Chapter 1

Basic curve theory

1.1 Hyperelliptic curves

As all of us learn in calculus, integrals involving square roots of quadratic polynomials can be evaluated by elementary methods. For higher degree polynomials, this is no longer true, and this was a subject of intense study in the 19th century. An integral of the form

$$\int \frac{p(x)}{\sqrt{f(x)}}dx \tag{1.1}$$

is called elliptic if $f(x)$ is a polynomial of degree 3 or 4, and hyperelliptic if f has higher degree, say d.

It was Riemann who introduced the geometric point of view, that we should really be looking at the algebraic curve X^o defined by

$$y^2 = f(x)$$

in $\mathbb{C}^2$. When $f(x) = \prod_0^d (x - a_i)$ has distinct roots (which we assume from now on), X^o is nonsingular, so we can regard it as a Riemann surface or one dimensional complex manifold. *Since surfaces will later come to mean two dimensional complex manifolds, we will generally refer to this as a (complex nonsingular) curve.* It is convenient to add points at infinity to make it a compact complex curve X called a (hyper)elliptic curve. One way to do this is to form the projective closure

$$\bar{X}^o = \{[x, y, z] \in \mathbb{P}^2 \mid F(x, y, z) = 0\}$$

where

$$F(x, y, z) = y^2 z^{d-2} - \prod (x - a_i z) = 0$$

is the homogenization of $y^2 - f(x)$. Unfortunately, $\bar{X}^o$ will usually be singular. To see this, let's switch to the affine chart $y = 1$. Then $\bar{X}^o$ is given by

$$z^{d-2} - \prod (x - a_i z) = 0$$

The partials vanish at $x = 0, y = 0$, as soon as $d > 3$.

We have to perform another operation on $\bar{X}^o$ which is called *resolving singularities* to obtain a nonsingular projective curve X containing X^o. We give two constructions, both with their advantages and disadvantages. The first is a general procedure called *normalization*. If A is an integral domain with fraction field K, its normalization or integral closure is the ring

$$\tilde{A} = \{a \in K \mid \exists \text{ monic } f(t) \in A[t], f(a) = 0\} \supseteq A$$

Suppose that X is an algebraic variety (or integral scheme) then X is obtained by gluing affine varieties U_i (or schemes) with coordinate rings A_i. Let $\tilde{U}_i = Spec\tilde{A}_i$. Then these can be glued to get a new variety/scheme $\tilde{X}$ called the normalization. See Mumford's Red Book for details. The normalization comes with a morphism $\tilde{X} \to X$.

Here is an example. Let $A = k[x, y]/(y^2 - x^3)$ be the coordinate ring of a cusp over a field k. Then $y/x \in \tilde{A}$ because it satisfies $t^2 - x = 0$. With more work, we can see that $\tilde{A} = k[y/x]$. In general, by standard commutative algebra

Theorem 1.1.1. *If A is a one dimensional integral domain, then $\tilde{A}$ is Dedekind domain; in particular the localizations of $\tilde{A}$ at maximal ideals are discrete valuation rings and therefore regular.*

Corollary 1.1.2. *If X is a curve i.e. one dimensional variety, then $\tilde{X}$ is a nonsingular curve with the same function field as X.*

Returning to the original problem. Given $\bar{X}^o$ as above, X can be simply be taken to be it's normalization. Unfortunately, this process is not very geometric. So we briefly describe another procedure. The blow up of the affine plane $\mathbb{A}^2_k$ at the origin is the quasiprojective variety

$$B = Bl_0\mathbb{A}^2 = \{(v, \ell) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid v \in \ell\}$$

$$= \{(x, y, [X, Y]) \mid \mathbb{A}^2 \times \mathbb{P}^1 \mid xY = Xy\}$$

This comes with a projection $\pi : B \to \mathbb{A}^2$ which is an isomorphism over $\mathbb{A}^2 - \{0\}$. The blow of the projective plane can be defined as the closure of B in $\mathbb{P}^2 \times \mathbb{P}^1$. Using the Segre embedding, we can see that this is a projective variety.

Given a curve $C \subset \mathbb{A}^2$ (or $\mathbb{P}^2$), the closure C_1 of $\pi^{-1}C - \{0\}$ is the blow up of C. This is also called the strict transform of C. Let's calculate this for the cusp $y^2 - x^3 = 0$ in $\mathbb{A}^2$. Then B is covered by open sets where $X = 1$ and $Y = 1$. The intersection of C_1 with $X = 1$ is the irreducible component of

$$\{y^2 = x^3, xY = y\} \Leftrightarrow \{x(Y^2 - x) = 0, y = xY\}$$

dominating C. This is the locus $Y^2 - x = 0$, which is nonsingular. In general, the process many steps. For the example, $y^2 = x^5$, the first step produces $C_1 : Y^2 - x^3 = 0$, which is "less singular" than before. Blowing it up a second time at $x = Y = 0$, yields a nonsingular curve C_2.

Theorem 1.1.3. *Given a curve X, after finite number of blow ups, we obtain a nonsingular curve $\tilde{X} \to X$. This coincides with the normalization.*

Corollary 1.1.4. *$\tilde{X}$ can be embedded into a projective space (generally bigger than $\mathbb{P}^2$).*

1.2 Topological genus

Let's work over $\mathbb{C}$. Then a nonsingular projective curve $X \subset \mathbb{P}^n_{\mathbb{C}}$ can be viewed as complex submanifold and therefore a Riemann surface.[1] In particular, X can be viewed a compact C^∞ 2 (real) dimensional manifold. The fact that X is complex implies that it is orientable. Before describing the classification, recall that the give two connected n-manifolds X_1, X_2, their connected sum $X_1 \# X_2$ is obtained by removing an n-ball from both and joining them by the cylinder $S^{n-1} \times [0,1]$.

