Chapter 4

Elliptic Surfaces

4.1 Fibered surfaces

Let us say that a (smooth projective) surface X is fibered if it admits a morphism $f : X \to C$ to (smooth projective) curve, such that all fibres are connected. Using Hartshorne Chap III lemma 10.5, we can see that all but finitely many of the fibres $X_q = f^{-1}(q)$ are smooth curves. Let $\Sigma = \{q_1, \ldots, q_n\} \subset C$ be the set of points for which X_{q_i} is singular, and $U = C - \Sigma$. We call Σ the discriminant. A singular fibre X_q is best viewed as a subscheme of X, but it's a bit simpler to view it as a divisor $\sum m_i F_i$ given as the pullback of the divisor $q \in C$.

Given a point $p \in X$, set $q = f(p)$. Then we have a morphism $\mathcal{O}_{C,q} \to \mathcal{O}_{X,p}$, which allows us to view the second ring as a module over the first. Since $\mathcal{O}_{X,p}$ is an integral domain, it is torsion free as an $\mathcal{O}_{C,q}$-module.

Lemma 4.1.1. *A torsion free module M over a PID R is flat.*

Proof. We can write $M = \varinjlim M_i$, where M_i are finitely generated submodules. Since M_i is necessarily torsion free, it is free by the structure theorem for finitely generated modules over a PID. Therefore M_i is flat. Since $(\varinjlim M_i) \otimes N = \varinjlim (M_i \otimes N)$, and direct limits preserve exactness, $M \otimes -$ is an exact functor. □

Corollary 4.1.2. *$\mathcal{O}_{X,p}$ is a flat $\mathcal{O}_{C,q}$-module for all points. Therefore $f : X \to Y$ is a flat map.*

The significance of this stems from the following fact (see Hartshorne, chap III 9.10).

Theorem 4.1.3. *Given a flat map over a connected base, the arithmetic genera of the fibres are constant.*

Let us refer to this constant number as the fibre genus. In particular, it follows that the usual genera of the smooth fibres $X_q, q \in U$ are the same. This can be proved by a differential topological argument. Set $X_U = f^{-1}U$.

Proposition 4.1.4. *All fibres in X_U are diffeomorphic.*

Sketch. Choose a Riemannian metric on X_U. This allows us to split the tangent spaces T_pX_U into a direct sum of the vertical space $T_p^V = \ker df$ and horizontal space $T_p^H = (T_p^V)^\perp$. Call a curve horizontal if all of its tangent vectors lie in the horizontal spaces. Given two points $q, q' \in U$, connect them by a path $\gamma : [0,1] \to U$. For each $p \in X_q$, let $\tilde{\gamma}_p : [0,1] \to X_U$ be unique horizontal lift starting at p. Then $\phi(p) = \tilde{\gamma}_p(1)$ gives a diffeomorphism $\phi : X_q \to X_{q'}$. □

Refining this idea gives more:

Theorem 4.1.5 (Ehresmann)**.** *$X_U \to U$ is a C^∞ fibre bundle, i.e. there exists a g and an open cover in the usual topology $\{U_i\}$ of U, such that $f^{-1}U_i$ is diffeomorphic to $U_i \times F$, where F is compact genus g curve.*

The last theorem says that all cohomology groups $H^1(X_q, \mathbb{Z}) = \mathbb{Z}^{2g}, q \in U$. However, they can fit together in a nontrivial way. This idea can be measured in precise way using the concept of monodromy. Given an element $\gamma \in \pi_1(U, q)$, we can represent it by a path $\gamma : [0,1] \to U$ with $\gamma(0) = \gamma(1) = q$. We have an associated diffeomorphism $\phi : X_q \cong X_q$ constructed as above. Let $\rho(\gamma) = \phi^* : H^1(X_q, \mathbb{Z}) \cong H^1(X_q, \mathbb{Z})$. This is generally a nontrivial automorphism. This gives a homomorphism

$$\rho : \pi_1(U, q) \to Aut(H^1(X_q, \mathbb{Z}))$$

called the monodromy representation. This can be constructed directly using sheaf theoretical methods using the direct image $R^1 f_* \mathbb{Z}|_U$.

