Notes on basic algebraic geometry

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These are my notes for an introductory course in algebraic geometry. I have trodden lightly through the theory and concentrated more on examples. Some examples are handled on the computer using Macaulay2, although I use this as only a tool and won't really dwell on the computational issues.

Of course, any serious student of the subject should go on to learn about schemes and cohomology, and (at least from my point of view) some of the analytic theory as well. Hartshorne [Ht] has become the canonical introduction to the first topic, and Griffiths-Harris [GH] the second.

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Chapter 1

Affine Geometry

1.1 Algebraic sets

Let k be a field. We write $\mathbb{A}^n_k = k^n$, and call this n dimensional affine space over k. Let

$$k[x_1, \dots x_n] = \{\sum c_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n} \mid c_{i_1 \dots i_n} \in k\}$$

be the polynomial ring. Given $a = (a_i) \in A^n$, we can substitute x_i by $a_i \in k$ in f to obtain an element denoted by $f(a)$ or $ev_a(f)$, depending on our mood. A polynomial f gives a function $ev(f) : \mathbb{A}^n_k \to k$ defined by $a \mapsto ev_a(f)$.

Given $f \in k[x_1, \dots x_n]$, define it zero set by

$$V(f) = \{a \in \mathbb{A}^n_k \mid f(a) = 0\}$$

At this point, we are going to need to assume something about our field. The following is easy to prove by induction on the number of variables. We leave this as an exercise.

Lemma 1.1.1. *If k is algebraically closed and f nonconstant, then $V(f)$ is nonempty.*

If $S \subset k[x_1, \dots x_n]$, then let $V(S)$ be the set of common zeros

$$V(S) = \bigcap_{f \in S} V(f)$$

A set of this form is called *algebraic*. I want to convince you that algebraic sets abound in nature.

Example 1.1.2. *The Fermat curve of degree d is $V(x_1^d + x_2^d - 1) \subset \mathbb{A}^2$. More generally, a Fermat hypersurface is given by $V(x_1^d + x_2^d + \dots x_n^d - 1)$.*

Example 1.1.3. *Let us identify $\mathbb{A}^{n^2}_k$ with the set $Mat_{n\times n}(k)$ of $n \times n$ matrices. The set of singular matrices is algebraic since it is defined by the vanishing of the determinant det which is a polynomial.*

Example 1.1.4. *Then the set $SL_n(k) \subset \mathbb{A}^{n^2}$ of matrices with determinant* 1 *is algebraic since it's just* $V(\det -1)$.

The set of nonsingular matrices $GL_n(k)$ is not an algebraic subset of $Mat_{n\times n}(k)$. However, there is useful trick for identifying it with an algebraic subset of $\mathbb{A}^{n^2+1} = \mathbb{A}^{n^2} \times \mathbb{A}^1$.

Example 1.1.5. *The image of $GL_n(k)$ under the map $A \mapsto (A, 1/\det(A))$ identifies it with the algebraic set*

$$\{(A,a) \in \mathbb{A}^{n^2+1} \mid \det(A)a = 1\}$$

Example 1.1.6. *Identify $\mathbb{A}_k^{mn}$ with the set of $m \times n$ matrices $Mat_{m\times n}(k)$. Then the set of matrices of rank $\leq r$ is algebraic. This is because it is defined by the vanishing of the $(r+1)\times(r+1)$ minors, and these are polynomials in the entries. Notice that the set of matrices with rank equal r is not algebraic.*

Example 1.1.7. *The set of pairs $(A, v) \in Mat_{n\times n}(k) \times k^n$ such that v is an eigenvector of A is algebraic, since the condition is equivalent to $rank(A, v) \leq 2$.*

Example 1.1.8. *Let $N_i \subseteq \mathbb{A}_k^{n^2}$ be the set of matrices which are nilpotent of order i, i.e matrices A such that $A^i = 0$. These are algebraic.*

Before doing the next example, let me remind you about resultants. Given two polynomials

$$f = a_n x^n + \dots a_0$$

and

$$g = b_m x^m + \dots b_0$$

Suppose, we wanted to test whether they had a common zero, say α. Then multiplying $f(\alpha) = g(\alpha) = 0$ by powers of α yields

$$
\begin{array}{llllll}
 & & a_n\alpha^n + & a_{n-1}\alpha^{n-1} + \dots & a_0 & = 0\\
 & & \dots & & & \\
a_n\alpha^{n+m} + & a_{n-1}\alpha^{n+m-1} + \dots & & & & = 0\\
 & & & b_m\alpha^m + \dots & b_0 & = 0\\
 & & \dots & & & \\
b_m\alpha^{n+m} + \dots & & & & & = 0
\end{array}
$$

We can treat this as a matrix equation, with unknown vector $(\alpha^{n+m}, \alpha^{n+m-1}, \dots, 1)^T$. For the a solution to exist, we would need the determinant of the coefficient matrix, called the resultant of f and g, to be zero. The converse, is also true (for $k = \bar{k}$) and can be found in most standard algebra texts. Thus:

Example 1.1.9. *Identify the set of pairs (f, g) with $\mathbb{A}^{(n+1)+(m+1)}$. The set of pairs with common zeros is algebraic.*

We can use this to test whether a monic polynomial (i.e. a polynomial with leading coefficient 1) f has repeated root, by computing the resultant of f and its derivative f'. This called the discriminant of f. Alternatively, if we write $f(x) = \prod(x - r_i)$, the discriminant $disc(f) = \prod_{i<j}(r_i - r_j)^2$. This can be written as a polynomial in the coefficients of f by the theorem on elementary symmetric functions.