Theorem 1.2.1. *Any compact connected orientable 2-manifold is homeomorphic to either the two sphere S^2 or connected sum of g 2-tori for some integer $g > 0$.*

The integer g is called the *genus*. However, we will refer to as the topological genus temporarily until we have established all the properties. We set $g = 0$ in the case of S^2. Let's relate this to more familiar invariants. Recall that the Euler characteristic $e(X)$ of a (nice) topological space is the alternative sum of Betti numbers. When the space admits a finite triangulation, then it is number of vertices minus the number of edges plus ... We can triangulate S^2 as a tetrahedron, therefore $e(S^2) = 4 - 6 + 4 = 2$. In general, we can compute the Euler characteristic using the following inclusion-exclusion formula:

Proposition 1.2.2. *If $X = U \cup V$ is a union of open sets, $e(X) = e(U) + e(V) - e(U \cap V)$.*

Proof. In general, given an exact sequence of vector spaces

$$\dots V^i \to V^{i+1} \to \dots$$

$$\sum (-1)^i \dim V^i = 0$$

Now apply this to the Mayer-Vietoris sequence

$$\dots H^i(X) \to H^i(U) \oplus H^i(V) \to H^i(U \cap V) \to \dots$$

where $H^i(X) = H^i(X, \mathbb{C})$. □

Proposition 1.2.3. *Given a compact orientable surface X of genus g*

(a) $e(X) = 2 - 2g$

[1] Recall that a Riemann surface is the same thing as a one dimensional complex manifold.

(b) If $D \subset X$ is a disk, $e(X - D) = 1 - 2g$

Proof. By the previous proposition have $e(X) = e(D') + e(X - D) - e(D' \cap (X - D))$ where $D' \supset D$ is a slightly larger disk. Since, Betti numbers, and therefore Euler characteristic, is invariant under homotopy, $e(D') = e(pt) = 0$ and $e(D' \cap (X - D)) = e(S^1) = 0$. Therefore $e(X - D) = e(X) - 1 = 1 - 2g$ assuming (a).

We prove (a) by induction. The $g = 0$ case is clear. We can write $X = Y\#T$, where Y has genus $g - 1$. Therefore

$$e(X) = e(Y - D) + e(T - D) - e(S^1 \times [0, 1]) = 1 - 2(g - 1) - 1 = 2 - 2g$$

□

We can use this to compute the genus for our (hyper)elliptic curve X obtained from $y^2 = f(x)$ as before. We have holomorphic map $p : X \to \mathbb{P}^1$ extending the projection of the affine curve to the x-axis. Note that this map is 2 to 1 for all but finitely many points called the branch points. These consist of the zeros of f and possibly ∞. Let r be the number of these points. Either $r = \deg f$ or $\deg f + 1$. Let us triangulate $\mathbb{P}^1$, making sure to include the branch points among the vertices and no edge connects two branch points. Let $V = r + V'$ be the number of vertices, E the number of edges and F the number of faces. Since $\mathbb{P}^1$ is S^2 as topological space,

$$V - E + F = e(\mathbb{P}^1) = 2$$

Take the preimage of this triangulation under p. This gives a triangulation of X, with $r + 2V'$ vertices, $2E$ edges and $2F$ faces. Therefore g is the genus of X, we have

$$2 - 2g = r + 2V' - 2E + 2F = 2(V - E + F) - R = 4 - r$$

Thus

Proposition 1.2.4.

$$g = \frac{1}{2}r - 1$$

Note that this forces R to be even, and this allows us to conclude that

$$r = \begin{cases} \deg f & \text{if } \deg f \text{ is even} \\ \deg f + 1 & \text{otherwise} \end{cases}$$

This result generalizes. Let $f : X \to Y$ be a surjective holomorphic map between compact curves. A point $p \in X$ is called a ramification point if the deriviative of p vanishes at it. An image of a ramification point will be called a branch point. (Some people reverse the terminology.) There are a finite number of branch points B After removing these we get a finite sheeted covering space $X - f^{-1}B \to Y - B$. The number of sheets is called the degree of p. Let us denote this by d. So for each nonbranch point $|f^{-1}(q)| = d$. Now suppose that q is a branch point. Choose a small disk D centered at q. The premiage $f^{-1}(D)$ is a union of disks D_i centered at $p_i \in f^{-1}(q)$. We can choose coordinates so $D_i \to D$ is given by $y = x^{e_i}$. The exponents $e_i = e(p_i)$ are called ramification indices. After perturbing q slightly, p_i splits into a union of e_i points. Therefore

$$\sum e_i = d$$

By the same sort of argument as before, we get

Theorem 1.2.5 (Riemann-Hurwitz)**.** *If $g(X)$ and $g(Y)$ denote the genera of X and Y,*

$$2(g(X) - 2) = 2d(g(X) - 1) + \sum_{p \in X} e(p) - 1$$

1.3 Degree of the canonical divisor

While the previous description of the genus is pretty natural. It is topological rather than algebro-geometric. We give an alternative description which actually works over any field.

Let X be a compact curve. A divisor on X is finite formal sum $D = \sum n_i p_i$, $n_i \in \mathbb{Z}$, $p_i \in X$. The degree $degD = \sum n_i$. If f is a nonzero meromorphic function, the associated principal divisor

$$div(f) = \sum ord_p(f)p$$

where $ord_p(f)$ is the order of the zero f at p or minus the order of the pole.

Proposition 1.3.1. $\deg div(f) = 0$

Proof. Let p_j be the set of zeros and poles of f, and let D_j be a small disk centered at p_j. We can express $ord_{p_j}(f)$ as the residue of df/f at p_j. Thus by Stokes

$$\deg div(f) = \frac{1}{2\pi i} \sum \int_{\partial D_j} \frac{df}{f} = -\frac{1}{2\pi i} \iint_{X - \cup D_j} d\left(\frac{df}{f}\right) = 0$$

□

Given a nonzero meromorphic 1-form α, set

$$div(\alpha) = \sum ord_p(\alpha)p$$

Such a divisor is called *a* canonical divisior. Inspite of the similarity of notation, it is usually not principal. However any two canonical divisors differ by a principal divisor. Consequently the degree of a canonical divisor depends only on X. We write it as $\deg K_X$. For this to make sense, we need to show that a nonzero constant meromorphic form actually exists. Let us avoid the issue for now by restricting our attention to smooth projective curves.[2]

[2]In fact, as we will see later, all compact Riemann surfaces are of this form.

