Chapter 2

Divisors on a surface

2.1 Bezout's theorem

Given distinct irreducible curves $C, D \subset \mathbb{P}^2_{\mathbb{C}}$, $C \cap D$ is finite. The naive guess is that the number of points is the product of the degrees of (the defining equations of) C and D. This is not literally true, unless the points are counted with multiplicity. Let us start by making this precise. Our goal is to define a local intersection number $(C \cdot D)_p$ for each $p \in C \cap D$, such that it equals 1 if C and D are smooth and transverse at p. We could try to count the number of points in neighbourhood of p after perturbing the curves slightly, but it's more efficient to define it algebraically. Consider the local ring $\mathcal{O}_p = \mathcal{O}_{\mathbb{P}^2,p}$, given by stalk of the sheaf of regular functions. As usual m denotes the maximal ideal. Then the ideals of C and D are defined by elements $f, g \in \mathcal{O}_p$. We define

$$(C \cdot D)_p = \dim \mathcal{O}_p/(f,g)$$

Note that this number is finite provided that (f,g) is an m-primary ideal, and this holds in the present case. Had we used the stalk $\mathcal{O}_p^{an}$ of the sheaf of holomorphic functions in the above definition, we would have gotten the same result.

Lemma 2.1.1. $(C \cdot D)_p = \dim \mathcal{O}_p^{an}/(f,g) = \dim \hat{\mathcal{O}}_p/(f,g)$

Proof. Since $I = (f,g)$ is m-primary $\mathcal{O}_p/I \cong (\mathcal{O}_p/m^N)/I \cong (\hat{\mathcal{O}}_p/m^N)/I \cong \hat{\mathcal{O}}_p/m$ The same argument shows that $\mathcal{O}_p^{an}/I \cong \hat{\mathcal{O}}_p/m$ □

Let us check that it works as advertised

Lemma 2.1.2. *If C and D are smooth and transverse at $p \in C \cap D$, then* $(C \cdot D)_p = 1$.

Proof. We can identify the differentials df and dg with the images of f and g in m/m^2. The assumptions imply that these are linearly independent. Therefore (f,g) generate m/m^2. Thus, by Nakayama, they generate m, so $\mathcal{O}_p/(f,g) = \mathbb{C}$. □

Theorem 2.1.3 (Bezout's theorem)**.** *Let C and D be possibly reducible curves having no irreducible components in common, then*

$$\sum_{p \in C\cap D} (C\cdot D)_p = (\deg C)(\deg D)$$

Although this is easy to prove by elementary methods, we will deduce it from something more general.

2.2 Divisors

By an algebraic surface X, we will mean a two dimensional nonsingular projective variety over an algebraically closed field. We will work almost exclusively over $\mathbb{C}$ for now. An irreducible closed curve $C \subset X$ is also called a prime divisor. We allow C to be singular. Let $\mathbb{C}(X)$ be the field of rational functions on X. Let $U \subset X$ is an affine open set with coordinate ring R, then $\mathbb{C}(X)$ is fraction field of R. Let $C \subset X$ be a prime divisor which meets U. Then the ideal I_C of $C \cap U$ is locally principle. This means that after shrinking U, $I_C = (f)$ for some nonunit $f \in R$. The ideal I_C is a height one prime ideal, therefore the localization R_{I_C} is discrete valuation ring. Let $ord_C : \mathbb{C}(X) \to \mathbb{Z}\cup\{\infty\}$ be the associated valuation. This depends only on C and not on R. We can think of ord_C as the measure the order of zero or pole along C.

Lemma 2.2.1. *Given $f \in \mathbb{C}(X)^*$, there exists only a finite number of irreducible curves for which $ord_C(f) \neq 0$.*

Proof. f is regular on some nonempty Zariski open set $U \subset X$. The curves for which $ord_C(f) < 0$ are irreducible components of $X - U$, and there can be finitely many of these. The same arguement applied to f^{-1} shows that there are finite many C with $ord_C(f) > 0$.