Example 1.1.10. *The set of monic polynomials with repeated roots is a an algebraic subset of* $\mathbb{A}^n$.

We call a map $F : \mathbb{A}^n \to \mathbb{A}^m$ a *morphism* if it is given by

$$F(a) = (f_1(a), \dots f_m(a))$$

for polynomials $f_i \in k[x_1, \dots x_n]$. Clearly the preimage under a regular map of an algebraic set is algebraic. Let us identify $\mathbb{A}^{n^2}$ with the set of $n \times n$ matrices once again. To every matrix, A we can associate its characteristic polynomial $\det(tI - A)$. We thus get a morphism $ch : \mathbb{A}^{n^2} \to \mathbb{A}^n$ given by taking the coefficients of this polynomial other than the leading coefficient which is just one. Therefore

Example 1.1.11. *The set of matrices in* $\mathbb{A}^{n^2}$ *with repeated eigenvalues is an algebraic set. More explicitly it is the zero set of the discriminant of the characteristic polynomial.*

Exercise 1.1.12.

1. *Identify* $\mathbb{A}^6 = (\mathbb{A}^2)^3$ *with the set of triples of points in the plane. Which of the following is algebraic:*

   a) *The set of triples of distinct points.*

   b) *The set of triples* (p_1, p_2, p_3) *with* $p_3 = p_1 + p_2$.

   c) *The set of triples of colinear points.*

2. *Check that* $V(S) = V(\langle S \rangle)$, *where*

$$\langle S \rangle = \{\sum h_i f_i \mid h_i \in k[x_1, \dots x_n], f_i \in S\}$$

   is the ideal generated by S. *Therefore by the Hilbert basis theorem, which says that* $k[x_1, \dots x_n]$ *is Noetherian, we find that any algebraic set is defined by a finite number of polynomials.*

1.2 Weak Nullstellensatz

Recall that the a (commutative) k-algebra is a commutative ring R with a ring homomorphism $k \to R$. For example, $k[x_1, \dots x_n]$ is a k-algebra. A homomorphism of k-algebras is a ring homomorphism $R \to S$ such that

$$\begin{tikzcd} k \arrow[d] \arrow[dr] & \\ R \arrow[r] & S \end{tikzcd}$$

commutes. Let's make a simple observation at this point:

Lemma 1.2.1. *The map $f \mapsto ev(f)$ is a homomorphism of k-algebras from $k[x_1, \ldots x_n]$ to the algebra of k-valued functions on $\mathbb{A}^n$.*

Exercise 1.2.2. *Show that this homomorphism is injective if k is infinite, but not in general.(In view of this, we will eventually stop distinguishing between f and $ev(f)$ when k is infinite.)*

Let's suppose that S is explicity given to us as a finite set of polynomials. We can now ask is there an algorithm to decide when $V(S)$ this nonempty? Here are some answers:

1. Algebraically closed fields: Yes by Hilbert's Nullstellensatz (see below).
2. Finite fields: Yes, since there are only a finite number of points to check.
3. $\mathbb{R}$: Yes, by Tarski.
4. $\mathbb{Q}$: Unknown! (However, Matjisevich proved that there is no algorithm for $\mathbb{Z}$, or equivalently Hilbert's 10th problem has a negative solution. So it's reasonable to expect that it would be no for $\mathbb{Q}$ as well.)

Theorem 1.2.3 (Weak Hilbert Nullstellensatz)**.** *If k is algebraically closed, then $V(S) = \emptyset$ iff there exists $f_1 \ldots f_N \in S$ and $g_1 \ldots g_N \in k[x_1, \ldots x_n]$ such that $\sum f_i g_i = 1$*

The German word nullstellensatz could be translated as "zero set theorem". The Weak Nullstellensatz can be rephrased as $V(S) = \emptyset$ iff $\langle S \rangle = (1)$. Since this result is central to much of what follows, **we will assume that k is algebraically closed from now on** unless stated otherwise. To get an algorithm as claimed above, we need an effective form of the nullstellensatz:

Theorem 1.2.4 (Hermann)**.** *If $(f_1, \ldots f_N) = (1)$, then there exists g_i, with degree bounded by a computable constant depending on $\max\{\deg f_i\}$, such that $\sum f_i g_i = 1$.*

Define the ring

$$R = k[x_1, \ldots x_n]/\langle S \rangle$$

Lemma 1.2.5. *(k any field.) $ev_a : k[x_1, \ldots x_n] \to k$ factors throught the canonical map $k[x_1, \ldots x_n] \to R$ iff $a \in V(S)$.*

Proof. ev_a factors through R iff $ev_a(\langle S\rangle) = 0$ iff $f(a) = 0, \forall f \in \langle S\rangle$ iff $a \in V(S)$. $\square$

In view of the lemma, we can view ev_a as a homomorphism of $R \to k$ when $a \in V(S)$.