Lemma 1.3.2. *The degree of a canonical divisor on $\mathbb{P}^1$ is -2.*

Proof. let z be the standard coordinate on $\mathbb{C}$. We can use $\zeta = z^{-1}$ as the coordinate around ∞. Since $dz = d\zeta^{-1} = -\zeta^{-2}d\zeta$. This proves the lemma. □

Theorem 1.3.3 (Riemann-Hurwitz II)**.** *Let $f : X \to Y$ be a branched cover of degree d, then*

$$\deg K_X = \deg K_Y + \sum_{p\in X} e(p) - 1$$

Proof. Let α be a nonzero meromorphic form on Y We can assume, after multiplying α by a suitable meromorphic function, that set of zeros or poles are disjoint from the branch points. If q is a zero or pole of α or order n, then $f^*\alpha$ has zero or pole at each $p \in f^{-1}q$ of of order n. A a branch point q, locally $\alpha = u(y)dy$, where u is holomorphic and nonzero at q. Then $f^*\alpha = e_i u(x^{e_i})x^{e_i-1}dy$ Therefore $\deg(div(f^*\alpha)$ is given by the right side of the formula, we are trying to prove. □

Corollary 1.3.4. *For any nonsingular projective curve,* $\deg K_X = 2g - 2$.

Observe that this says that $\deg K_X = -e(X)$. A more conceptual explanation can be given using Chern classes discussed later. The Chern number of the tangent bundle is $e(X)$, whereas $\deg K_X$ is the Chern number of the cotangent bundle.

1.4 Line bundles

Given X be as above. The divisors from a group $Div(X)$, and the principal divisors form a subgroup $Princ(X)$. The divisor class group

$$Cl(X) = Div(X)/Princ(X)$$

We have a surjective homorphism

$$\deg : Cl(X) \to \mathbb{Z}$$

induced by the degree. To understand the structure of the kernel $Cl^0(X)$, we bring in modern tools. Recall that a holomorphic line bundle is a complex manifold L with a holomorphic map $\pi : L \to X$ which " locally looks like" the projection $\mathbb{C} \times X \to X$. More precisely, there exists an open cover $\{U_i\}$ and holomorphic isomorphisms $\psi_i : \pi^{-1}U_i \to U_i \times \mathbb{C}$, which are linear on the fibres. Such a choice called a local trivialization. Given a line bundle $\pi : L \to X$, the sheaf of sections

$$\mathcal{L}(U) = \{f : U \to \pi^{-1} \mid f \text{ holomorphic and } \pi \circ f = id\}$$

is a locally free $\mathcal{O}_X$-module of rank 1. Conversely, any such sheaf arises this way from a line bundle which is unique up to isomorphism. We will therefore identify the two notions.

To be a bit more explicit, fix L with a local trivialization. Set $U_{ij} = U_i \cap U_j$ etc., and let

$$\Phi_{ij} : U_{ij} \times \mathbb{C} \xrightarrow{\psi_j^{-1}} \pi^{-1} U_{ij} \xrightarrow{\psi_i} U_{ij} \times \mathbb{C}$$

It is easy to see that this collection of maps satisfies the 1-cocycle conditions:

$$\Phi_{ik} = \Phi_{ij}\Phi_{jk}, \text{ on } U_{ijk}$$

$$\Phi_{ij} = \Phi_{ji}^{-1}$$

$$\Phi_{ii} = id$$

We can decompose

$$\Phi_{ij}(x) = (x, \phi_{ij}(x))$$

where $\phi_{ij} : U_{ij} \to \mathbb{C}^*$ is a collection of holomorphic functions satisfying the 1-cocycle conditions. This determines a Čech cohomology class in $\check{H}^1(\{U_i\}, \mathcal{O}_X^*)$. Conversely, such a class determines a line bundle. We can summarize everything as follows.

Theorem 1.4.1. *There is an bijection between*

1. *the set of isomorphism classes of line bundles on X*
2. *the set of isomorphism classes of rank one locally free $\mathcal{O}_X$-modules*
3. $H^1(X, \mathcal{O}_X^*)$

The last set is of course a group, called the Picard group. It is denoted by $Pic(X)$. The group operation for line bundles, or sheaves is just tensor product. The inverse of the dual line bundle $\mathcal{L}^{-1} = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$. We have an exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{e^{2\pi i}} \mathcal{O}_X^* \to 1$$

This yields a long exact sequence

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to Pic(X) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

The last map is called the first Chern class. This is can viewed as an integer because $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$. The kernel of c_1 is denoted by $Pic^0(X)$. We will show later that $H^1(X, \mathbb{Z})$ sits as a lattice in $H^1(X, \mathcal{O}_X)$. Therefore $Pic^0(X)$ has the structure of a complex torus.

We can relate this to the divisor class group. Let $K = \mathbb{C}(X)$ be the field meromorphic functions on X. Given $D = \sum n_i p_i \in Div(X)$, define the sheaf

$$\mathcal{O}_X(D)(U) = \{f \in K \mid ord_{p_i} f + n_i \geq 0\}$$

This is a locally principal fractional ideal sheaf, and therefore a line bundle. A straight forward verification shows that

Lemma 1.4.2.

$$\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2) \cong \mathcal{O}_X(D_1 + D_2)$$

and

$$\mathcal{O}_X(D) \cong \mathcal{O}_X$$

when D is principal.

Therefore map $D \mapsto \mathcal{O}_X(D)$ induces a homomorphism $Cl(X) \to Pic(X)$.

Theorem 1.4.3. *The above map gives an isomorphism $Cl(X) \cong Pic(X)$.*

Proof of injectivity. Let $D = \sum n_i p_i$. Suppose that $\mathcal{O}_X(D) \cong \mathcal{O}_X$. Then the section 1 on the right corresponds to a global section $f \in H^0(X, \mathcal{O}_X(D))$ which generates all the stalks of $\mathcal{O}(D)_p$. We can identify $\mathcal{O}(D)_{p_i}$ with $x^{-n_i}\mathcal{O}_{p_i}$, where x is a local parameter. For f to generate $x^{-n_i}\mathcal{O}_{p_i}$, we must have $ord_{p_i} f = -n_i$. Therefore $D = div(f^{-1})$. Thus the homomorphism $Cl(X) \to Pic(X)$ is injective. We will prove surjectivity in the next section. $\square$

We have an isomorphism $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$, therefore $c_1(\mathcal{L})$ can be viewed as a number. Let us explain how to compute it in terms of a 1-cocycle ϕ_{ij} for $\mathcal{L}$ on $\mathcal{U} = \{U_i\}$. We have another sequence

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \to 0$$

We can map the exponential sequence to this

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}^*_X & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \frac{1}{2\pi\sqrt{-1}} d\log} & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \Omega^1_X & \longrightarrow & 0 \end{array}$$

