Chapter 9

Differential Forms

9.1 Differential forms on affine varieties

We can extend standard constructions from calculus to algebraic varieties. Let k be an algebraically closed field. A differential or a 1-form on $\mathbb{A}^n_k$ is an expression

$$\alpha = \sum_{i=1}^{n} f_i dx_i$$

where $f_i \in R = \mathcal{O}(X) = k[x_1, \dots x_n]$. Let $\Omega^1_{R/k} = \Omega_R$ denote the set of these. It is an R-module in an obvious way:

$$\sum f_i dx_i + \sum g_i dx_i = \sum (f_i + g_i) dx_i$$

$$r(\sum f_i dx_i) = \sum r f_i dx_i$$

In fact, it is a free module with basis dx_i. Ω_R is called the module of Kähler differentials

Given$f \in R$, define the (exterior) derivative

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

The following properties are easy to check

Lemma 9.1.1. *d is a k-linear derivation, which means that:*

(a) d is k-linear.

(b) The Leibnitz rule holds, i.e. $d(fg) = fdg + gdf$.

To generalize this, we explain the universal property:

Lemma 9.1.2. *Given an R-module M and a k-linear derivation $\delta : R \to M$, there exists, a unique R-linear map $\phi : \Omega_R \to M$ such that $\delta(r) = \phi \circ dr$. To put it more succinctly, $d : R \to \Omega_R$ is the universal k-linear derivation.*

Proof. One can see that $\phi(\sum f_i dx_i) = \sum f_i \delta x_i$ works. ◻

Theorem 9.1.3. *Given a k-algebra R, there exists a R-module Ω_R with universal k-linear derivation $d : R \to \Omega_R$. (Ω_R is called the module of Kähler differentials.)*

Proof. There are a few different ways to do this. We give a construction, which works for affine algebras, and is actually more complicated than the general constructions. However, it gives a bit more information, which will be useful. Let $R = S/(f_1, \dots, f_m)$, where $S = k[x_1, \dots x_n]$. Let $q : S \to R$ be the quotient map. Define

$$\Omega'_R = \Omega_S/\langle df_1, \dots, df_n\rangle$$

where the notation $\langle df_1, \dots\rangle$ means the S-submodule generated by these elements. We can force this to be an R-module by setting

$$\Omega_R = R \otimes_S \Omega'_R = \Omega_S/\langle df_i, f_i dx_j\rangle \tag{9.1}$$

Given an element $r \in R$, we can find a polynomial $s \in S$ mapping to it under q. Then

$$dr = \text{image of } ds$$

can be easily seen to be a well defined operation.

Note that d has two different meanings, but it should be clear which is which. It is easy to see that the newly defined d is k-linear and a derivation. Universality is a bit less straightforward, so we will check it. Suppose that $\delta : R \to M$ is a k-linear derivation to an R-module. We can view M as an S-module via restriction of scalars. Let $\delta' = \delta \circ q$. This is a k-linear derivation $\delta' : S \to M$ with the special property

$$\delta' f_i = 0 \tag{9.2}$$

By the universal property for Ω_S, we get a homomorphism $\phi' : \Omega_S \to M$ such that $\delta' = \phi' \circ d$. Using (9.2) and the fact that M is an R-module, we obtain that ϕ' factors through a homomorphism $\phi : \Omega_R \to M$ such that $\delta = \phi \circ d$. ◻

Using equation (9.1), we an explicit method for calculating Ω_R.

Corollary 9.1.4. *With notation as in the proof. Let $J = (\frac{\partial f_i}{\partial x_j})$ be the Jacobian matrix arranged as an $n \times m$ matrix. Then there is an exact sequence*

$$R^m \xrightarrow{J} R^n \to \Omega_R \to 0$$

9.2 Tangent bundle for affine varieties

Let $X = V(f_1, \dots, f_m) \subset \mathbb{A}^n_k$, $R = \mathcal{O}(X)$, and $J = (\frac{\partial f_i}{\partial x_j})$ be the $n \times m$ Jacobian. Let $p \in X$. We also denote the associated maximal ideal by $p \in \operatorname{Max} R$. This should hopefully not cause too much confusion. Given $f \in R$, evaluation $f(p) \in$

k can be identified with the image of f in the residue field $k(p) = R/p \cong k$. The last isomorphism follows from the Nullstellensatz because k is algebraically closed.

Proposition 9.2.1. *The tangent space T_pX is isomorphic to the dual of $k(p) \otimes_R \Omega_R$*

First Proof. Earlier, we saw that the tangent space T_pX is the kernel of $J(p)^T$, or in other words, that

$$0 \to T_pX \to k^n \xrightarrow{J(p)^T} k^m \tag{9.3}$$

Tensoring the sequence in corollary by $k(p)$, gives

$$k^m \xrightarrow{J(p)} k^n \to k(p) \otimes_R \Omega_R \to 0$$

which is dual to (9.3). □

Second Proof. We can give a more conceptual proof as follows. First recall that the $T_pX \cong (p/p^2)^*$. By the universal property, $(k(p) \otimes_R \Omega_R)^*$ is the space of k-linear derivations from $\delta : R \to k(p)$. Given such a δ, it extends to a derivation from the local ring $R_p \to k(p)$. If $f, g \in p$, then $\delta(fg) = f(p)\delta g + g(p)\delta f = 0$. Therefore $\delta|_{p^2} = 0$. So δ induces a linear map $p/p^2 \to k$, or in other words an element of T_pX. Thus we have defined a map

$$(k(p) \otimes_R \Omega_R)^* \to T_pX$$

To see that this is an isomorphism, we construct the inverse. Given $\lambda \in T_pX$, $\delta(f) = \lambda(f - f(p))$ is an element of $(k(p) \otimes_R \Omega_R)^*$. □

We define the tangent module $T_R = Hom_R(\Omega_R, R)$. By the universal property, we can see that

Lemma 9.2.2. *T_R is the R-module of k-linear derivations from $\delta : R \to R$.*

Earlier, we defined the notion of locally free module M For example, it means that all the localizations M_p are free. We want to give a numerical criterion.