4.2 Ruled surfaces

Let us start with the simplest case of a fibered surface, $f : X \to C$ where the fibre genus is zero. We say that X is ruled if f has no singular fibres. Here is a basic construction. Let V be a rank 2 algebraic vector bundle over C. Recall that V is given by an open covering $\{U_i\}$ of C in the Zariski topology and a collection of regular functions $g_{ij} \in GL_2(\mathcal{O}(U_{ij}))$ satisfying the cocycle identity $g_{ik} = g_{ij} g_{jk}$. This tells us how to glue $U_i \times \mathbb{C}^2$ to $U_j \times \mathbb{C}^2$ to get V. Since GL_2 acts on the projective line, we can use g_{ij} to glue $U_i \times \mathbb{P}^1$ to $U_j \times \mathbb{P}^1$. This gives a ruled surface that we denote by $\mathbb{P}(V) \to C$[1] In more geometric terms,

$$\mathbb{P}(V) = \{(q, \ell) \mid q \in C, \ell \subset V_q \text{ a 1 dim. subspace}\}$$

Theorem 4.2.1. *Every ruled surface is given by $\mathbb{P}(V)$ for some V. Two vector bundles, V, V', give isomorphic ruled surfaces if $V = V' \otimes L$, for some line bundle L.*

Proof. We give a sketch. The first step is to show that a ruled surface $f : X \to C$ is locally trivial in the analytic topology. See Beauville's Complex Algebraic

[1] Beware that many authors use the dual convention that this should be $\mathbb{P}(V^*)$. This is important to keep in mind when comparing formulas from different sources.

Surfaces for a proof. From this, it follows that X is given by gluing $U_i\times\mathbb{P}^1$ to $U_j\times\mathbb{P}^1$ by a cocycle $h_{ij}\in PGL_2(\mathcal{O}(U_{ij})$. We define $\check{H}^1(\{U_i\},PGL_2(\mathcal{O}_C))$ as the set of cocycles modulo the relation $h'_{ij}\sim h_{ij}$ if there exists $\eta_i\in PGL_2(\mathcal{O}(U_i))$ such that $h'_{ij}=\eta_i h_{ij}\eta_j^{-1}$. This is a set with a distinguished element corresponding to the trivial cocycle $h_{ij}=1$. We have an central extension of groups

$$1\to\mathcal{O}_C^*\to GL_2(\mathcal{O}_C)\to PGL_2(\mathcal{O}_C)\to 1$$

which gives an exact sequence of "pointed sets"

$$\check{H}^1(C,GL_2(\mathcal{O}_C))\to\check{H}^1(C,PGL_2(\mathcal{O}_C))\to H^2(C,\mathcal{O}_C^*)$$

We can see that the group on the right vanishing from the exponential sequence

$$0=H^2(C,\mathcal{O}_C)\to H^2(C,\mathcal{O}_C^*)\to H^3(C,\mathbb{Z})=0$$

Therefore h_{ij} lifts to a cocycle g_{ij} in GL_2. One can see directly that g'_{ij} is another lift of the h_{ij}, then $g_{ij}=a_{ij}g'_{ij}$ for some some cocycle $a_{ij}\in\mathcal{O}^*(U_{ij})$. These two statements translate exactly to what the theorem says.

□

We can use this to classify ruled surfaces over $C=\mathbb{P}^1$.

Theorem 4.2.2 (Grothendieck)**.** *Every vector bundle on $\mathbb{P}^1$ is a sum of line bundles.*

Corollary 4.2.3. *Any ruled surface is given by $F_n=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n)\oplus\mathcal{O}_{\mathbb{P}^1})$ for a unique integer $n\geq 0$.*

The surface $F_0=\mathbb{P}^1\times\mathbb{P}^1$ and F_1 can be shown to be the blow up of $\mathbb{P}^2$ at some (any) point.