This shows that c_1 can be factored as

$$H^1(X, \mathcal{O}^*_X) \to H^1(X, \Omega^1_X)$$

composed with the connecting map

$$H^1(X, \Omega^1_X) \to H^2(X, \mathbb{C})$$

We need to make the last part explicit. Let $\alpha_{ij} \in Z^1(\mathcal{U}, \Omega^1)$ be the cocycle $\frac{1}{2\pi\sqrt{-1}} d\log\phi_{ij}$. We can view this as a 1-cocycle in $Z^1(\mathcal{U}, \mathcal{E}^1_X)$, where $\mathcal{E}^1_X$ are C^∞ 1-forms. But we know that $H^1(X, \mathcal{E}^1_X) = 0$ because $\mathcal{E}^1_X$ is soft. Therefore we can find C^∞ 1-forms α_i such that $\alpha_{ij} = \alpha_i - \alpha_j$. Since $d\alpha_{ij} = \bar{\partial}\alpha_{ij} = 0$, $\beta = d\alpha_i$ is a globally defined 2-form. The de Rham class of β is precisely what we are after. Under the isomomorphism

$$H^2(X, \mathbb{C}) \cong \mathbb{C}$$

given by integration,

$$c_1(\mathcal{L}) = \int_X \beta$$

This is really only a first step. A more satisfying answer is given by the next theorem.

Theorem 1.4.4. *After identifying $H^2(X, \mathbb{Z}) = \mathbb{Z}$, $c_1(\mathcal{O}_X(D)) = \deg D$.*

Proof. It's enough to prove that $c_1(\mathcal{O}_X(-p)) = -1$ for every $p \in X$. Let U_0 be a coordinate disk around p with coordinate z, and let $U_1 = X - \{p\}$. $\mathcal{O}_X(-p) \subset \mathcal{O}_X$ is the ideal of p. On U_0, z gives a trivializing section; on U_1 we can use 1. The change of basis function $\phi_{01} = z^{-1}$ is the cocycle for $\mathcal{O}(p)$. So our task is to compute $\int_X \beta$, where β is obtained from $\phi_{01} = z^{-1}$ by the above process. We split $X = E \cup B$, where $B = \bar{U}_0$ is the closed disk and $E = X - U_0$. Let $C = \partial B$ oriented counterclockwise around p. By Stokes and the residue theorem

$$\int_X \beta = \int_E \beta + \int_B \beta = \int_E d\alpha_1 + \int_B d\alpha_0 = \int_C \alpha_0 - \alpha_1$$

$$= \frac{-1}{2\pi i} \int_C \frac{dz}{z} = -1$$

This concludes the proof.

□

Corollary 1.4.5. *$Cl^0(X) \cong Pic^0(X)$.*

1.5 Serre duality

Let X be a connected compact complex curve of genus g. Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions and let Ω^1_X denote the sheaf of holomorphic 1-forms on X. We have an exact sequence of sheaves

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \to 0$$

which gives a long exact sequence

$$0 \to H^0(X, \mathbb{C}) \to H^0(X, \mathcal{O}_X) \to H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C})$$

Since X is connected $H^0(X, \mathbb{C}) = H^0(X, \mathcal{O}_X) = \mathbb{C}$, where the last equality follows from the maximum principle. Therefore we have an injection

$$0 \to H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C})$$

In particular, $\dim H^0(X, \Omega^1_X) \leq 2g$. In fact, we do much better. If we identify $H^1(X, \mathbb{C})$ with de Rham cohomology, then we have a pairing

$$H^1(X, \mathbb{C}) \times H^1(X, \mathbb{C}) \to \mathbb{C}$$

given by

$$\langle\alpha,\beta\rangle = \int_X \alpha\wedge\beta \tag{1.2}$$

A suitably strong form of Poincaré duality shows that this is a nondegenerate pairing. It is clearly also skew symmetric i.e.

$$\langle\alpha,\beta\rangle = -\langle\beta,\alpha\rangle$$

We quote the following fact from linear algebra

Theorem 1.5.1. *If V is finite dimensional vector with a nondegenerate skew symmetric pairing $\langle,\rangle$, $\dim V$ is even. If $W\subset V$ is subspace such that $\langle\alpha,\beta\rangle = 0$ for all $\alpha,\beta\in W$ (W is called isotropic), then $\dim W\le \frac{1}{2}\dim V$.*

Corollary 1.5.2. $\dim H^0(X,\Omega^1_X)\le g$.

Proof. The formula (1.2) makes it clear that this is isotropic. □

Recall last semester, we constructed a Dolbeault resolution

$$0\to\mathcal{O}_X\to\mathcal{E}^{00}_X\xrightarrow{\bar\partial}\mathcal{E}^{01}_X\to 1$$

where $\mathcal{E}^{00}_X$ (resp $\mathcal{E}^{(0,1)}_X$) is the sheaf of C^∞ functions (resp locally C^∞ multiples of $d\bar z$.), $\bar\partial f$ is $(0,1)$-part of df. Since this gives a soft resolution of $\mathcal{O}_X$, we can use this to compute sheaf cohomology

$$H^0(X,\mathcal{O}_X) = \ker[\mathcal{E}^{00}_X(X)\xrightarrow{\bar\partial}\mathcal{E}^{01}_X(X)]$$

$$H^1(X,\mathcal{O}_X) = \mathrm{coker}[\mathcal{E}^{00}_X(X)\xrightarrow{\bar\partial}\mathcal{E}^{01}_X(X)]$$
$$H^i(X,\mathcal{O}_X) = 0, i\ge 2$$

Suppose that $\alpha\in\Omega^1(X)$ and $\beta\in\mathcal{E}^{(0,1)}(X)$. Define

$$\langle\alpha,\beta\rangle = \int_X\alpha\wedge\beta$$

as before.

