Chapter 10

More about Permutations and Symmetry Groups

Our ultimate goal in this chapter is study some more complicated symmetry groups than what we did previously. First recall that a permutation of the set $\{1, 2, \ldots, n\}$ is a one to one onto function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. We have used the notation

$$\begin{pmatrix} 1 & 2 & \ldots \\ f(1) & f(2) & \ldots \end{pmatrix}$$

previously, but this can get a little tedious if we have to write a lot of them (and we will). The most efficient notation is cycle notation, which we will explain. A permutation p is called a *cycle* of length k, or a k-cycle, if there exists a subset $\{a_1, a_2, \ldots, a_k\} \subseteq \{1, 2, \ldots, n\}$ such that $p(a_1) = a_2, p(a_2) = a_3, \ldots p(a_k) = a_1$, and $p(i) = i$ for all $i \notin \{a_1, a_2, \ldots, a_k\}$. In cycle notation, we write $p = (a_1 a_2 \ldots a_k)$. Note that we can start anywhere, so for example we could also write $p = (a_2 a_3 \ldots, a_k a_1)$. Two cycles are disjoint if their entries are disjoint as sets. Since cycles are permutations, we are allowed to multiply them.

Theorem 10.1. *Any permutation can be expressed as a product of disjoint cycles.*

We will omit the proof, but describe the conversion procedure in an informal way. Given a permutation p, start with 1, then compute $p(1)$, $p(p(1))$ and so on until you return to 1. This gives the first cycle $(1p(1)\ldots)$. Now repeat for numbers not contained in the first cycle, to construct the remaining cycles. Here is an example

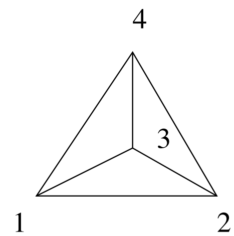
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 4 & 6 & 1 & 3 \end{pmatrix} = (15)(2)(346)$$

We usually omit the 1-cycles, so we would write this as (15)(346).

One thing that is easy to see from cycle notation is the order of a permutation p Recall that this is the smallest integer $N > 0$, so that p^N is the identity.

Theorem 10.2. *The order of a product of disjoint cycles is the least common multiple of the lengths of the cycles.*

Let us analyse the (rotational) symmetries of the regular tetrahedron



which is might be thought of as the 3D analogue of an equilateral triangle. Let us call the symmetry group T. We view it as a subgroup of the group S_4 of permutations of $\{1, 2, 3, 4\}$. Let's try and list the elements. There is the identity I. We have two 120° rotations which involve turning the base and keeping vertex 4 fixed:

$$(123), (132)$$

There are more rotations keeping 1 fixed:

$$(234), (243)$$

2 fixed:

$$(134), (143)$$

and 3 fixed:

$$(124), (142)$$

But this isn't all. We can rotate 180° about the line joining the midpoint of the lines $\overline{13}$ and $\overline{24}$ to get (13)(24). We can do the same thing with other pairs of lines to get

$$(14)(23), (12)(34)$$

This gives 12 elements of T so far. We claim that we're done. To see this, we need

Theorem 10.3 (Lagrange's theorem)**.** *If H is a subgroup of a finite group G, then the number of elements $|H|$ of H divides the number of element $|G|$ of G.*

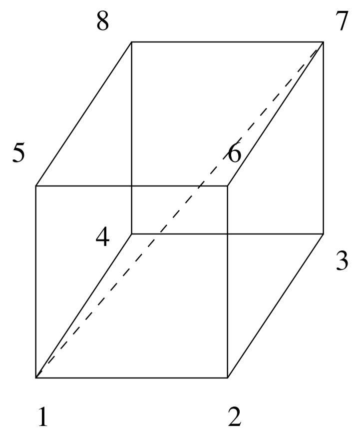
We will give the proof later on. The number $|G|$ is traditionally called the order of the group.[1][2] Returning to our example, the order $|T|$ of T must divide $|S_4| = 4! = 24$. Since $|T| \geq 12$, the only possibilities are $|T| = 12$ or 24. However, it cannot be 24 becomes some permutations such as (12) cannot be achieved by rotating the tetrahedron. Therefore

The symmetry group of the regular tetrahedron T consists of the 12 *elements listed above*

Next, we want to analyze the group C of rotational symmetries of the cube

[1] This might seem to clash with the previous usage but it doesn't. The order of an element g is the same thing as the order of the cyclic group generated by it.