4.3 Elliptic surfaces: examples

An ellptic surface is a surface with fibre genus 1. For example, if E is an elliptic curve, the product $E\times C\to C$ is an elliptic surface. Although this is somewhat trivial. Let us describe a more interesting class of examples.

Example 4.3.1. *Let $f,g\in\mathbb{C}[x,y,z]$ be homogeneous cubic polynomial such that $V(f)$ and $V(g)$ are distinct nonsingular cubics in $\mathbb{P}^2$. By Bezout's theorem $V(f)\cap V(g)$ meet in 9 points counting multiplicity. For simplicity, assume that there are no multiplicities. In other words, that there are really* 9 *points $p_1,\dots,p_9$. For $(s,t)\neq 0$, let $E_{[t,s]}=V(tf+sg)$. This gives a family of cubic curves parameterized by $\mathbb{P}^1$. This is called a pencil. For all but finitely many values, $E_t=E_{[t,1]}$ is a nonsingular cubic, and therefore an ellipitic curve. It is easy to see that $E_t\neq E_{t'}$ unless $t=t'$. Note that all of these curves contain p_i. We can separate these by considering the surface*

$$X=\{(p,t)\in\mathbb{P}^2\times\mathbb{P}^1\mid p\in E_t\}$$

Projection

$$\pi : X \to \mathbb{P}^1$$

makes this a surface having E_t as the fibre over t. To get a sense of what this looks like, let us project on the other factor

$$\psi : X \to \mathbb{P}^2$$

We have $\psi^{-1}(p) = \{t \mid p \in E_t\}$. So $\psi^{-1}(p_i) = \mathbb{P}^1$. Suppose that $p \neq p_i$, for any i. Then we claim $\phi^{-1}(p)$ consists of exactly one point. If not, then we would have $E_t \cap E_{t'}$ consisting of at least 10 points. But this is impossible by Bezout. It follows that ψ is a birational map to $\mathbb{P}^2$. In fact, X is precisely the blow up of $\mathbb{P}^2$ at $p_1, \ldots, p_9$.

To proceed further, recall that any elliptic curve can be put in Weierstrass form

$$y^2 = x^3 + ax + b$$

where $\Delta = 4a^3 + 27b^2 \neq 0$. Of course, we describing the affine equation; the projective curve is defined by the corresponding homogeneous equation. Also recall that the isomorphism class of the curve is determined by the j-invariant

$$j = \frac{a^3}{\Delta}$$

(where we drop the usual normalization factor for simplicity.) It follows that if we substitute $a \mapsto \lambda^4 a, b \mapsto \lambda^6 b$, we get an isomorphic curve. The isomorphism is simply given by the change of variable $x \mapsto \lambda^2 x, y \mapsto \lambda^3 y$.

Example 4.3.2. *Let*

$$a(t) = a_n t^n + \ldots + a_0$$
$$b(t) = b_m t^m + \ldots + b_0$$

be polynomials, then

$$y^2 = x^3 + a(t)x + b(t) \tag{4.1}$$

defines an elliptic surface over the affine t-line. We can try complete to a surface over $\mathbb{P}^1$. Let $s = t^{-1}$, be the coordinate at ∞, and consider

$$Y^2 = X^3 + \alpha(s)X + \beta(s) \tag{4.2}$$

where

$$\alpha(s) = a_0 s^n + \ldots a_n = s^n a(t)$$
$$\beta(s) = b_0 s^m + \ldots a_m = s^m b(t)$$

Suppose $n = 4d, m = 6d$ for some integer $d \geq 0$. Then by the above discussion, (4.1) and (4.2) can be patched using $X = s^{2d}x, Y = s^{3d}y$.