Lemma 1.5.3. *This gives a well defined pairing*

$$\langle,\rangle : H^0(X,\Omega^1_X)\times H^1(X,\mathcal{O}_X)\to\mathbb{C}$$

Proof. It is enough to show that $\langle\alpha,\bar\partial f\rangle = 0$. This follows from Stokes' theorem because $\alpha\wedge\bar\partial f = \pm d(f\alpha)$. □

Theorem 1.5.4 (Serre duality I)**.** *The above pairing is perfect, i.e. it induces an isomorphism*

$$H^1(X,\mathcal{O}_X)\cong H^0(X,\Omega^1_X)^*$$

Corollary 1.5.5. $\dim H^0(X, \Omega^1_X) = \dim H^1(X, \mathcal{O}_X) = g$

Proof. The theorem gives $\dim H^0(X, \Omega^1_X) = \dim H^1(X, \mathcal{O}_X)$. Call his number h. Earlier we saw that $h \leq g$. The exact sequence

$$0 \to H^0(X, \Omega_X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X)$$

forces $2g \leq 2h$. □

What we stated above is really a special case of Serre duality:

Theorem 1.5.6 (Serre duality II). *If $\mathcal{L}$ is a line bundle then there is a natural pairing inducing an isomorphism*

$$H^1(X, \mathcal{L}) \cong H^0(X, \Omega^1_X \otimes \mathcal{L}^{-1})^*$$

We can use this finish the proof of theorem 1.4.3. It remains to prove that the map $Cl(X) \to Pic(X)$ is surjective that is:

Proposition 1.5.7. *Any line bundle is isomorphic to $\mathcal{O}_X(D)$ for some divisor D.*

We first need a few lemmas which are useful by themselves.

Lemma 1.5.8. *If $\deg \mathcal{L} < 0$, then $H^0(X, \mathcal{L}) = 0$.*

Proof. A nonzero section would correspond to a nonzero map $\sigma : \mathcal{O}_X \to \mathcal{L}$. Dualizing gives a nonzero map

$$\sigma^* : \mathcal{L}^{-1} \to \mathcal{O}_X$$

Consider $\mathcal{K} = \ker \sigma^*$. Then $\mathcal{K} \otimes \mathcal{L} \subset \mathcal{O}_X$ is ideal sheaf. It must by either 0 or of the form $\mathcal{O}_X(-D)$ for some effective divisor D (effective means that the coefficients are nonnegative). It follows that either $\mathcal{K} = \mathcal{L}^{-1}(-D) := \mathcal{L}^{-1} \otimes \mathcal{O}(-D)$ if it isn't zero. If $D = 0$, then $\sigma^* = 0$ which is impossible. If $D > 0$, then OO_X would contain the nonzero torsion sheaf $\mathcal{L}^{-1}/\mathcal{K}$ which is also impossible. Therefore $\mathcal{K} = 0$, and consequently σ^* is injective. Thus $\mathcal{L}^{-1} \subset \mathcal{O}_X$ can be identified with $\mathcal{O}_X(-D)$ with D effective. This implies $\deg \mathcal{L} = \deg D \geq 0$, which is a contradiction. □

Corollary 1.5.9. *If $\deg \mathcal{L} > 2g - 2$ then $H^1(X, \mathcal{L}) = 0$.*

Proof. By corollary 1.3.4 $\deg \Omega^1_X \otimes \mathcal{L}^{-1} = 2g - 2 - \deg \mathcal{L} < 0$. This implies that $H^0(X, \Omega^1_X \otimes \mathcal{L}^{-1}) = 0$. □

Lemma 1.5.10. *If $\deg \mathcal{L} > 2g - 1$, then $H^0(X, \mathcal{L}) \neq 0$*

Proof. Choose $p \in X$, and let $\mathbb{C}_p$ be the skyskraper sheaf

$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{otherwise} \end{cases}$$

Then we have an exact sequence

$$0 \to \mathcal{O}_X(-p) \to \mathcal{O}_X \to \mathbb{C}_p \to 0$$

Tensoring with $\mathcal{L}$ gives a sequence

$$0 \to \mathcal{L}(-p) \to \mathcal{L} \to \mathbb{C}_p \to 0$$

because $\mathbb{C}_p \otimes \mathcal{L} \cong \mathbb{C}_p$. Therefore we get an exact sequence

$$H^0(X, \mathcal{L}) \to \mathbb{C} \to H^1(X, \mathcal{L}(-p)) = 0$$

This proves the lemma. □

Proof of proposition 1.5.7. Let $\mathcal{L}$ be a line bundle. By the previous $\mathcal{L}(F)$ has a nonzero section for $\deg F \gg 0$. This gives a nonzero map $\mathcal{O}_X \to \mathcal{L}(D)$ or dually a nonzero map $\mathcal{L}^{-1}(-F) \to \mathcal{O}_X$. Arguing as above, we see that $\mathcal{L}^{-1}(-F) = \mathcal{O}_X(-E)$ for some effective divisor E. Therefore $\mathcal{L} \cong \mathcal{O}_X(E - F)$. □

1.6 Harmonic forms

We outline the proof of theorem 1.5.4. We start with a seemingly unrelated problem. Recall that de Rham cohomology

$$H^1_{dR}(X, \mathbb{C}) = \frac{\{\alpha \in \mathcal{E}^1(X) \mid d\alpha = 0\}}{\{df \mid f \in C^\infty(X)\}}$$

So an element of it is really an equivalence class. *Does such a class have a distinguished representative?* We can answer the analogous problem in finite dimensional linear algebra by the method of least squares: given a finite dimensional inner product space V with a subspace W, there is an isomorphism $f : V/W \cong W^\perp$ where $f(X) \in X$ is the element with smallest norm. We introduce a norm on our space as follows. In local analytic coordinates $z = x + yi$, define $*dx = dy$, $*dy = -dx$. This is amounts to multiplication by i in the cotangent plane, so it is globally well defined operation. We extend this to $\mathbb{C}$-linear operator. Then

$$(\alpha, \beta) = \int_X \alpha \wedge *\bar{\beta}$$

gives an inner product on $\mathcal{E}^1(X)$ and therefore a norm $||\alpha||^2 = (\alpha, \alpha)$.

Theorem 1.6.1 (Hodge theorem)**.** *Every cohomology class has unique representative which minimizes norm.*

We want to explain the uniqueness part. First we need to understand the norm minimizing condition in more explicit terms. We define a 1-form α to be *harmonic* if $d\alpha = d(*\alpha) = 0$. (This is a solution of a Laplace equation, as the terminology suggests, but we won't need this.)

Lemma 1.6.2. *A harmonic form is the unique element of smallest norm in its cohomology class. Conversely, if a closed form minimizes norm in its class, then it is harmonic.*

Proof. For simplicity, let's prove this for real forms.

