Chapter 3

Abelian varieties

3.1 Elliptic curves

An elliptic curve is a curve X of genus one with a distinguished point 0. Topologically it is looks like a torus. A basic example is given as follows. A subgroup $L \subset \mathbb{C}$ generated by a real basis of $\mathbb{C}$ is called a lattice. As an abstract group $L \cong \mathbb{Z}^2$. The group quotient $\mathbb{C}/L$ has the structure of genus one curve with the image of 0 as the distinguished point. This has a commutative group law such that the group operations are holomorphic. Therefore these operations are regular by GAGA. In fact, one can see this more directly by embedding $\mathbb{C}/L \subset \mathbb{P}^2$ as a cubic so that 0 maps to an inflection point. Then $p + q + r = 0$ precisely when p, q, r are collinear. The details can be found in any book on elliptic curves such as Silverman.

We want to show that all examples of elliptic curves are given as above.

Theorem 3.1.1. *Any elliptic curve is isomorphic to $\mathbb{C}/L$, for some lattice.*

There are a number of ways to prove this. We use an argument which will generalize. We have that $H^0(X, \Omega^1_X) = 1$. Pick a nonzero element ω in it. Also choose a basis α, β for $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^2$.

Proposition 3.1.2. *The integrals $\int_\alpha \omega, \int_\beta \omega$ are linearly independent over $\mathbb{R}$. Therefore they generate a lattice $L \subset \mathbb{C}$ called the period lattice.*

Proof. We first redefine L in a basis free way. We have a map $I : H_1(X, \mathbb{Z}) \to H^0(X, \Omega^1_X)^*$ which sends $\gamma \mapsto \int_\gamma$, and L is just the image. The proposition is equivalent to showing that the $\mathbb{R}$-linear extension

$$I_\mathbb{R} : H_1(X, \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{R} \to H^0(X, \Omega^1_X)^*$$

is an injection of real vector spaces. It is enough to show that it is a surjection, because they have the same dimension. The dual map

$$I^*_\mathbb{R} : H^0(X, \Omega^1_X) \to Hom(H_1(X, \mathbb{Z}), \mathbb{R}) \cong H^1(X, \mathbb{R})$$

factors through the natural map

$$H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C})$$

which, as we saw earlier, is an inclusion. This implies that $I_{\mathbb{R}}$ is surjective.

□

We define the Jacobian $J(X) = \mathbb{C}/L$. Let $\int_0^p \omega$ denote the integral with respect to some path connecting 0 to p. Let $AJ(p)$ denote the image in $J(X)$. Since we are dividing out by periods, this is independent of the chosen path. The map $AJ : X \to J(X)$ is called the Abel-Jacobi map.

Proposition 3.1.3. *AJ is holomorphic.*

Proof. Let $\tilde{X}$ denote the universal cover of X. This is naturally a complex manifold, and it is enough to show that the induced map $\widetilde{AJ} : \tilde{X} \to J(X)$ is holomorphic. Since $\tilde{X}$ is simply connected, $H^1(\tilde{X}, \mathbb{C}) = 0$. Therefore the pullback of ω to $\tilde{X}$ equals df for some C^∞ function f. This is holomorphic because ω is. Therefore

$$\widetilde{AJ}(p) = \int_0^p df = f(p) - f(0)$$

is holomorphic. □

Proposition 3.1.4. *AJ is a holomorphic isomorphism.*

Proof. Since we already know that AJ is holomorphic, it is enough to show that is a diffeomorphism. We saw that $H^0(X, \Omega^1_X) \cong H^1(X, \mathbb{R})$. Under this identification, we can describe AJ as given by integration of C^∞ closed 1-forms. Note any two 2-tori are diffeomorphic, so we can assume that $X = \mathbb{R}^2/\mathbb{Z}^2$. If x, y are standard coordinates on $\mathbb{R}^2$, then dx, dy give a basis for $H^1(X, \mathbb{R})$. We may write

$$AJ(a, b) = (\int_{(0,0)}^{(a,b)} dx, \int_{(0,0)}^{(a,b)} dy) = (a, b) \mod \text{periods}$$

using the straight line path, and the period lattice can be identified with the standard lattice.

□

This concludes the proof of theorem 3.1.1.