Example 4.3.3. *The previous construction can be generalized by replacing $\mathbb{P}^1$ by any curve C, choosing a line bundle L, such that $H^0(L^i) \neq 0$ for $i = 2, 3$, and viewing x, y, a, b as global sections of L^2, L^3, L^4, L^6 in (4.1)*

4.4 Singular fibres

Most elliptic surfaces will have singular fibres. Kodaira, and independently Neron, gave a complete classification of the singular fibres. The problem is local, so we can imagine a family of elliptic curves E_t, $0 < |t| < \epsilon$ degenerating to a singular, possible reducible curve $E_0 = \sum n_i C_i$. If we insist that E_0 stay a cubic, then the problem is easy:

(1) E_0 is either an irreducible cubic with a node,

(2) an irreducible cubic with a cusp,

(3) a union of line and an irreducible conic, which may or may not be tangent,

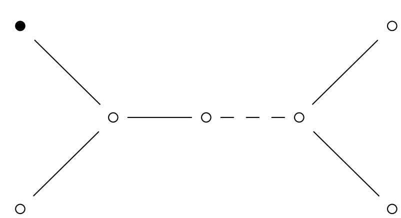
(4) or a union of three lines (there are several subcases).

However, in general, E_0 will not remain a cubic, so there are many more cases. In fact, there are infinitely many possibilities for the simple reason that we can always blow up the surface along E_0. We can recognize a blow up, because we will see an exceptional curve i.e. $\mathbb{P}^1$ with self intersection -1 as one of the components. From now on we will insist the no such curve appears among the fibres. Such an elliptic surface is called (relatively) *minimal.* In order to visualize the result, we use a dual graph: a vertex corresponds to a curve C_i; two vertices if the curves meet.

Theorem 4.4.1 (Kodaira-Neron)**.** *The set of possible singular fibres of a minimal elliptic surface forms two infinite families:*

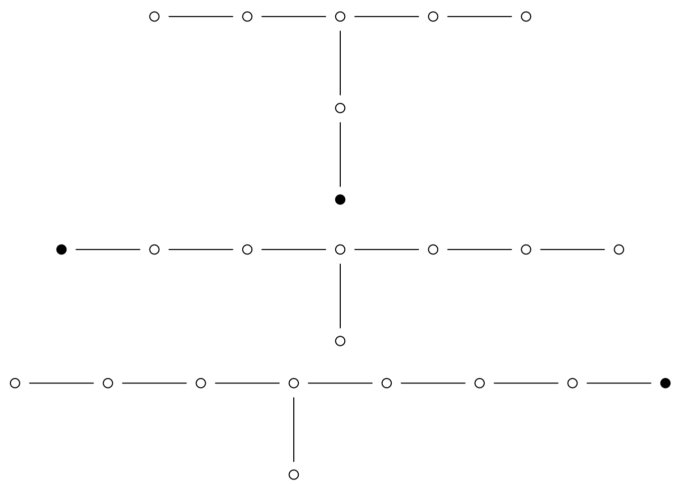
($_M I_N$) $M > 0, N \geq 0$. $N = 0$ *corresponds to a smooth curve,* $N = 1$ *is a nodal rational curve, and otherwise an* N*-gon of* $\mathbb{P}^1$*'s. All curves occur with multiplicity* M*.*

(I_N^*) $N \geq 0$. *A collection of* $N+5$ $\mathbb{P}^1$*'s with dual graph* $\tilde{D}_{N+4}$



A finite list of exceptional cases described below.

People familiar with Dynkin diagrams, which arise in the study of Lie algebras, will recognize $\tilde{D}_{N+4}$ as the usual Dynkin diagram D_{N+4} plus an extra vertex marked in black. Similarly type (I_N) corresponds to the extend Dynkin diagram $\tilde{A}_N$ given by an $N+1$ cycle. Removing one of the vertices gives A_N. The exceptional cases are the cuspidal rational curve (II), a union of two $\mathbb{P}^1$'s meeting at a point (III), three $\mathbb{P}^1$'s meeting at point (IV) and configurations corresponding to the extended Dynking diagrams $\tilde{E}_6$ (IV*) $\tilde{E}_7$ (III*), $\tilde{E}_8$ (II*) listed below. In brackets, we have indicated Kodaira's notation.