As a first step, we can establish the identity

$$\langle df, \alpha\rangle = -\int_X f d{*}\alpha \tag{1.3}$$

by observing that

$$\int_X d(f \wedge {*}\alpha) = 0$$

by Stokes' and then expanding this out. Therefore

$$||\alpha + df||^2 = ||\alpha||^2 + 2\langle df, \alpha\rangle + ||df||^2 = ||\alpha||^2 - 2\int_X f d{*}\alpha + ||df||^2 \tag{1.4}$$

If α is harmonic, then

$$||\alpha + df||^2 = ||\alpha||^2 + ||df||^2 > ||\alpha||^2$$

when $df \neq 0$.

Now suppose that $||\alpha||^2$ is minimal in its class. Then for any f, we must have

$$\frac{d}{dt}||\alpha + t||^2|_{t=0} = 0$$

Using (1.4) we conclude that

$$\int_X f d{*}\alpha = 0$$

Since f is arbitrary, we must have $d^*\alpha = 0$.

□

A special case of the uniqueness statement, which follows directly from (1.3), is

Corollary 1.6.3. 0 *is the only exact harmonic form.*

The proof of the existence statement for theorem 1.6.1 is where most of the analytic subtleties lie. We won't give the proof, but refer the reader to Griffiths-Harris or Wells. The only thing we want to observe is that the proof yields an extension of theorem 1.6.1 which applies to non closed forms.

Theorem 1.6.4 (Hodge theorem II)**.** *Any form $\mathcal{E}^1(X)$ can be uniquely decomposed into a sum $\beta + df + {*}dg$, where β is harmonic and f, g are C^∞ functions.*

Proposition 1.6.5.

(a) *A harmonic* 1*-form is a sum of a* $(1,0)$ *harmonic form and* $(0,1)$ *harmonic form.*

(b) *A* $(1,0)$*-form is holomorphic if an only if it is closed if and only it is harmonic.*

(c) *A* $(0,1)$*-form is harmonic if and only if it is antiholomorphic i.e. its complex conjugate is holomorphic.*

Proof. If α is a harmonic 1-form, then $\alpha = \alpha' + \alpha''$, where $\alpha' = \frac{1}{2}(\alpha + i * \alpha)$ is a harmonic $(1,0)$-form and $\alpha'' = \frac{1}{2}(\alpha - i * \alpha)$ is a harmonic $(0,1)$-form.

If α is $(1,0)$, then $d\alpha = \bar{\partial}\alpha$. This implies the first half (b). For the second half, use the identity

$$*dz = *(dx + idy) = dy - idx = -idz$$

Finally, note that the harmoncity condition is invariant under conjugation, so the (c) follows from (b).

□

Proposition 1.6.6. $H^1(X, \mathcal{O}_X)$ *is isomorphic to the space of antiholomorphic forms.*

Proof. Let $H \subset \mathcal{E}^{01}(X)$ denote the space of antiholomorphic forms. We will show that

$$\pi : H \to \mathcal{E}^{01}(X)/\operatorname{im}\bar{\partial}$$

is an isomorphism. Suppose that $\alpha \in \mathcal{E}^{01}(X)$. By theorem 1.6.4, we may choose a harmonic form β such that $\beta = \alpha + df + *dg$ for some $f, g \in C^\infty(X)$. Then the $(0,1)$ part of β gives an element $\beta' \in H$ such that $\beta' = \alpha + \bar{\partial}(f + ig)$. This shows that π is surjective.

Suppose that $\alpha \in \ker \pi$. Then $\alpha = \bar{\partial} f$ for some f. Therefore $\alpha + \bar{\alpha} = df$. Consequently $\alpha + \bar{\alpha}$ is exact and harmonic, so $\alpha + \bar{\alpha} = 0$. This implies $\alpha = 0$. □

Corollary 1.6.7. $\alpha \mapsto \bar{\alpha}$ *gives a conjugate linear isomorphism*

$$H^0(X, \Omega^1_X) \cong H^1(X, \mathcal{O}_X)$$

Proof of theorem 1.5.4. We have a linear map

$$\sigma : H^0(X, \Omega^1_X) \to H^1(X, \mathcal{O}_X)^*$$

which assigns to α the functional $\langle \alpha, - \rangle$. If $\alpha \in H^0(X, \Omega^1_X)$ is nonzero then

$$\sigma(\alpha)(\bar{\alpha}) = \int_X \alpha \wedge \bar{\alpha} \neq 0$$

Therefore σ is injective. Since these spaces have the same dimension, σ is an isomorphism. □

The general Serre duality can also be proved by a similar method.

1.7 Riemann-Roch

Let $h^i(\mathcal{L}) = \dim H^i(X, \mathcal{L})$, and $\chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L})$.

Theorem 1.7.1 (Riemann-Roch)**.** *If X is compact curve of genus g, and D is a divisor*

$$\chi(\mathcal{O}_X(D)) = \deg D + 1 - g$$

Proof. Let $D = \sum n_i p_i$. We prove this by induction on the "mass" $M(D) = \sum |n_i|$. When $M(D) = 0$, then this is just the equality

$$\chi(\mathcal{O}_X) = 1 - g$$

We follows from the facts $h^0(\mathcal{O}_X) = 1, h^1(\mathcal{O}_X) = g$ established earlier.

Given p, we have an exact sequence

$$0 \to \mathcal{O}_X(-p) \to \mathcal{O}_X \to \mathbb{C}_p \to 0$$

Tensoring by $\mathcal{O}_X(D)$ gives

$$0 \to \mathcal{O}_X(D-p) \to \mathcal{O}_X(D) \to \mathbb{C}_p \to 0$$

Observe that $H^0(X, \mathbb{C}_p) = \mathbb{C}$ by definition and $H^1(X, \mathbb{C}_p) = 0$ because $\mathbb{C}_p$ is flasque. Thus

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D-p)) + \chi(\mathbb{C}_p) = \chi(\mathcal{O}_X(D-p)) + 1$$

Therefore

$$\chi(\mathcal{O}(D)) - \deg D = \chi(\mathcal{O}(D-p)) - \deg(D-p)$$

or by changing variable and writing the formula backwards

$$\chi(\mathcal{O}(D)) - \deg D = \chi(\mathcal{O}(D+p)) - \deg(D+p)$$

Since we can choose p so that $M(D \pm p) < M(D)$. One of these two formulas shows that $\chi(\mathcal{O}_X(D))$ equals $1 - g$.