Corollary 3.1.5. *Any elliptic curve is isomorphic to $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$, where $\tau \in \mathbb{H}$.*

Proof. We know that $X = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1/\omega_2 \notin \mathbb{R}$. Then $\tau = (\omega_1/\omega_2)^{\pm 1} \in \mathbb{H}$ and multiplication by some constant induces an isomorphism $X = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. □

One can show that

$$\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau \cong \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau'$$

if and only if τ and τ' lie in the same orbit of $SL_2(\mathbb{Z})$ if and only if $j(\tau) = j(\tau')$, where

$$j(\tau) = 1728 \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2}$$

with

$$g_2(\tau) = 60 \sum_{(m,n)\in\mathbb{Z}^2 - \{(0,0)\}} \frac{1}{(m\tau+n)^4}$$

$$g_3(\tau) = 140 \sum_{(m,n)\in\mathbb{Z}^2 - \{(0,0)\}} \frac{1}{(m\tau+n)^6}$$

3.2 Abelian varieties and theta functions

A complex torus is quotient V/L of a finite dimensional complex vector space $V \cong \mathbb{C}^n$ by a lattice $L \cong \mathbb{Z}^{2n}$. This is a complex Lie group, that it is a complex manifold with a group structure whose operations are holomorphic. We say that a complex torus is an abelian variety if it can be embedded into some complex projective space $\mathbb{P}^N$ as a complex submanifold. GAGA would imply that it is a projective variety and the that the group operations are regular.

Our first goal is to find a more convenient criterion for a torus to be an abelian variety. Let us start with an elliptic curve $X = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. We know that X can embedded into projective space in principle, but we would like to do this as explicitly as possible. So need to construct functions $f_i : \mathbb{C} \to \mathbb{C}$ such that $p \mapsto [f_0(p), \ldots, f_N(p)] \in \mathbb{P}^N$ is well defined and gives an embedding. In order for it to be well defined, we need "quasiperiodicity"

$$f_i(p+\lambda) = (\text{some factor}) f_i(p), \quad \forall \lambda \in \mathbb{Z} + \mathbb{Z}\tau$$

where the factor in front is the same for all i and nonzero. To begin with, we construct the Jacobi θ-function is given by the Fourier series

$$\theta(z) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau) \exp(2\pi i n z)$$

Writing $\tau = x + iy$, with $y > 0$, shows that on a compact subset of the z-plane the terms are bounded by $O(e^{-n^2 y})$. So uniform convergence on compact sets is guaranteed. This is clearly periodic

$$\theta(z+1) = \theta(z) \tag{3.1}$$

In addition it satifies the function equation

$$\begin{aligned} \theta(z+\tau) &= \sum \exp(\pi i n^2 \tau + 2\pi i n(z+\tau)) \\ &= \sum \exp(\pi i (n+1)^2 \tau - \pi i \tau + 2\pi i n z) \\ &= \exp(-\pi i \tau - 2\pi i z)\theta(z) \end{aligned} \tag{3.2}$$

It turns out that θ spans the space of solutions to these equations. We get more solutions by relaxing these conditions. Let $N > 0$ be an integer, and consider the space V_N of holomorphic functions satisfying

$$\begin{aligned} f(z+N) &= f(z) \\ f(z+N\tau) &= \exp(-\pi i N^2\tau - 2\pi i N z) f(z) \end{aligned} \tag{3.3}$$

Any function in V_N can be expanded in a Fourier series by the first equation, and the second equation gives recurrence conditions for the coefficients. This leads to

Lemma 3.2.1. $\dim V_N = N^2$.

Proof. See Mumford's Lectures on Theta 1, page 9, for details. □

The functions

$$\theta_{a,b}(z) = \exp(\pi i a^2\tau + 2\pi i a(z+b))\theta(z+a\tau+b), \quad a,b \in \frac{1}{N}\{0,\dots,N-1\}$$

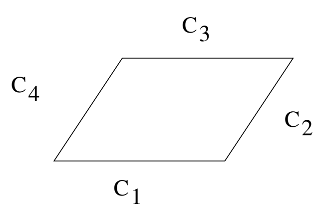
lie in V_N and are independent. Therefore they form a basis.