We will be content to prove the weaker statement that the dual graphs of the singular fibres are the ones given above. The proof can be reduced to the study of quadratic forms. Given a graph Γ (allowing loops and multiple edges) with vertices labelled 1 through n, we associate an $n \times n$ matrix $Q = Q_\Gamma$ given by

$$q_{ij} = \begin{cases} \#\text{edges joining } i \text{ and } j & \text{if } i \neq j \\ -2 + 2(\text{number of loops at } i) & \text{if } i = j \end{cases}$$

This represents an integer valued quadratic form. Conversely, we see that any symmetric integer matrix with nonnegative off-diagonal entries and diagonal entries in $\{-2, 0, 2, \ldots\}$ comes from a graph. The key fact, whose proof can be found in Miranda's Basic Theory of Elliptic Surfaces pp 13-14, is:

Proposition 4.4.2. *If Γ is a connected graph such that Q_Γ is negative semidefinite with one dimensional kernel, then Γ must be an $\tilde{A}, \tilde{D}, \tilde{E}$ graph. The converse is true, and Q_Γ is negative definite for the ADE graphs.*

We want to apply this to elliptic surfaces. First, we define the Neron-Severi group of a surface X as the image of the divisor class group $NS(X) = \mathrm{im}[Cl(X) \to H^2(X, \mathbb{Z})]$. The cup product pairing on H^2 restricts to a pairing on NS which is compatible with the intersection pairing. The proof of the following basic fact can be found in Griffiths-Harris, or it can be deduced from the Riemann-Roch theorem for surfaces (see Hartshorne, Chap V).

Theorem 4.4.3 (Hodge index theorem)**.** *If H is ample, then $H^2 > 0$. If $D \in NS(X)$ is a nonzero class such that $D \cdot H = 0$, then $D^2 < 0$*

Corollary 4.4.4. *The form restricted to the orthogonal complement $H^\perp$ is negative definite. More generally, the form on $D^\perp$ is negative definite for any D with $D^2 > 0$.*

Let X be a minimal elliptic surface with a singular fibre $E_0 = \sum n_i C_i$. Let $V \subset NS(X)$ be the span of the C_i.

Theorem 4.4.5. *The restriction of the intersection pairing to V is negative semidefinite with a one dimensional kernel. More precisely, $D \in V$ satisfies $D^2 = 0$, if and only if $D = NE_0$ for some integer N.*

Proof. Note that E_0 is equivalent in $H^2(X)$ to E_t for any t, because the classes of 0 ant t are equal in $H^2(C)$. By choosing $t \neq 0$, we see that $C_i \cdot E_0 = C_i \cdot E_0 \cdot E_t = 0$. In particular, $E_0^2 = 0$. If $D \in V$ satisfies $D^2 > 0$, then the fact that $D \cdot E_0 = E_0^2 = 0$ contradicts the Hodge index theorem. Therefore $D^2 \leq 0$, which says that the form is negative semidefinite.

Now suppose that $D \in V$ satisfies $D^2 = 0$, but D is not a multiple of E_0. Then we can find a rational number r, so that $G = D + rE_0$ is a linear combination of C_i all nonzero coefficients, and at least one positive coefficient and at least one negative. Write $G = P - N$, where coefficients of $,N$ are positive. Since E_0 is connected $P \cdot N > 0$. Since $D^2 = 0$ and the previous identities $G^2 = 0$. But

$$G^2 = P^2 - 2P \cdot N + N^2 < P^2 + N^2 \leq 0$$

which is a contradiction.

□

We are now almost ready to prove the Kodaira-Neron theorem. The additional thing we need is strong form of the adjunction formula.

Theorem 4.4.6. *If C is a possibly singular curve on surface X, then*

$$(K + C) \cdot C = 2p_a(C) - 2$$

where the arithmetic genus $p_a(C)$ is defined by $1 - p_a(C) = \chi(\mathcal{O}_C)$.

We prove the following special case of Kodaira-Neron.