Proposition 9.2.3. *Suppose that M is a finitely module over a noetherian local domain R, with residue field k and fraction field K. Then*

$$\dim k \otimes_R M \geq \dim K \otimes_R M$$

with equality if and only if M is free.

Proof. Choose elements m_i such that the reductions a basis $\bar{m}_i$ for $k \otimes M$. Then m_i will be generate M by Nakayama's lemma. Therefore m_i will span $K \otimes M$, and this implies the inequality.

If $M = R^n$, then

$$\dim k \otimes_R M = n = \dim K \otimes_R M$$

Conversely, suppose that

$$\dim k \otimes_R M = \dim K \otimes_R M$$

Choose elements $m_1, \ldots, m_n$ as above, where $n = \dim k \otimes M$. Then we have a surjection $f : R^n \to M$ sending ith basis vector $e_i \mapsto m_i$. By assumption $K \otimes f$ is injective. Therefore f is injective, and consequently an isomorphism. $\square$

Corollary 9.2.4. *A finitely generated module M over a noetherian domain R is locally free if an only if $p \mapsto \dim k(p) \otimes_R M$ is constant for all $p \in \operatorname{Spec} R$.*

Remark 9.2.5. *One can also show that it is enough check this for maximal ideals $p \in \operatorname{Max} R$.*

The value of $\dim k(p) \otimes_R M$ is called the rank. Putting together the previous corollary with proposition 9.2.1 implies

Proposition 9.2.6. *Let X be an d dimensional affine variety with coordinate ring R. Then X is nonsingular if and only if Ω_R is locally free of rank d.*

When X is nonsingular, it follows that T_R is also locally free of rank d, and that $\Omega_R = Hom_R(T_R, R)$. The Zariski-Lipman conjecture asserts that when $char\, k = 0$, T_R locally free should imply X nonsingular. It's false in positive characteristic (see exercises).

Let's assume $X \subseteq \mathbb{A}^n$ is nonsingular. We define the tangent bundle

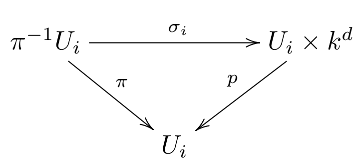
$$TX = \{(p, v) \in X \times k^n \mid v \in T_pX\}$$

We have a morphism $\pi : TX \to X$ given by projection, and the fibre over p is precisely T_pX. A section is morphism $s : X \to TX$ such that $\pi \circ s = id$. Thus we can view s as assignment of a vector $s(p) \in T_pX$ to each $p \in X$. In other words, s is a vector field. The set of sections is an R-module: Given two sections s_1, s_2 and $r_1, r_2 \in R$, $(r_1 s_1 + r_2 s_2)(p) = r_1(p)s_1(p) + r_2(p)s_2(p)$.

Theorem 9.2.7. *Suppose that X is nonsingular of dimension d.*

(a) T_R is isomorphic to the module of sections of TX.

(b) There exists an open cover $\{U_i\}$ of X and linear isomorphisms σ_i as indicated such that



commutes.

Sketch. Given $\delta \in T_R$, define $\delta(p) \in T_R$ as the composite $R \to R \to k(p)$. This defines an isomorphism. Since X is nonsingular, there exists an open cover $U_i = D(f_i)$ of X with isomorphisms $T_R[1/f_i] \cong R^d$. These can be converted to isomorphisms σ_i above.

□

A rank n vector bundle over a variety X is a variety a V together with a morphism $\pi : V \to X$ satisfying the conditions such that the fibres are n dimensional vector spaces and condition (b) of the theorem (called local triviality) holds. Thus $TX \to X$ is a rank d a vector bundle.

9.3 Exercises

Exercise 9.3.1. *In the exercises below $R = \mathcal{O}(X)$.*

1. *Let $X = \mathbb{A}^1_k - \{0\}$. Calculate Ω_R.*
2. *Define the* 0*th de Rham cohomology $H^0_{dR}(X) = \{f \in R \mid df = 0\}$. Show that if char $k = 0$, then $\dim H^0_{dR}(X) = 1$ when $X = \mathbb{A}^n_k$ or $X = \mathbb{A}^1_k - \{0\}$. But $\dim H^0_{dR}(X) = \infty$, for these same examples in positive characteristic.*
3. *Calculate Ω_R for the cusp $X = V(y^2 - x^3)$ (assume char $k \neq 2, 3$ if necessary), and show that it has torsion, and is consequently not reflexive.*
4. *When char $k = 2$ and $R = k[x, y, z]/(z^2 + xy)$, calculate Ω_R and $T_R = \Omega^*_R$. Show that the last module is locally free.*
5. *Prove that for a* variety *X, Ω_R locally free implies that X is nonsingular.*