x_1, \dots x_n)$. The second operation is a homomorphism of rings, but the first isn't. We have $(fg)^H = f^H g^H$ for any f, g, but $(f+g)^H = f^H + g^H$ only holds if f, g have the same degree. We extend this to ideals

$$I^H = \{f^H \mid f \in I\}$$
$$I^D = \{f^D \mid f \in I\}$$

For a principal ideal I^H is obtained by homogenizing the generator, but the corresponding statement is usually *not true* for more general ideals (see exercises).

Theorem 3.1.5. $\overline{V(I)} = V_{\mathbb{P}}(I^H)$.

3.2 Quasiprojective varieties

A *projective variety* is an irreducible algebraic subset of some $\mathbb{P}^n$. An open subset of a projective variety is called a *quasiprojective variety.* For example, an affine variety is quasiprojective because it's an open subset of its closure in projective space.

Given $\mathbb{P}^n$ with homogeneous coordinates, let U_i the set where $x_i \neq 0$. Let us call this the standard open cover. We can identify

$$\mathbb{A}^n \cong U_i,$$

$$(t_1, \dots, t_n) \mapsto [t_1, \dots, 1 \text{ (ith place)}, \dots, t_n]$$

$$[x_0, \dots, x_n] \mapsto (\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}) \text{ (skip ith term)}$$

This is in fact a homeomorphism. Given a projective variety $X \subset \mathbb{P}^n$, we can restrict this to get an open cover $X_i = X \cap U_i$. Notice that under the above identification, $X_i \subset \mathbb{A}^n$ is closed and irreducible. More generally, if $X \subset \overline{X} \subset \mathbb{P}^n$ is quasiprojective, $X_i \subset \overline{X_i}$ is open. So we can cover X_i by basic open sets $D(f_{ij})$, which are affine. Therefore a quasiprojective variety is locally an affine variety.

We can use this to define regular functions quasiprojective varieties. We temporally distinguish between new and old meaning of regular. A function $f : X \to k$ is locally regular if and only X has open affine cover, as above, such that $f|_{U_i}$ is regular.

Lemma 3.2.1. *When X is affine, f is locally regular if and only its regular.*

Proof. Clearly regular implies locally regular. Suppose f is locally regular. There is an open affine cover $U_i = D(g_i)$ (which we can assume finite) such that $f = \mathcal{O}(U_i) = \mathcal{O}(X)[1/g_i]$. This means that $f = f_i/g_i^{n_i}$ for some $f_i \in \mathcal{O}(X)$. We can assume that all the $n_i = 1$ by replacing g_i by $g_i^{n_i}$. Since $\bigcup D(g_i) = X$, there exists $h_i \in \mathcal{O}(X)$ such that $\sum h_i g_i = 1$ by the Nullstellensatz. Therefore

$$f = \sum \frac{h_i g_i f_i}{g_i} = \sum h_i f_i$$

is regular. □

From now on, we treat regular and locally regular as the same. We write $\mathcal{O}(X)$ for the algebra – which it clearly is – of regular functions on a quasiprojective variety X.

Example 3.2.2. *Let us calculate $\mathcal{O}(\mathbb{P}^1_k)$. A regular function restricts to a regular function on U_0 and U_1, which agree on the intersection. Therefore*

$$\mathcal{O}(\mathbb{P}^1) = k[x_0/x_1] \cap k[x_1/x_0] = k$$

This might seem strange, but it is consistent with what you get from complex analysis when $k = \mathbb{C}$. $\mathbb{P}^1_{\mathbb{C}}$ is just the Riemann sphere. A regular function would be entire and bounded (by compactness of the sphere), and therefore constant by Liouville's theorem. More generally,

Example 3.2.3. $\mathcal{O}(\mathbb{P}^n_k) = k$. *The quickest way to see this is to observe that any two points* p, q *can be connected by a line* $\cong \mathbb{P}^1$*, so a regular function must take the same value at* p *and* q*. (On might complain that this really only a sketch, so see the exercises.)*