Lemma 3.2.2. *Given nonzero $f \in V_N$, it has exactly N^2 zeros, counted with multiplicities, in the parallelogram with vertices $0, N, N\tau, N+\tau$ (where we translate if necessary so no zeros lie on the boundary).*

Proof. Complex analysis tells us that the number of zeros is given by the integral

$$\frac{1}{2\pi i}\int_{C_1+C_2+C_3+C_4} \frac{f'(z)dz}{f(z)}$$

over the boundary of the parallelogram.



Using $f(z+N) = f(z)$, we obtain

$$\int_{C_1+C_3} \frac{f'(z)dz}{f(z)} = 0$$

and from $f(z+N\tau) = Const.\exp(-2\pi i N z)f(z)$, we obtain

$$\int_{C_2+C_4} \frac{f'(z)dz}{f(z)} = 2\pi i N^2$$

□

Theorem 3.2.3. *Choose an integer $N > 1$ and basis f_i of V_N. The the map of $\phi : \mathbb{C}/L \to \mathbb{P}^{N^2-1}$ by $z \mapsto [f_i(z)]$ is an embedding.*

Proof. Suppose that ϕ is not one to one. Say that $f(z_1) = f(z_1')$ for some $z_1 \neq z_1'$ in $\mathbb{C}/L$ and all $f \in V_N$. By translation by $(a\tau + b)/N$ for $a, b \in \frac{1}{N}\mathbb{Z}$, we can find another such pair z_2, z_2' with this property. Choose additional points, so that $z_1, \ldots z_{N^2-1}$ are distinct in $\mathbb{C}/NL$. We define a map $V_N \to \mathbb{C}^{N^2-1}$ by $f \mapsto (f(z_i))$. Since $\dim V_N = N^2$, we can find a nonzero $f \in V_N$ so that

$$f(z_1) = f(z_2) = f(z_3) = \ldots f(z_{N^2-1}) = 0$$

Notice that we are forced to also have $f(z_1') = f(z_2') = 0$ which means that f has at least $N^2 + 1$ zeros which contradicts the lemma.

A similar argument shows that the derivative $d\phi$ is nonzero.

□

Now let us consider a general g dimensional complex torus $X = \mathbb{C}^g/L$. In order to generalize the previous arguments, let us assume that the lattice is a special form. Suppose that Ω is a $g \times g$ symmetric matrix with positive definite imaginary part. This will play the role of τ. The set of such matrices is called the "Siegel upper half plane", and we denote it by $\mathbb{H}_g$. Suppose that

$$L = \mathbb{Z}^g + \underbrace{\mathbb{Z}^g\Omega}_{\text{integer column space of } \Omega}$$

Then we now construct the Riemann theta function on $\mathbb{C}^g$

$$\theta(z) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i n^t \Omega n + 2\pi i n^t z)$$

which generalizes Jacobi's theta function. We note because of the assumptions, we made about Ω, this series will converge to a holomorphic function on $\mathbb{C}^g$ which is quasiperiodic. With this function hand we can construct a large class of auxillary functions $\theta_{a,b}$ as before, and use these to construct an embeding of X into projective space.

Theorem 3.2.4 (Riemann)**.** *If $\Omega \in \mathbb{H}_g$ then $\mathbb{C}^g/\mathbb{Z}^g + \mathbb{Z}^g\Omega$ is an abelian variety.*

We want to ultimately rephrase the condition in a coordinate free language. But first we have to do some matrix calculations. Let $\Pi = (I, \Omega)$ and let

$$E = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

be $2g \times 2g$. E determines a symplectic pairing

$$L \times L \to \mathbb{Z}$$

One can check that the following identity holds

$$E(iu, iv) = E(u, v) \tag{3.4}$$

Then $\Omega \in \mathbb{H}_g$ is equivalent to

1. $\Pi E^{-1}\Pi^T = 0$ and
2. $i\Pi E^{-1}\bar{\Pi}^T$ is positive definite.

Set

$$H(u,v) = E(iu,v) + iE(u,v) \tag{3.5}$$

Then the following is easy to check directly, or see Birkenhake-Lange §4.2.

Lemma 3.2.5. *The above conditions are equivalent to H being positive definite hermitian.*

An integer valued nondenerate symplectic pairing E on L is called a *polarization* if (3.4) holds and H, defined by (3.5), is positive definite hermitian. Note that $E = ImH$, so E and H determine each other. The polarization is called principal if in addition, E is equivalent to

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

A slight refinement of the previous theorem.