□

A divisor is a finite integer linear combination $D = \sum n_i C_i$ of prime divisors. As with curves, given $f \in \mathbb{C}(X)^*$, we can associate a divisor

$$div(f) = \sum_C ord_C(f) C \quad (\text{ sum over irreducible curves})$$

Such a divisor is called principal. Since div is a homomorphism, the set of princpal divisors forms a subgroup $Princ(X)$ of all divisors $Div(X)$. The quotient is the class group $Cl(X)$. Two divisors are called linearly equivalent if they have the same image in $Cl(X)$. We will use the symbol $\sim$ for linear equivalence.

Locally a divisor is given by an equation $f = 0$. Consequently, the ideal sheaf $\mathcal{I}_C$ is locally principal, and therefore locally free of rank 1. We let $\mathcal{O}_X(nC) = \mathcal{I}_C^{-n}$, where the notation means tensor $\mathcal{I}_C^{\pm 1}$ with itself $|n|$ times. We define

$$\mathcal{O}_X(D) = \mathcal{O}_X(n_1C_1) \otimes \mathcal{O}_X(n_2C_2) \otimes \dots$$

This can also be identified with sheaf of fractional ideals

$$\mathcal{O}_X(D)(U) = \{f \in \mathbb{C}(X) \mid \forall C, C \cap U \neq 0 \Rightarrow ord_C(f) + D \geq 0\}$$

The second description shows that a principal divisor gives a principal fractional ideal, which is isomorphic to $\mathcal{O}_X$. There $D \mapsto \mathcal{O}_X(D)$ gives a homomorphism from $Cl(X)$ to the group of line bundle $Pic(X)$. This is an isomorphism (Hartshorne, chap II, 6.16). A standard computation (loc. cit. 6.17) shows that $Pic(\mathbb{P}^n) = \{\mathcal{O}_{\mathbb{P}^n}(i)\} \cong \mathbb{Z}$.

2.3 Intersection Pairing

Let X be an algebraic surface. Given a line bundle $\mathcal{L}$, let $h^i(\mathcal{L}) = \dim H^i(X, \mathcal{L})$. These are numbers are known to be finite, and zero when $i > 2$. Set

$$\chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L}) + h^2(\mathcal{L})$$

Theorem 2.3.1. *There exists a symmetric bilinear pairing*

$$Cl(X) \times Cl(X) \to \mathbb{Z}$$

such that

(a)

$$(C \cdot D) = \sum_{p \in C \cap D} (C \cdot D)_p \tag{2.1}$$

whenever C and D are distinct irreducible curves.

(b) Let C, D be irreducible curves, and let $\pi : \tilde{C} \to C$ be the normalization and $\iota : \tilde{C} \to X$ the composition with the inclusion. Then

$$C \cdot D = \deg(\iota^* \mathcal{O}_X(D))$$

(c)

$$(C \cdot D) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_X(-C - D)) \tag{2.2}$$

Proof. We will use slick approach due to Mumford, Lectures on curves on an algebraic surface, pp 84-85. Define

$$(C \cdot D) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_X(-C - D))$$

so that (c) holds by definiton. This gives a symmetric pairing on $Cl(X) \cong Pic(X)$. The bilinearity will be proven later. If C and D are distinct irreducible curves, we have a Koszul resolution

$$0 \to \mathcal{O}_X(-C - D) \to \mathcal{O}_X(-C) \oplus \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_{C \cap D} \to 0$$

where $C \cap D$ refers to the scheme defined by $\mathcal{I}_C + \mathcal{I}_D$. $\mathcal{O}_{C\cap D}$ is a sum of sky scraper sheaves of length $(C \cdot D)_p$ at each $p \in C \cap D$. Hence $h^0(\mathcal{O}_{C\cap D})$ is the sum on the right of (2.1), and there are no higher cohomologies. Therefore

$$\sum_{p\in C\cap D} (C\cdot D)_p = \chi(\mathcal{O}_{C\cap D})$$
$$= \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_X(-C-D))$$

This proves (2.1).