□

Using Serre duality, we get the more classical form of Riemann-Roch

Corollary 1.7.2. *Let K be a canonical divisor, then*

$$h^0(\mathcal{O}(D)) - h^0(\mathcal{O}(K-D)) = \deg D + 1 - g$$

In particular

$$h^0(\mathcal{O}(D)) \geq \deg D + 1 - g \quad \textit{(Riemann's inequality)}$$

The Riemann-Roch theorem is a fundamental tool in the study of curves. One consequence is that any compact complex curve carries a nonconstant meromorphic function. From this, it is not difficult to deduce the following important fact:

Theorem 1.7.3. *Every compact Riemann surface is a nonsingular projective algebraic curve.*

So from here on, we won't make a distinction between these notions. We also make use of GAGA (Serre, Géometrie algébrique et géometrie analytique) to switch between algebraic and analytic viewpoints whenever convenient.

Theorem 1.7.4 (Chow's theorem/GAGA)**.**

(a) Meromorphic functions on a projective varieties are rational

(b) Holomorphic maps between projective varieties are regular.

(c) Submanifolds of nonsingular projective varieties are subvarieties.

(d) Holomorphic vector bundles on projective varieties are algebraic, and their cohomology groups can be computed algebraically i.e. as the cohomology of the corresponding sheaves on the Zariski topology

As an easy application of Riemann-Roch, let us classify curves of genus ≤ 2, where by curve we mean a nonsingular projective curve below.

Lemma 1.7.5. *The only curves of genus* 0 *is* $\mathbb{P}^1_{\mathbb{C}}$.

Proof. Let X be a curve of genus 0. Let $p \in X$. By Riemann-Roch $h^0(\mathcal{O}(p)) \geq 2$. This implies that there exists a nonconstant meromorphic function f having a pole of order 1 at p and no other singularities. Viewing f as a holomorphic map $X \to \mathbb{P}^1$, we have $f^{-1}(\infty) = p$ and the ramification index $e_p = 1$. This means that the degree of f is 1, and one can see that this must be an isomorphism. □

From the proof, we obtain the following useful fact.

Corollary 1.7.6. *If for some* p, $h^0(\mathcal{O}(p)) > 1$, *then* $X \cong \mathbb{P}^1$.

We saw earlier that if $f(x)$ is a degree 4, polynomial then $y^2 = f$ has genus 1. Such a curve is called elliptic.

Lemma 1.7.7. *Conversely, any genus* 1 *curve can be realized as degree* 2 *cover of* $\mathbb{P}^1$ *branched at* 4 *points.*

Proof. Let X be a curve of genus 1. Let $p \in X$. By Riemann-Roch $h^0(\mathcal{O}(np)) \geq n$ for $n \geq 0$. Thus we can find a nonconstant function $f \in H^0(\mathcal{O}(2p))$. Either f has a pole of order 1 or order 2 at p. The first case is ruled out by the last corollary, so we are in the second case. Viewing f as a holomorphic map $X \to \mathbb{P}^1$, we have $f^{-1}(\infty) = p$ with $e_p = 2$. Therefore f has degree 2. Now apply the Riemann-Hurwitz formula to conclude that f has 4 branch points. □

Lemma 1.7.8. *Any genus* 2 *curve can be realized as degree* 2 *cover of* $\mathbb{P}^1$ *branched at* 6 *points.*

Proof. Either using Riemann-Roch or directly from corollary 1.5.5, we can see that $h^0(\Omega^1_X) = 2$. Choose two linearly independent holomorphic 1-forms ω_i. The ratio $f = \omega_2/\omega_1$ is a nonconstant meromorphic function. Since $\deg div(\omega_2) = 2(2)-2 = 2$, f has most two poles counted with multiplicity. Therefore $f : X \to \mathbb{P}^1$ has degree at most 2, and as above we can see that it is exactly 2. Again using the Riemann-Hurwtiz formula shows that f has 6 branch points. □

1.8 Genus 3 curves

Continuing the analysis, we come to genus 3 curves. Here things become more complicated. Some of these curves are hyperelliptic, but not all.

Theorem 1.8.1. *A genus* 3 *curve is either a hyperelliptic curve branched at* 8 *points or a nonsingular quartic in* $\mathbb{P}^2$. *These two cases are mutually exclusive.*

Fix a genus 3 curve X. Then $h^0(\Omega^1_X) = 3$. We can choose a basis $\omega_0, \omega_1, \omega_2$.

Lemma 1.8.2. *For every* $p \in X$, *one of the* $\omega_i(p) \neq 0$. *(This condition says that* Ω^1_X *is generated by global sections, or in classical language that the canonical linear system is based point free.)*

Proof. Suppose that all $\omega_i(p) = 0$ for some p. Then ω_i would define sections of $\Omega^1_X(-p)$. So we must have $h^0(\Omega^1_X(-p)) \geq 3$. But by Riemann-Roch

$$h^0(\Omega^1(-p)) - h^0(\mathcal{O}(p)) = 2(3) - 2 - 1 + (1 - 3) = 1$$

Since $h^0(\mathcal{O}(p)) = 1$, we obtain $h^0(\Omega^1(-p)) = 2$. This is a contradiction. □

Choosing a local coordinate z, we can write $\omega_i = f_i(z)dz$. By the previous lemma, the point $[f_0(z), f_1(z), f_2(z)] \in \mathbb{P}^2_{\mathbb{C}}$ is defined at all points of the coordinate chart. If we change coordinates to ζ, then $\omega_i = uf_i d\zeta$, where $u = \frac{dz}{d\zeta}$. This means that the point $[f_0(z), f_1(z), f_2(z)] \in \mathbb{P}^2_{\mathbb{C}}$ is globally well defined. In this way, we get a nonconstant holomorphic map

$$\kappa : X \to \mathbb{P}^2$$

called the canonical map. The image is a curve in $\mathbb{P}^2$, which we denote by Y. We can factor κ as $\pi : X \to Y$ followed by the inclusion $Y \subset \mathbb{P}^2$. Let d_1 be the degree of π. Let x_i denote the homogenous coordinates of $\mathbb{P}^2$. Consider the line $\ell \subset \mathbb{P}^2$ defined by $a_0x_0 + a_1x_1 + a_2x_2 = 0$. If we assume that a_i are chosen generically, then $\ell \cap Y$ is a union, of say $d_2 = \deg Y$ "distinct points" (more precisely, the scheme theoretic intersection is a reduced subscheme of length d_2). If we pull this back to X we get d_1d_2 points which coincides with the zero set of $\sum a_i\omega_i$. The degree of a canonical divisor is 4. Thus we get three cases:

(1) $\deg Y = 4$ and π has degree 1,

(2) $\deg Y = 2$ and π has degree 2,

(3) $\deg Y = 1$ and π has degree 4.