Theorem 3.2.6 (Riemann)**.** *$\mathbb{C}^g/L$ is an abelian variety if L carries a polarization.*

We will see later the converse is true.

3.3 Jacobians

Let X be a smooth projective curve of genus g. Then

$$H^1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$$

We also proved earlier that $H^1(X,\mathcal{O}_X) \cong \mathbb{C}^g$. The natural map $\mathbb{Z} \to \mathcal{O}_X$ of sheaves induces a map

$$H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X)$$

Proposition 3.3.1. *This map is injective and the image is lattice.*

Proof. Since $H^1(X,\mathbb{Z})$ sits a lattice inside $H^1(X,\mathbb{R})$, it is enough to show that the natural map

$$r : H^1(X,\mathbb{R}) \to H^1(X,\mathcal{O}_X)$$

is injective. It α is a harmonic form representing a nonzero element of $H^1(X,\mathbb{R})$. Then we can uniquely decompose $\alpha = \alpha^{0,1} + \alpha^{1,0}$ into a sum of a holomorphic and antiholomorphic forms. Note that $r(\alpha) = \alpha^{0,1}$. Since α is real, $\alpha^{0,1} = \overline{\alpha^{1,0}}$. Therefore $(\alpha) \neq 0$.

□

We define the Jacobian

$$J(X) = \frac{H^1(X, \mathcal{O}_X)}{H^1(X, \mathbb{Z})}$$

This is a g dimensional complex torus. Form the exponential sequence, we see that this fits into a sequence

$$J(X) \to H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$$

In other words, $J(X) = Pic^0(X)$ as groups.

Next, we construct a polarization on $J(X)$. We have a cup product pairing $-E$ on $H^1(X, \mathbb{Z})$ which coincides with the intersection pairing under the Poincaré duality isomorphism

$$H^1(X, \mathbb{Z}) \cong H_1(X, \mathbb{Z})$$

In terms of the embedding $H^1(X, \mathbb{Z}) \subset H^1(X, \mathbb{R})$, this given by integration

$$E(\alpha, \beta) = -\int_X \alpha \wedge \beta$$

The key point is that E is skew symmetric with determinant $+1$. By linear algebra, we can find a basis for L, called a symplectic basis, so that E is represented by the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

We can identify $H^1(X, \mathcal{O}_X) \subset H^1(X, \mathbb{C})$ with the subspace spanned by antiholomorphic 1-forms. The are forms given locally by $\alpha = \overline{f(z)} d\bar{z}$ where f is holomorphic. Since

$$\alpha \wedge \bar{\alpha} = +2i|f(z)|^2 dx \wedge dy$$

we conclude that

$$H(\omega, \eta) = i \int_X \omega \wedge \bar{\eta}$$

is a positive definite Hermitian form on $H^1(X, \mathcal{O}_X)$. Thus E defines a principal polarization. Therefore

Theorem 3.3.2. *The Jacobian $J(X)$ is a g dimensional principally polarized abelian variety.*

In the first section, we used the dual description of $J(X)$. Given a complex torus $T = V/L$, let $V^\dagger$ denote the space of antilinear maps $\ell : V \to \mathbb{C}$ and let

$$L^\dagger = \{\ell \in V^\dagger \mid Im\ell(L) \subseteq L\}$$

The dual torus $T^* = V^\dagger/L^\dagger$. If H is a principal polarization, then $v \mapsto H(v, -)$ gives an isomorphism $V \cong V^\dagger$ mapping L isomorphically to $L^\dagger$. Therefore $T \cong T^*$. We can apply this to the Jacobian to obtain

$$J(X) \cong \frac{H^1(X, \mathcal{O})^\dagger}{H^1(X, \mathbb{Z})^\dagger} \cong \frac{H^0(X, \Omega^1_X)^*}{H_1(X, \mathbb{Z})}$$

Using the last description, we define the Abel-Jacobi map as follows. Fix a base point x_0. Define

$$AJ : X \to J(X)$$

by sending x to the functional $\int_{x_0}^{x}$. We extend this to a map

$$AJ : X^n \to J(X)$$

by $AJ(x_1, \dots, x_n) = \sum AJ(x_i)$. This factors through quotient $S^n X := X/S_n$ by the symmetric group. This can be given the structure of a smooth projective variety. The points of $S^n X$ can be viewed as degree n effective divisors.