We will write $\mathcal{O}_C(\pm D)$ instead of $\iota^*\mathcal{O}_X(\pm D)$ below. We have exact sequences

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$
$$0 \to \mathcal{O}_C \to \pi_*\mathcal{O}_{\tilde{C}} \to \pi_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C \to 0$$

Tensor both with $\mathcal{O}_X(-D)$ to obtain

$$0 \to \mathcal{O}_X(-C-D) \to \mathcal{O}_X(-D) \to \mathcal{O}_C(-D) \to 0$$

and

$$0 \to \mathcal{O}_C(-D) \to \pi_*(\mathcal{O}_{\tilde{C}}) \otimes \mathcal{O}_X(-D) \to (\pi_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C) \otimes \mathcal{O}_X(-D) \to 0$$

We can simplify the second sequence to

$$0 \to \mathcal{O}_C(-D) \to \pi_*(\pi^*\mathcal{O}_C(-D)) \to \pi_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C \to 0$$

using the projection formula

$$\pi_*\pi^*(\mathcal{O}_C(-D)) \cong \pi_*(\mathcal{O}_{\tilde{C}})(-D))$$

and the fact that $\pi_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C$ is a sum of sky scraper sheaves. Using these sequences, we obtain

$$\chi(\mathcal{O}_{\tilde{C}}) - \chi(\pi_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C))$$
$$\chi(\pi^*\mathcal{O}_C(-D)) - \chi(\pi_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C) = \chi(\mathcal{O}_X(-D)) - \chi(\mathcal{O}_X(-C-D))$$

Subtract these and apply (2.2) to obtain

$$C\cdot D = \chi(\mathcal{O}_{\tilde{C}}) - \chi(\pi^*\mathcal{O}_C(-D))$$

Now use Riemann-Roch. This proves (b).

(b) implies additivity when one of the divisors is prime. For the general case, we need the following fact:

Lemma 2.3.2 (Moving lemma)**.** *Any divisor D is linearly equivalent to a difference $E - F$, where E and F are smooth curves in general position.*

A proof of the lemma can be found on pp 358-359 of Hartshorne, although this isn't stated as a lemma there. Let

$$S(C_1, C_2, C_3) = (C_1 \cdot C_2 + C_3) - (C_1 \cdot C_2) - (C_1 \cdot C_3)$$

We need to show that s is identically zero. This is zero if C_1 is prime as noted above. The expression s is easily verified to be symmetric in C_1, C_2, C_2. Therefore it is zero if any of the C_i are prime.

Given divisors C and D, apply the lemma to write $D \sim E - F$ as above. Since $s(C, D, F) = 0$, we obtain

$$(C \cdot D) = (C \cdot E) - (C \cdot F)$$

By what we said earlier, the right side is additive in first variable.

□

Corollary 2.3.3. *Bezout's theorem holds.*

Proof. We give two proofs.

First using the fact that $Pic(\mathbb{P}^2) = Z$ with a generator corresponding to a line. Clearly, $L \cdot L' = 1$ for 2 distinct lines. This together with the other properties forces $C \cdot D = \deg C \deg D$.

For the second proof, combine formulas (2.2) and (1.5) and simplify to obtain $C \cdot D = \deg C \deg D$.

□

We can give the intersection pairing a topological interpretation. We can define the first Chern class

$$c_1 : Pic(X) \to H^2(X, \mathbb{Z})$$

as before as the connecting map associated to the exponential sequence. Given an irreducible curve $C \subset X$, $H_2(C, \mathbb{Z}) = \mathbb{Z}$ with a preferred generator determined by the orientation coming from the complex structure. Let $[C] \in H_2(X, \mathbb{Z})$ be the push forward of this class.

Theorem 2.3.4. *Given irreducible curves $C, D \subset X$*

$$C \cdot D = \langle c_1(\mathcal{O}_X(D), [C] \rangle$$

where $\langle , \rangle$ here denotes evaluation of a cohomology class on a homology class.

Proof. Let $\iota : \tilde{C} \to X$ be the composition of the normalization $\tilde{C} \to C$ and the inclusion. If $[\tilde{C}] \in H_2(\tilde{C}, \mathbb{Z})$ is the preferred generator, then $\iota_*[\tilde{C}] = [C]$. Then

$$\langle c_1(\mathcal{O}_X(D), [C] \rangle = \langle \iota^* \mathcal{O}_X(D), [\tilde{C}] \rangle = \deg \mathcal{O}_C(D)$$

Now apply (b) of the previous theorem. □

This can be put in more symmetric form with the help of Poincaré duality, which gives an isomorphism

$$H_2(X, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$$

Under this isomorphism one has that $[C]$ corresponds to $c_1(\mathcal{O}(C))$. There is a natural pairing

$$H^2(X, \mathbb{Z}) \otimes H^2(X, \mathbb{Z}) \xrightarrow{\cdot} \mathbb{Z}$$

given by cup product followed by evaluation to the fundamental class. Then we have