In fact, case (3) is impossible because it would imply a linear dependence between the ω_i. In case (2), Y is an irreducible conic, which is necessarily isomorphic to $\mathbb{P}^1$. So this is the hyperelliptic case. Using Hartshorne, chap IV, prop 5.3, we see that conversely a hyperelliptic genus 3 curve necessarily falls into case (2). By Riemann-Hurwitz this must be branched at 8 points.

To complete the analysis, let us take closer look at case (1). Here π is birational. So X is necessarily the normalization of Y.

Lemma 1.8.3. *Y is nonsingular. Therefore $X \to Y$ is an isomorphism.*

Proof. Suppose that Y is singular. Consider the sequence

$$0 \to \mathcal{O}_Y \to \pi_*\mathcal{O}_X \to C \to 0$$

where the cokernel C is a sum of sky scraper sheaves

$$\bigoplus_p \Big(\bigoplus_{q \in \pi^{-1}(p)} \mathcal{O}_{X,q} \Big)/\mathcal{O}_{Y,p}$$

supported a the singular points of Y. Let δ_p denote the dimension of each summand above. At least one of these numbers is positive because Y is singular. C is flasque, therefore $H^1(Y, C) = 0$. Also since π is finite, $H^i(X, \mathcal{O}_X) = H^i(Y, \pi_*\mathcal{O}_X)$ (c.f. Hartshorne, Algebraic Geometry, chap III ex 4.1). Consequently

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) + \sum \delta_p$$

and therefore

$$\chi(\mathcal{O}_X) > \chi(\mathcal{O}_Y)$$

The arithmetic genus p_a of Y is defined so $\chi(\mathcal{O}_Y) = 1 - p_a$. The inequality says that p_a is strictly less that the genus of X. It is worth pointing that the argument is completely general; it applies to any singular curve. To obtain a contradiction, we show that $p_a = 3$. This will follow from the next result. $\square$

Theorem 1.8.4. *If $C \subset \mathbb{P}^2$ is a curve of degree d, then the arithmetic genus (= the ordinary genus when C is nonsingular)*

$$p_a = \frac{(d-1)(d-2)}{2}$$

Before starting the proof, we summarize some basic facts about projective space. Proofs of these statements can be found in Hartshorne. Let $x_0, \ldots, x_n$ be homogeneous coordinates of $\mathbb{P}^n_{\mathbb{C}}$. We have an open cover consisting of $U_i = \{x_i \neq 0\}$. $U_i \cong \mathbb{A}^n$ with coordinates $\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots \frac{x_n}{x_i}$. Let $\mathcal{O}_{\mathbb{P}^n}(d)$ be the line bundle determined by the cocycle $\left(\frac{x_i}{x_j}\right)^d$. Suppose that $d \geq 0$. Given a homogeneous degree d polynomial $f(x_0, \ldots, x_n)$, the transformation rule

$$f\left(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right) = \left(\frac{x_i}{x_j}\right)^d f\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

shows that it determines a section of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$. In fact, all sections are of this form. Therefore

$$h^0(\mathcal{O}_{\mathbb{P}^n}(d)) = \binom{n+d}{n}, \quad d \geq 0$$

All the other cohomology groups are zero. We also have a duality

$$h^i(\mathcal{O}_{\mathbb{P}^n}(-d)) = h^{n-i}(\mathcal{O}_{\mathbb{P}^n}(-n-1+d))$$

Therefore

$$\chi(\mathcal{O}_{\mathbb{P}^2}(-d)) = h^2(\mathcal{O}(-d)) = \frac{(d-1)(d-2)}{2} \tag{1.5}$$

Under the this identification

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong Hom(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(d))$$

a nonzero polynomial f determines an injective morphism $\mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(d)$. Tensoring with $\mathcal{O}_{\mathbb{P}^n}(-d)$ yields an injective morphism $\mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n}$. The image can be identified with the ideal sheaf generated by f.

Proof of theorem. We have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0$$

Therefore

$$\begin{aligned} \chi(\mathcal{O}_C) &= \chi(\mathcal{O}_{\mathbb{P}^2}) - \chi(\mathcal{O}_{\mathbb{P}^2}(-d)) \\ &= 1 - \frac{(d-1)(d-2)}{2} \end{aligned}$$

□

All of these results taken together implies theorem 1.8.1.

1.9 Automorphic forms

We give one more application of the Riemann-Roch. The group $SL_2(\mathbb{R})$ acts on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid Im\, z > 0\}$ by fractional linear transformations. Suppose that $\Gamma \subset SL_2(\mathbb{R})$ is a discrete subgroup which acts freely on $\mathbb{H}$ (or more precisely suppose that $\Gamma/\{\pm I\}$ acts freely). Then the quotient $X = \mathbb{H}/\Gamma$ is naturally a Riemann surface. Let us assume that X is compact. The genus g can be shown to be at least two by the Gauss-Bonnet theorem. An automorphic form of weight $2k$ is a holomorphic function $f(z)$ on $\mathbb{H}$ satisfying

$$f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right)$$

for each

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

When $f(z)$ has weight 2, $f(z)dz$ is invariant under the group. Therefore it defines a holomorphic form on X. It follows that

Proposition 1.9.1. *The dimension of the space of weight two automorphic forms is g.*

If $f(z)$ has weight $2k$, then $f(z)dz^{\otimes k}$ defines a section of $(\Omega^1_X)^{\otimes k} = \mathcal{O}_X(kK)$. From Riemann-Roch

$$h^0(\mathcal{O}(kK)) - h^0(\mathcal{O}((1-k)K)) = \deg(kK) + (1-g) = (2k-1)(g-1)$$

If $k > 1$, then $\deg(1-k)K < 0$. Therefore:

Proposition 1.9.2. *If $k > 1$, the dimension of the space of weight $2k$ automorphic forms is $(g-1)(2k-1)$.*