Theorem 4.4.7. *The dual graph of a singular fibre of a minimal elliptic surface is of $\tilde{A}, \tilde{D}, \tilde{E}$ type.*

Proof. Let $E_0 = \sum n_i C_i$ be a singular fibre, and E_t a smooth fibre. We may suppose that E_0 is reducible, otherwise there is nothing to prove. Since E_t is an elliptic curve with $E_t^2 = 0$, the adjunction formula implies

$$K \cdot E_0 = K \cdot E_t = 0$$

Therefore

$$\sum n_i(2p_a(C_i) - 2 - C_i^2) = 0 \tag{4.3}$$

Suppose that $2p_a(C_i) - 2 - C_i^2 < 0$. Then

$$-2 \leq 2p_a(C_i) - 2 < C_i^2 \leq -1$$

which forces $C_i^2 = -1$ and $p_a(C_i) = 0$. The last equation implies $C_i = \mathbb{P}^1$, so we get a contradiction to minimality. Therefore all $2p_a(C_i) - 2 - C_i^2 \geq 0$ and so they must be 0 by (4.3). But this only possible if $C_i \cong \mathbb{P}^1$ and $C_i^2 = -2$. This already shows that the intersection matrix on V comes from a graph, which is necessarily connected. Therefore theorem 4.4.5 proves the graph is of $\tilde{A}, \tilde{D}, \tilde{E}$ type.

□

4.5 The Shioda-Tate formula

Given a field K, an elliptic curve over K, is a genus one curve defined over K with distinguished K-rational point $0 \in E(K)$. Here "defined over K" can be taken to mean the equation is defined over K, and $E(K)$ is the set of solutions in this field. The condition $0 \in E(K)$ allows us to define a group law. This is sometimes called the Mordell-Weil group. If K is a number field, such as $\mathbb{Q}$, the we have the following important result that this group is finitely generated. Understanding the rank of this group is an important problem in number theory.

Now let us return to the world of complex surfaces and look for an analogue. We start with an elliptic surface $f : X \to C$ with a section σ_0. If we remove the singular fibres and possibly some additional fibres form X, it can be described by a Weierstrass equation

$$y^2 = x^3 + a(t)x + b(t), \quad a(t), b(t) \in \mathbb{C}(C) \tag{4.4}$$

This equation can be viewed as defining an elliptic curve over $K = \mathbb{C}(C)$ that we call the generic fibre. For people familiar with schemes, this is the same thing as the generic fibre in the sense of scheme theory $E = X \times_C \operatorname{Spec} \mathbb{C}(C)$ The set $E(K)$ is the set rational sections, i.e. $\sigma : C \dashrightarrow X$ such that $f \circ \sigma = id$. Since C is a smooth projective curve any rational section can be completed to a regular section. Since we assumed $E(K) \neq \emptyset$, it forms a group with σ_0 as the identity. So now we can ask does Mordell-Weil hold? The answer is not always. Given the product $X = E_0 \times C$, then $E(K)$ is simply E_0 which uncountable! However, in a sense this is the only problem. Let us say that $X \to C$ is non-isotrivial, if the j-functions of the smooth fibres are not all the same.

Theorem 4.5.1. *If $f : X \to C$ is non-isotrivial, then there is an exact sequence*

$$0 \to T \to NS(X) \to E(K) \to 0$$

where T is the span of $\sigma_0(C)$ and the fibres.

Corollary 4.5.2. *$E(K)$ is finitely generated.*

Corollary 4.5.3 (Shioda-Tate formula)**.** *If r_t the number components of the fibre X_t,*

$$\operatorname{rank} E(K) = \operatorname{rank} NS(X) - 2 - \sum_t (r_t - 1)$$

Before getting to the proof, we need to say a bit more about the $NS(X)$. From the exponential sequence we obtain the exact sequence

$$H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to Pic(X) \to H^2(X,\mathbb{Z})$$

Therefore

$$0 \to Pic^0(X) \to Pic(X) \to NS(X) \to 0$$

where $Pic^0(X)$ is the quotient $H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$. In fact, this quotient can be shown to be complex torus, and even an abelian variety called the Picard variety. This is a generalization of the Jacobian of a curve. In fact, $Pic^0(C) = J(C)$. One can see that we have a homomorphism of tori $f^* : J(C) \to Pic^0(X)$. The key fact that we is that

Proposition 4.5.4. *If $X \to C$ is non-isotrivial, $J(C) \to Pic^0(X)$ is an isomorphism.*

This can be easily proved using the Leray spectral sequence, but we won't go into details.