Theorem 2.3.5. $C \cdot D = c_1(\mathcal{O}_X(C)) \cdot c_1(\mathcal{O}_X(D))$

Let $f : X \to Y$ be a surjective morphism of smooth projective varieties of the same dimension n. Then we get a finite extension of function field $\mathbb{C}(Y) \subset \mathbb{C}(X)$. Its degree is (by definition) the degree of f. Let's call this d. We can understand this more geometrically as follows. After restricting to a Zariski open set $U \subset Y$, $f^{-1}U \to U$ becomes étale. Topologically, this is a covering space, and d is the number of sheets. Given a divisor $D \subset Y$, locally given by an equation $g = 0$. f^*D is the locally defined by $f^*g = 0$. Note that even if D is prime, f^*D need not be. This operation is compatible with pullback of line bundles.

Corollary 2.3.6. *$f : X \to Y$ be a surjective morphism smooth projective surfaces. Then $f^*C \cdot f^*D = (\deg f)(C \cdot D)$*

Proof. We have

$$f^*C \cdot f^*D = \int_X f^*[c_1(\mathcal{O}_X(C)) \cup c_1(\mathcal{O}_X(D))] = \deg f \int_Y c_1(\mathcal{O}_X(C)) \cup c_1(\mathcal{O}_X(D))$$

□

2.4 Adjunction formula

Given a surface X, a two form α is given locally by $f(x,y)dx \wedge dy$, where x, y are coordinates. Given a prime divisor C, the number $ord_C(\alpha) = ord_C(f)$ is well defined. Therefore we can define a divisor

$$div(\alpha) = \sum_C ord_C(\alpha) C$$

The class in $Cl(X)$ is well defined and referred to as the canonical divisor (class) $K = K_X$.

Theorem 2.4.1 (Adjunction formula)**.** *Let $C \subset X$ be a smooth curve of genus g, then*

$$2g - 2 = (K + C) \cdot C$$

The proof will be broken down into a couple of lemmas. Fix $\iota : C \to X$ as above. Given a vector bundle (= locally free sheaf) V of rank r, let $\det V = \wedge^r V$. This is a line bundle.

Lemma 2.4.2. *Given an exact sequence of vector bundles*

$$0 \to V \to W \to U \to 0$$

$$\det W \cong \det V \otimes \det U$$

Proof. Exactly as for line bundles, a rank r vector bundle is determined by an open covering $\{U_i\}$ and a collection of holomorphic maps $g_{ij} : U_{ij} \to GL_r(\mathbb{C})$ satisfying the cocycle identity. This is constructed from a local trivialization. Choose compatible local trivializations for V and W. Then the cocycle for W is of the form

$$g_{ij} = \begin{pmatrix} h_{ij} & * \\ 0 & \ell_{ij} \end{pmatrix}$$

where h_{ij}, ℓ_{ij} are cocycles for V and U. This implies that

$$\det g_{ij} = \det h_{ij} \det \ell_{ij}$$

and therefore that $\det W \cong \det V \otimes \det U$.

□

The theorem will follow from theorem 2.3.1(b) and the next result (which also called the adjunction formula).

Lemma 2.4.3. $\Omega^1_C \cong \Omega^2_X|_C \otimes \mathcal{O}_C(C)$ *where* $\Omega^2_X|_C = \iota^* \Omega^2_X$.

Proof. We have an exact sequence of locally free sheaves

$$0 \to \mathcal{O}_C(-C) \to \Omega^1_X|_C \to \Omega^1_C \to 0$$

By the previous lemma, we obtain

$$\Omega^1_C \otimes \mathcal{O}_C(-C) \cong \Omega^2_X|_C$$

Tensor both sides by $\mathcal{O}_C(C)$ to obtain the lemma. □

The adjunction formula gives a very efficient method for computing the genus. As a first application, let us reprove theorem 1.8.4 for smooth curves. First we observe that:

Lemma 2.4.4. *The canonical bundle of projective space is*

$$\Omega^2_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$$

Proof. Let x_0, x_1, x_2 homogeneous coordinates, then the ratios x_i/x_j give coordinates on the patch $U_j = \{x_j \neq 0\}$. A direct computation gives

$$d\left(\frac{x_1}{x_0}\right) \wedge d\left(\frac{x_2}{x_0}\right) = -\left(\frac{x_0}{x_1}\right)^{-3} d\left(\frac{x_0}{x_1}\right) \wedge d\left(\frac{x_2}{x_1}\right)$$

which shows that the form on the left has singularity of order 3 on the line $x_0 = 0$. □

Remark 2.4.5. *More generally, we have*

$$\Omega^n_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$$

Example 2.4.6. *Suppose that $C \subset \mathbb{P}^2$ is a smooth curve of degree d, then*

$$2g - 2 = (K + C) \cdot C = (-3 + d)(d)$$

solving for g gives us the earlier formula

$$g = \frac{(d-1)(d-2)}{2}$$

Example 2.4.7. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. We can see from Künneth's formula that $H^2(X, \mathbb{Z}) = \mathbb{Z}^2$, with two generators corresponding to the lines $L = \mathbb{P}^1 \times \{0\}$ and $M = \{0\} \times \mathbb{P}^1$. It follows from theorem 2.3.1 that $L^2 = M^2 = 0$ and $L \cdot M = 1$. Let $K = aL + bM$ be the canonical divisor. By the adjunction formula,*

$$-2 = L \cdot (K + L) = b$$

$$-2 = M \cdot (K + M) = a$$

A curve C defined by a bihomogeneous polynomial of bigree (d, e), can be seen to be linearly equivalent to $dL + eM$. The genus is given by

$$2g - 2 = C \cdot (K + C) = (dL + eM)((d-2)L + (e-2)M) = d(e-2) + e(d-2)$$

2.5 Riemann-Roch

Let X be a smooth projective surface as before.

Theorem 2.5.1 (Riemann-Roch for Surfaces)**.** *Given a divisor*

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2} D \cdot (D - K) + \chi(\mathcal{O}_X)$$

As in the curve case, the key ingredient is Serre duality.

Theorem 2.5.2 (Serre duality)**.** *Given an n dimensional compact complex manifold X, there is an isomorphism*

$$H^i(X, \mathcal{L}) \cong H^{n-i}(X, \omega_X \otimes \mathcal{L}^{-1})^*$$

where $\omega_X = \Omega^n_X$ is the canonical sheaf.

Proof of theorem 2.5.1. Using the formula in (2.2), we obtain

$$(-D) \cdot (D - K) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X(K - D)) + \chi(\mathcal{O}_X(K))$$

Note that $\omega_X = \mathcal{O}_X(K)$, so by Serre duality, $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_X(K))$ and $\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(K - D))$. Therefore

$$-D \cdot (D - K) = 2(\chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(D))$$

and the theorem follows.

□

Corollary 2.5.3. $h^0(\mathcal{O}(D)) + h^0(\mathcal{O}(K - D)) \geq \frac{1}{2} D \cdot (D - K) + \chi(\mathcal{O}_X)$

In order to sharpen the above inequality we need to eliminate the second term on the left. Here is a simple way to do it.

Lemma 2.5.4. *If some positive multiple of a divisor E is linearly equivalent to an nonzero effective divisor, then $H^0(\mathcal{O}(-E)) = 0$.*

Proof. Suppose that nE is equivalent to E' effective for some $n > 0$. Embed $X \subset \mathbb{P}^N$, and let $C = X \cap H$ for some general hyperplane. By Bertini, C is nonsingular. Then $\deg \mathcal{O}(E)|_C = \frac{1}{n}(C \cdot E') > 0$ because H must meet E'. If $f \in H^0(X, \mathcal{O}(-E))$ is nonzero, then f restricts to a nonzero section of $\mathcal{O}(-E)|_C$. But this impossible, since the degree is negative. Therefore $H^0(X, \mathcal{O}(-E)) = 0$. □

Corollary 2.5.5. *If $D - K$ is linearly equivalent to an effective divisor then* $h^0(\mathcal{O}(D)) \geq \frac{1}{2} D \cdot (D - K) + \chi(\mathcal{O}_X)$

Proof. If $D - K$ is linearly equivalent to 0, then the corollary reduces to equality $\chi(\mathcal{O}(K)) = \chi(\mathcal{O}_X)$ noted earlier. Otherwise apply the lemma to obtain $H^0(X, \mathcal{O}(K - D)) = 0$. The together with the previous corollary implies the result. □

When Riemann-Roch is combined with Kodaira's vanishing theorem or one of its refinements, we can get an exact formula for the dimension of global sections. Kodaira's theorem is discussed in Griffiths-Harris. For the more recent results, see Lazarsfeld, Positivity in Algebraic Geometry.