[2] And if you find footnotes confusing and distracting, you don't have to read them.



We can view this as a subgroup of S_8. Note that S_8 has $8!$, or over $40,000$ elements. So this is a bit more daunting. Let us start by writing down all the obvious ones. Of course, we have the identity I. We can rotate the cube $90°$ about the axis connecting the top and bottom faces to get

$$(1234)(5678)$$

More generally, we have 3 rotations of $90°, 180°, 270°$ fixing each pair of opposite faces. Let us call these type I. There are 2 rotations, other than I, fixing each diagonally opposite pair of vertices such as 1and 7 (the dotted line in picture). Call these type II. For example

$$(254)(368)$$

is type II. We come to the next type, which we call type III. This is the hardest to visualize. To each opposite pair of lines, such as $\overline{12}$ and $\overline{78}$, we can connect their midpoints to a get a line L. Now do a $180°$ rotation about L. Let's count what we have so far:

identity: 1

type I: 3 (rotations) $\times$ 3 (pairs of faces) $= 9$

type II: 2 (rotations) $\times$ 4 (pairs of vertices) $= 8$

type III: 6 (pairs of lines) $= 6$

making 24. We claim that this is a complete list.

Theorem 10.4. *C has exactly 24 elements.*

This is not as easy to prove as the for the tetrahedron. We need a new tool. The key point is that the cube, like the tetrahedron, has perfect symmetry in the sense that it is possible to rotate any vertex to any other vertex. We can turn this into a definition.

Definition 10.5. *A subgroup $G \subseteq S_n$ is called transitive if for each pair $i, j \in \{1, \dots n\}$, there exists $f \in G$ such that f takes i to j i.e. $f(i) = j$.*

Definition 10.6. *Given subgroup $G \subseteq S_n$ and $i \in \{1, \dots n\}$, the stabilizer of i, is the set of permutations of G which leaves i fixed i.e. $\{f \in G \mid f(i) = i\}$*

Theorem 10.7 (Orbit-Stabilizer theorem I)**.** *Given a transitive subgroup $G \subseteq S_n$, let H be the stabilizer of some element i, then $|G| = n|H|$.*

Let us re-analyse the symmetry group $T \subseteq S_4$ of the tetrahedron The stablizer of 4 for T is the set

$$\{I, (123), (132)\}$$

with 3 elements. Therefore we recover $|T| = 3 \times 4 = 12$

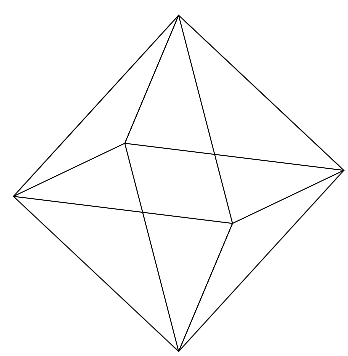
Now we can prove theorem 10.4. The symmetry group C can be viewed as a subgroup of S_8. We need to calculate the stabilizer of 1. Aside from the identity, the only rotations which keep 1 fixed are those with the line joining 1 and 7 as its axis. Thus the stabilizer consists of

$$\{I, (254)(368), (245)(386)\}$$

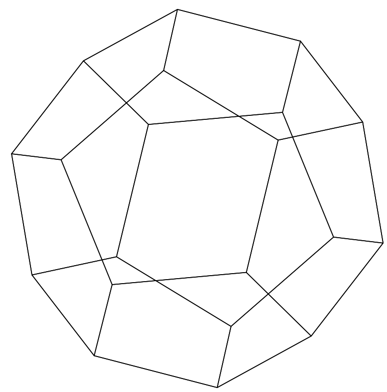
Therefore $|C| = 3 \times 8 = 24$

10.8 Exercises

1. Calculate the set of all numbers that occur as orders of elements of S_7.
2. Show that T is not Abelian.
3. Calculate the order of the symmetry group for the octrahedron (which most people would call a diamond)



4. Calculate the order of the symmetry group for the dodecahedron



(There are 20 vertices, and 12 pentagonal faces.)

5. Let $G \subseteq S_n$ be a subgroup. Prove that the stablizer H of an element i is a subgroup of G.

6. Using the Orbit-Stabilizer theorem, prove that $|S_n| = n|S_{n-1}|$, and use this to give a new proof that $|S_n| = n!$.