Finally, we need to say a few words about divisors on E. Let $\bar{K}$ be the algebraic closure. The Galois group $G = Gal(\bar{K}/K)$ acts on the group $E(\bar{K})$, and $E(K) \subset E(\bar{K})$ is the subgroup of elements fixed by this action. A divisor is a finite sum $D = \sum n_i p_i \in Div(E(\bar{K}))$, $\sigma \in G$ acts on D by sending it to $\sum n_i \sigma(p_i)$. We say that D is defined over K if it is fixed by G. Let $Div(E)(K)$ be the group of these. It is sufficient but *not necessary* that all $p_i \in E(K)$. If g is a rational function on E, then $div(g) \in Div(E)(K)$. To be explicit, if E is given by (4.4), $g \in Frac(\mathbb{C}(C)[x,y]/(y^2 - x^3 - ax - b))$, so g is a rational function on X itself. Since $E(\bar{K})$ is a group, we have a homomorphism $Div(E(\bar{K})) \to E(\bar{K})$ which restricts to $Div(E)(K) \to E(K)$. This takes principal divisors to 0 by Abel's theorem (in a slightly more general form than we stated earlier).

Proof of theorem. Given a divisor D on X, we can take its restriction $D|_E \in Div(E)(K)$. Therefore we get a homomorphism $r : Div(X) \to E(K)$, which factors through $Pic(X)$. The generic fibre E is also the generic fibre of the $f^{-1}U \to U$ for any nonempty Zariski open set $U \subset C$. It follows that if D is sum of components of the fibres, then $D|_E$ is trivial. In particular, r is trivial on $f^*J(C)$. Therefore r factor through $NS(X) = Pic(X)/f^*J(C)$. We denoting resulting by r also. Given $\sigma \in E(K)$, it can be viewed as section $\sigma : C \to X$. One can see that $r(\sigma(C)) = \sigma$. Therefore r is surjective.

We saw that anything supported on the fibres lies in $\ker r$. Also the zero section $Z = \sigma_0(C)$ lies in $\ker r$ by definition. So $T \subseteq \ker r$. Suppose that $r(D) = 0$, i.e. suppose D maps to σ_0. Then, Abel, $D|_E - n\sigma_0 = div(g)$ for some $n \in \mathbb{Z}$ and rational function g on E (in fact, $n = (D \cdot E_t)$). Since, as observed above, g is a rational function on X, which, to avoid confusion, we denote by G. Then

$$D - n\sigma_0(D) - div(G)$$

is zero on E, it must be zero on $f^{-1}U$ for some Zariski open U. But this says that the class of $D \in T$.

□

Proof of Shioda-Tate. By the theorem

$$\operatorname{rank} E(K) = \operatorname{rank} NS(X) - \operatorname{rank} T$$

So it suffices to show that

$$\dim T \otimes \mathbb{R} = 2 + \sum (r_t - 1)$$

We can of course sum over the t's for which E_t has at least two components. For each such t, let $C_{t,1}, \dots, C_{t,r_t-1}$ be the set of components of E_t which don't meet Z. Let F denote a smooth fibre. It is easy to see that $Z, F, C_{t,i}$ spans T. It suffices to check that these are linearly independent. We do this by checking that the intersection matrix

$$A = \begin{pmatrix} Z^2 & Z \cdot F & \dots \\ F \cdot Z & F^2 & \dots \\ \dots & & \end{pmatrix}$$

is nonsingular. With respect to the intersection form, we can decompose $T \otimes \mathbb{R}$ into an orthogonal direct sum

$$\langle Z, F \rangle \oplus \bigoplus_t \langle C_{t,1}, C_{t,2}, \dots \rangle$$

so A decomposes accordingly into blocks. For the first block, the matrix is $\begin{pmatrix} * & 1 \\ 1 & 0 \end{pmatrix}$. The remaining blocks come from ADE graphs, so they are negative definite.

□