It is clear from these examples, that the ring of regular functions on all of X is not a particularly useful invariant for a nonaffine variety. What is more useful is to consider all the $\mathcal{O}(U)$'s as U runs over open sets. Let us introduce some terminology. A concrete presheaf of rings of k-valued functions on a topological space X, is a collections of subrings $F(U) \subset Hom_{sets}(U, k)$ which is closed under restriction. We say that F is a concrete sheaf of rings if it has the following patching property: for any open cover U_i of U, $f \in F(U)$ if and only if $f|_{U_i} \in F(U_i)$. We will consider more general presheaves and sheaves later on, but for now we use these words to mean (pre)sheaves of this type.

Example 3.2.4. *Given any topological space* X*, the collection of continuous real or complex valued functions forms a sheaf. Here* $k = \mathbb{R}$ *or* $\mathbb{C}$.

Example 3.2.5. *Given* $X = \mathbb{R}^n$ *or a* C^∞ *manifold, the collection of* C^∞ *real or complex valued functions forms a sheaf.*

The main example for us, which follows from the previous discussion, is:

Example 3.2.6. *If* X *is a quasiprojective variety over* k*, the collection of regular functions forms a sheaf. We usually denote this by* $\mathcal{O}_X$ *to indicate the variety.*

3.3 Morphisms

Earlier we defined a regular map or morphism between varieties, as a map given by a single polynomial formula.

$$F(a) = (f_1(a), \ldots, f_n(a))$$

If we tried to exactly copy this definition for quasiprojective varieties, we end up with something too restrictive.

Example 3.3.1. *Suppose that* $char\, k \neq 2$*. Consider the conic* $X = V_{\mathbb{P}}(x_0^2 + x_1^2 - x_2^2) \subset \mathbb{P}^2$*. Let* $V_1 = X - \{[0,1,1]\}$ *and* $V_2 = X - \{[0,-1,1]\}$*. Consider the projection* $p : X \to \mathbb{P}^1$ *given by*

$$p([x_0, x_1, x_2]) = \begin{cases} [x_0, x_2 - x_1] & \text{if } [x_0, x_1, x_2] \in V_1 \\ [x_2 + x_1, x_0] & \text{if } [x_0, x_1, x_2] \in V_2 \end{cases}$$

To see that this gives a well defined function, observe that on the intersection on their domains $V_1 \cap V_2$

$$[x_0, x_2 - x_1] = [x_0(x_2 + x_1), x_2^2 - x_1^2] = [x_0(x_2 + x_1), x_0^2] = [x_2 + x_1, x_0]$$

The previous example, which is fairly typical, suggests we must allow a definition by cases. To be precise, we could define $f : X \to Y$ to be regular, if there is an open cover of $\{X_i\}$ of X and another cover $\{Y_j\}$ of Y, such that each $f(X_i)$ lies in some Y_j and the map $X_i \to Y_j$ is defined by polynomials. This is what we do in practice, but it is not very elegant. So let us give somewhat cleaner definition which is equivalent. First, we point out:

Lemma 3.3.2. *If* $F : X \to Y$ *is a map between affine varieties, and* $F^* f := f \circ F$ *is regular for every* $f \in \mathcal{O}(Y)$*, then* F *is regular.*

Proof. Exercise. □

Here is our official definition: a map $F : X \to Y$ between quasiprojective varieties is *regular* or a *morphism* if

1. F is continuous.
2. For every open set $U \subseteq Y$, $F^*(\mathcal{O}_Y(U)) \subset \mathcal{O}_X(F^{-1}U)$.

One see almost immediately that:

Lemma 3.3.3. *The composite of two regular maps is regular.*

Thus we can form a category of quasiprojective varieties. In particular, we can define an isomorphism as before. Pretty much by definition, $X \mapsto \mathcal{O}(X)$ gives a contravariant functor from the category of quasiprojective varieties to k-algebras. We also have products, as before:

Theorem 3.3.4. *The cartesian product* $X \times Y$ *of two quasiprojective varieties can be made into a quasiprojective variety satisfying the standard universal property.*

We will discuss special cases in the exercises.