Theorem 3.3.3 (Abel's theorem)**.** *$AJ(D) = AJ(D')$ if and only if D is linearly equivalent to D'.*

We will prove the reverse direction. We need:

Lemma 3.3.4. *Any holomorphic map $f : \mathbb{P}^N \to T$ to a complex torus is constant.*

Proof. Since $\mathbb{P}^N$ is simply connected, f lifts to a holomorphic map from $\mathbb{P}^N$ to the universal cover $\mathbb{C}^m$. Since $\mathbb{P}^n$ is compact, this is constant. □

Proof of theorem. Fix $D \in S^n X$. The set of effective divisors linearly equivalent to a given degree n divisor D forms a projective space $|D| = \mathbb{P}(H^0(X, \mathcal{O}_X(D))$. Furthermore, this is a subvariety of $S^n X$. Therefore $AJ(|D|)$ is constant by the lemma.

□

Theorem 3.3.5 (Jacobi's inversion theorem)**.** *If $n \geq g$, then $AJ : S^n X \to J(X)$ is surjective.*

We won't prove this, but we will draw some conclusions. Jacobi implies that $AJ : S^g X \to J(X)$ is a surjective map between varieties of the same dimension. Moreover, Abel's theorem tells us that the fibres of AJ are projective spaces and therefore connected. Therefore $AJ : S^g X \to J(X)$ is birational. Weil gave a purely algebraic construction of $J(X)$ by working backwards from this. First he observed that $S^g X$ has a "birational group law" i.e. it has rational maps which satisfy the identities of a group. Then he showed that any birational group can always be completed to an algebraic group, that is a variety with morphisms that make it into a group. Carrying out the procedure for $S^g X$ yields $J(X)$. This construction works over any algebraically closed field.

3.4 More on polarizations

Earlier we showed that polarized tori are abelian varieties. Now we want to show the converse. To do this, we need to understand the geometric meaning of polarizations. Let us start with analyzing the de Rham cohomology of a torus For the moment, we can ignore the complex structure and work with a real torus $X = \mathbb{R}^n/\mathbb{Z}^n$. Let $e_1, \ldots, e_n$ (resp. $x_1, \ldots, x_n$) denote the standard basis (resp. coordinates) of $\mathbb{R}^n$. A k-form is an expression

$$\alpha = \sum f_{i_1,\ldots i_k}(x_1, \ldots, x_n)dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

where the coefficient are L-periodic C^∞-functions. We let $\mathcal{E}^k(X)$ denote the space of these. As usual

$$d\alpha = \sum \frac{\partial f_{i_1,\ldots i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

This satisfies $d^2 = 0$. So we defined k-th de Rham cohomology as

$$H^k(X, \mathbb{R}) = \frac{ker[\mathcal{E}^k(X) \stackrel{d}{\to} \mathcal{E}^{k+1}(X)]}{im[\mathcal{E}^{k-1}(X) \stackrel{d}{\to} \mathcal{E}^k(X)]}$$

The constant form $dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ certainly defines an element of this space, which is nonzero because it has nonzero integral along the subtorus spanned by e_{i_j}. (Integration can be interpreted as pairing between cohomology and homology.) These form span by the following special case of Künneth's formula:

Theorem 3.4.1. *A basis is given by cohomology classes of constant forms* $\{dx_{i_1} \wedge \ldots \wedge dx_{i_k}\}$. *Thus* $H^k(X, \mathbb{R}) \cong \wedge^k \mathbb{R}^n$.

It is convenient to make this independent of the basis. Let us suppose that we have a torus $X = V/L$ given as quotient of real vector space by a lattice. Then we can identify $\alpha \in H^k(X, \mathbb{R})$ with the alternating k-linear map

$$L \times \ldots \times L \to \mathbb{R}$$

sending $(\lambda_1, \ldots, \lambda_k)$ to the integral of α on the torus spanned by λ_j. Thus we have an natural isomorphism

$$H^k(X, \mathbb{R}) = \wedge^k Hom(L, \mathbb{R})$$

This works with any choice of coefficients such as $\mathbb{Z}$. For our purposes, we can identify $H^k(X, \mathbb{Z})$ the group of integer linear combinations of constant forms. Then