Theorem 2.5.6 (Kawamata-Viehweg/Kodaira)**.** *If D is a divisor such that $D^2 > 0$ and $D \cdot C \geq 0$ for every effective divisor C,*

$$H^i(X, \mathcal{O}_X(K + D)) = 0, i > 0$$

In particular, this holds when D is ample i.e. a postive multiple is equivalent to the pullback of a hyperplane for some projective embedding of X.

Corollary 2.5.7. *If D is as above,*

$$h^0(\mathcal{O}(K + D)) = \frac{1}{2} D(K + D) + \chi(\mathcal{O}_X)$$

The Riemann-Roch theorem as stated above, gives no information about $\chi(\mathcal{O}_X)$. In fact there is a more precise form of Riemann-Roch due to Hirzebruch which yields

$$\chi(\mathcal{O}_X) = \frac{K^2 + e(X)}{12}$$

where the expressions are the self intersections of the canonical divisor and the topological Euler characteristic. In fact an equivalent statement was obtained by Max Noether over 100 years ago, consequently this often refered to as "Noether's formula".

Now we give an application automorphic forms in two variables. Let $\Gamma \subset SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ be discrete subgroup. It acts on the product of two copies of the upper half plane $\mathbb{H}^2$. We assume that the action is fixed point free and that the quotient $X = \mathbb{H}^2/\Gamma$ is compact. This is a complex manifold. In fact, using Kodaira's embedding theorem (see Griffiths-Harris) we can see that X is a smooth projective surface and K is ample. A section of $\mathcal{O}(mK)$ can be interpreted as an automorphic form of weight $2m$, i.e. a holomorphic function on $\mathbb{H}^2$ satisfying

$$f\left(\frac{a_1z_1 + b_1}{c_1z_1 + d_1}, \frac{a_2z_2 + b_2}{c_2z_2 + d_2}\right) = (c_1z_1 + b_1)^{2m}(c_2z_2 + b_2)^{2m} f(z_1, z_2)$$

for every element of Γ. From Riemann-Roch together with Kodaira vanishing, we obtain

$$\text{Dim of wt } 2m \text{ automorphic forms} = \frac{1}{2}m(m-1)K^2 + \chi(\mathcal{O}_X)$$

2.6 Blow ups and Castelnuovo's theorem

The blow up of the plane at the origin

$$Bl_0\mathbb{A}^2 = \{(x, \ell) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid x \in \ell\}$$

This is easily seen to be a quasiprojective variety. This comes with a map $\pi : Bl_0\mathbb{A}^2 \to \mathbb{A}^2$ given by projection. It is an isomorphism over $\mathbb{A}^2 - \{0\}$. The fibre $\pi^{-1}0 \cong \mathbb{P}^1$ is called the exceptional divisor. More generally, one can blow a point on any smooth projective surface X to get a new smooth projective surface Bl_pX with a map $\pi : Bl_pX \to X$ which is an isomorphism over $X - p$ and with $\pi^{-1}p = E \cong \mathbb{P}^1$. Let us denote Bl_pX by Y.

Proposition 2.6.1.

(a) $\pi^*C \cdot \pi^*D = C \cdot D$

(b) $\pi^*C \cdot E = 0$

(c) $E^2 = -1$

Proof. Then $\pi : Y \to X$ has degree 1, therefore by corollary 2.3.6 implies (a). By lemma 2.3.2, we can assume that $p \notin C$. Therefore $\pi^* C \cdot E = 0$ because they are disjoint. For (c), choose a smooth curve C containing p. The set theoretic preimage $\pi^{-1}C$ is a union of E and a curve $C' = \overline{C - \{p\}}$ called the strict transform. This means that $\pi^* C$ is a linear combination of these two divisors. To calculate the coefficients, we can choose local analytic coordinates x, y about p so that x gives the local equation of C. On Y we have a chart with coordinates $y, t = x/y$. Then $\pi^* C$ is defined by $yt = x = 0$. So locally it factors into $y = 0$ and $t = 0$ which define E and C respectively. This means that $\pi^* C = C' + E$. We also see from the local calculation that C' and E meet transversally at one point. Therefore