3.4 Projective coordinate ring

Given a projective variety $i : X \subset \mathbb{P}^n$. The projective coordinate ring

$$S(X, i) = \mathcal{O}(Cone(X)) = k[x_0, \ldots x_n]/\mathcal{I}(Cone(X))$$

We can identify $\mathcal{I}(Cone(X)) = \mathcal{I}(X)$ with the ideal generated by all homogeneous polynomials vanishing on X. We will usually just write $S(X)$, when i is understood. Since $\mathcal{I}(X)$ is a homogeneous ideal, $S(X)$ has a graded ring such that elements of degree 1 generate it as an algebra.

Unlike the affine case, $S(X, i)$ really depends on the embedding (see exercises), so there is no (contravariant) functor $X \mapsto S(X)$ on projective varieties, let alone an anti-equivalence of any kind. Although there is something that we can say.

Lemma 3.4.1. *Given $S = S(X, i)$ as a graded algebra, we can recover X up to isomorphism.*

Proof. By definiton, S comes with a surjective map $f : k[x_0, \dots x_n] \to S$. If we just know S as a graded algebra, then we can proceed by choosing a basis of the degree 1 part of S. This induces a surjective map of graded algebras $f' : k[x_0, \dots x_n] \to S$. The two maps f, f' would agree after a linear change of variables

$$x_i = L_i(x_0, \dots, x_n)$$

and this shows that there is an automorphism of $\mathbb{P}^n$ taking $V_{\mathbb{P}}(\ker f)$ to X. In fact, we proved more than what was stated. We can essentially recover the embedding $X \subset \mathbb{P}^n$ as well. □

We can describe the points of X more directly. Let $\operatorname{Proj} S(X)$ denote the set of homogeneous prime ideals of $S(X)$, different from the irrelevant ideal, ordered by inclusion.

Lemma 3.4.2. *The points of $S(X)$ correspond to maximal elements of $\operatorname{Proj} S(X)$.*

Proof. A point corresponds to a line through 0 in $Cone(X)$, or a minimal irreducible subcone. This is maximal element of $\operatorname{Proj} S(X)$. □

3.5 Exercises

1. Check that a Noetherian topological space is compact[1], and conclude that every quasiprojective variety has a finite open affine cover.

2. Consider the map $f : \mathbb{A}^1 \to \mathbb{A}^3$ given by $f(t) = [t, t^2, t^3]$. The image is closed and the ideal is clearly given by $(x_2 - x_1^2, x_3 - x_2^3)$. Show if we homogenize these generators, then they *do not* generate the ideal of the closure $\overline{f(\mathbb{A}^1)} \subset \mathbb{P}^3$.

3. Show that map $p : X = V_{\mathbb{P}}(x_0^2 + x_1^2 - x_2^2) \to \mathbb{P}^1$ constructed in example 3.3.1 is an isomorphism. Thus we have an inclusion $i : \mathbb{P}^1 \to \mathbb{P}^2$, with image X. Let $id : \mathbb{P}^1 \to \mathbb{P}^1$ denote the identity. Show that the rings $S(\mathbb{P}^1, i)$ and $S(\mathbb{P}^1, id)$ are not isomorphic (a) as graded rings, or (b) in the ungraded sense.

4. Given $\mathbb{P}^n$ and $\mathbb{P}^m$ with homogeneous coordinates $x_0, \dots, x_n$ and $y_0, \dots, y_m$, embed the cartesian product $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$ by

   $$\sigma([x_0, x_1 \dots], [y_0, \dots]) = [x_0 y_0, x_1 y_0, \dots] \text{ (the order is unimportant)}$$

   Check that this is injective, and that the image $S = \operatorname{im} \sigma$ is a projective variety. This is called the Segre embedding.

[1] For us this means that any open cover has a finite subcover. Some people, especially in algebraic geometry, call this quasicompactness, and reserve "compact "for spaces which are also Hausdorff.