$$H^k(X, \mathbb{Z}) = \wedge^k L^*, \; L^* = Hom(L, \mathbb{Z})$$

Now we return to the complex case. Let $V = \mathbb{C}^n$ be a complex vector space, and $X = V/L$ a complex torus. Let $z_1, \ldots, z_n$ be coordinates on V. Then $x_1 = Re\, z_1, y_1 = Im\, z_1, \ldots$ give real coordinates. Suppose that $E : L \times L \to \mathbb{Z}$ is a polarization. By the above, we can view $E \in H^2(X, \mathbb{Z})$. Under the inclusion $H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{C})$, we can express

$$E = \sum a_{pq} dz_p \wedge dz_q + \sum b_{pq} dz_p \wedge d\bar{z}_q + \sum c_{pq} d\bar{z}_p \wedge d\bar{z}_q$$

Identifying $V = \mathbb{R} \otimes_{\mathbb{Z}} L$ gives a real endomorphism J on the right corresponding to i on the right. Recall that to be a polarization E is required to satisfy

$$E(Ju, Jv) = E(u, v) \tag{3.6}$$

$$H(u, v) = E(Ju, v) + iE(u, v) \text{ is positive definite} \tag{3.7}$$

Using the fact that $Jdz_p = -idz_p$ etc., these can be translated into conditions on the differential form. Condition (3.6) says

$$E = -\sum a_{pq} dz_p \wedge dz_q + \sum b_{pq} dz_p \wedge d\bar{z}_q - \sum c_{pq} d\bar{z}_p \wedge d\bar{z}_q$$

Therefore

$$E = \sum b_{pq} dz_p \wedge d\bar{z}_q$$

is a $(1, 1)$-form. If we normalize the coefficients as

$$E = \frac{i}{2} \sum h_{pq} dz_p \wedge d\bar{z}_q$$

then condition (3.7) is equivalent to (h_{pq}) being a postive definite hermitian matrix. A $(1, 1)$-form locally of this type is called positive.

Now let us now turn to the cohomology of $\mathbb{P}^N$. It is known that $H^2(\mathbb{P}^N, \mathbb{Z}) \cong \mathbb{Z}$. Let us represent the generator by a differential form. Let $z_0, \ldots, z_N$ denote homogeneous coodinates. These represent true coordinates on $\mathbb{C}^{N+1}$. We have a projection $\pi : \mathbb{C}^{N+1} - \{0\} \to \mathbb{P}^N$. Let

$$\tilde{\omega} = \frac{i}{2\pi} \partial\bar{\partial} \log ||(z_0, \ldots, z_N)||^2 = \frac{i}{2\pi} \sum \frac{\partial^2}{\partial z_p \partial \bar{z}_q} \log ||(z_0, \ldots, z_N)||^2$$

We now summarize facts that can be checked by direct calculation. See Griffiths-Harris, pp 31-32 for the first two.

1. $\tilde{\omega}$ is expressible in terms of $z_0/z_i, \ldots, z_N/z_i$ which means that $\tilde{\omega} = \pi^*\omega$ for some 2-form on $\mathbb{P}^N$.
2. ω is positive.
3. If $\mathbb{P}^1 \subset \mathbb{P}^N$ is the line $z_2 = z_3 = \ldots 0$. Then $\int_{\mathbb{P}^1} \omega = 1$.

The form ω is usually called "the Kähler form associated to the Fubini-Study metric", although we will not explain what these words mean. The key point for us is that ω is a positive $(1, 1)$-form which represents an element of $H^2(\mathbb{P}^N, \mathbb{Z})$. Now suppose that $X \subset \mathbb{P}^N$ is a complex submanifold. Then $\omega|_X$ is also represents and integral cohomology class. Furthermore ω is locally $\frac{i}{2}\partial\bar{\partial} f$ for some C^∞-function f for which the so called Levi form $\frac{\partial^2 f}{\partial z_p \partial \bar{z}_q}$ is positive definite. We see that $\omega|_X$ is locally $\frac{i}{2}\partial\bar{\partial} f|_X$ and the Levi form of $f|_X$ has the same property. Therefore $\omega|_X$ is again positive. In the case where X is a complex torus, this proves:

Theorem 3.4.2. *An abelian variety must carry a polarization.*