$$0 = E \cdot (C' + E) = 1 + E^2$$

□

Theorem 2.6.2 (Castelnuovo)**.** *If Y is a surface with an curve $E \cong \mathbb{P}^1$ with $E^2 = -1$, then it must be the blow up of a smooth surface X such that E is the exceptional divisor.*

Sketch. We outline the main step of the proof. First let us indicate the broad strategy. If $\pi : X \to Y$ exists and because our surfaces are projective, we have an embedding $\iota : Y \subset \mathbb{P}^N$. The restriction $\mathcal{O}_Y(1) = \iota^* \mathcal{O}_{\mathbb{P}^N}(1)$ gives a line bundle which is generated by global sections say $f_0, \ldots, f_N$. We can pull the bundle and it sections back to X to get $\pi^* f_i$. Since these are locally just functions, it makes sense to map $x \mapsto [\pi^* f_0(x), \ldots, \pi^* f_N(x)] \in \mathbb{P}^N$. In this way, recover Y as the image of this map in $\mathbb{P}^N$. The trick is to do this without knowing Y. Let us start with a very ample divisor H on X. This means that H is the restriction of hyperplane to X under an embedding $X \subset \mathbb{P}^M$. It is technically can convenient that $H^1(X, \mathcal{O}(H)) = 0$. This can always be acheived by replacing H by a positive multiple, if necessary, by a theorem of Serre. Now let $D = H + nE$, where $n = E \cdot H$. Observe that $D \cdot E = 0$.

We claim that $H^1(X, \mathcal{O}_X(H + iE)) = 0$ for $i = 0, \ldots, n$. For $i = 0$, this is by assumption. Suppose it's true for $i < n$. Then from the sequence

$$0 \to \mathcal{O}_X(H + iE) \to \mathcal{O}_X(H + (i+1)E) \to \mathcal{O}_E(H + (i+1)E) \to 0 \qquad (2.3)$$

Observe that $E = \mathbb{P}^1$ and $D \cdot E = 0$. Therefore

$$\mathcal{O}_E(H + (i+1)E) \cong \mathcal{O}_{\mathbb{P}^1}(n - (i+1))$$

So we get

$$H^1(X, \mathcal{O}_X(H + iE)) \to H^1(X, \mathcal{O}(H + (i+1)E)) \to H^1(\mathbb{P}^1, \mathcal{O}(d))$$

where $d = n - (i+1) \geq 0$. By induction, $H^1(X, \mathcal{O}_X(H + iE)) = 0$. By Serre duality

$$H^1(\mathbb{P}^1, \mathcal{O}(d)) \cong H^0(\mathbb{P}^1, \mathcal{O}(-2 - d)) = 0$$

This implies the claim.

Next, we claim that $\mathcal{O}_X(D)$ is generated by global sections. This means that for any $p \in X$, we can find global section which is nonzero at p. For $p \notin E$ this is easy. Since H is very ample, $\mathcal{O}_X(H)$ is generated by generated by global sections. Under the inclusion $\mathcal{O}_X(H) \subset \mathcal{O}_X(D)$, these give sections $\mathcal{O}(D)$. For any $p \notin E$, we can find such a section nonzero at p. From (2.3), we obtain

$$H^0(\mathcal{O}_X(D)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \to H^1(\mathcal{O}_X(H + (n-1)E)) = 0$$

This implies that we can lift the constant section 1 on E can be lifted to a global section.

Now we can map $\pi : X \to \mathbb{P}^N$, where $N = \dim H^0(X, \mathcal{O}_X(D))$, using global sections as above. Along E, these sections are all constant. Therefore $\pi(E)$ is a point. Let $Y = \pi(X)$. This will turn out to be the desired variety. It remains to check that Y is smooth, and that X is the blow up of Y at the image $\pi(E)$. For these details, we refer to Hartshorne, pp 415-416.

□