Corollary 1.2.6. *The map $a \mapsto ev_a$ gives a bijection*

$$V(S) \cong Hom_{k-algebras}(R, k)$$

Proof. Given $h \in Hom(R, k)$, let $a(h)$ be the vector whose ith component is $ev_a(\bar{x}_i)$, where $\bar{x}_i = image(x_i)$ in R. Then $h \mapsto a(h)$ gives the inverse. $\square$

We are now, almost already to prove WN, we need the following which is really an algebraic form of the Nullstellensatz:

Theorem 1.2.7 (Algebraic Nullstellensatz)**.** *Let $k_1 \subset k_2$ be a field extension, such that k_2 is finitely generated as a k_1-algebra, then k_2 is a finite field extension of k_1.*

Proof. See Atiyah-MacDonald [AM, prop. 7.9]. $\square$

Proof of W. Nullstellensatz. If $\sum f_i g_i = 1$ for $f_i \in S$, then clearly these polynomials have no common zeros.

Conversely, suppose that $\langle S\rangle$ is a proper ideal. Therefore $R = k[x_1, \ldots x_n]/\langle S\rangle$ is nonzero, so it has a maximal ideal m. R/m is a field containing k which is finitely generated as a k-algebra. By the previous, theorem R/m is a finite, hence algebraic, extension of k. Thus $k = R/m$. The homomorphism, $R \to R/m = k$ corresponds to by above, to a point of $V(S)$. $\square$

1.3 Zariski topology

One nice feature of working over $\mathbb{R}$ or $\mathbb{C}$ is that affine carries natural topology defined by the Euclidean metric. It turns out that one can define a topology over any field which is sometimes just as good. Given a subset $X \subset \mathbb{A}^n_k$, let

$$\mathcal{I}(X) = \{f \in k[x_1, \ldots x_n] \mid f(a), \forall a \in X\}$$

This is an ideal of $k[x_1, \ldots x_n]$. We now have two operations

$$\{\text{subsets of } \mathbb{A}^n\} \underset{V}{\overset{\mathcal{I}}{\rightleftarrows}} \{\text{subsets of } k[x_1, \ldots x_n]\}$$

which we need to compare.

Proposition 1.3.1. *1. $X \subseteq Y \Rightarrow \mathcal{I}(X) \supseteq \mathcal{I}(Y)$.*

2. $S \subseteq T \Rightarrow V(S) \supseteq V(T)$.

3. $V(\mathcal{I}(X)) \supseteq X$.

4. $\mathcal{I}(V(S)) \supseteq S$.

5. If I *and* J *are ideals,* $V(I \cap J) = V(I) \cup V(J)$.

6. If $\{I_a\}$ *is a family of ideals, then*

$$V(\sum I_a) = \bigcap V(I_a)$$

Proof. We prove (5) assuming the previous parts. We have $I \cap J \subseteq I, J$ which implies $V(I\cap J) \supseteq V(I), V(J)$. Therefore $V(I\cap J) \supseteq V(I)\cup V(J)$. If $a \in V(I\cap J)$ is not contained in the union, then $f(a) \neq 0 \neq g(a)$. Therefore $fg(a) \neq 0$ which contradicts the fact that $fg \in I \cap J$. □

Corollary 1.3.2. $V(\mathcal{I}(V(S))) = V(S)$.

Proof. $\supseteq$ follows by (3). (4) gives $S \subseteq \mathcal{I}(V(S))$ which implies the opposite inclusion. □

Corollary 1.3.3. *The complements of algebraic sets forms a topology on* $\mathbb{A}^n$ *called the Zariski topology. In other words the algebraic sets are the precisely the closed sets for this topology.*

Proof. A collection of subsets is a topology if it is closed under arbitrary unions, finite intersections, and contains $\emptyset$ and $\mathbb{A}^n$. The collection of complements of algebraic sets satisfies all of the above. □

While it's nice to have a topology, you have to be careful about your intuition. It's much coarser than the ones you may encounter in an analysis class. For example, the nonempty open sets of $\mathbb{A}^1$ are the complements of finte sets. This is often called the cofinite topology.

A function $f : \mathbb{A}^n \to \mathbb{A}^1$ is called *regular* if it is a morphism, i.e. if it is defined by a polynomial.

Lemma 1.3.4. *All regular functions* $f : \mathbb{A}^n_k \to \mathbb{A}^1_k$ *are continuous with respect to the Zariski topology on* $\mathbb{A}^n_k$ *and the cofinite topology on* $\mathbb{A}^1_k$*. This is the weakest topology with this property.*

Proof. Continuity means that the preimage of open sets are open, or equivalently the preimage of closed sets are closed. The preimage of $\{a_1, \ldots a_N\} \subset \mathbb{A}^1_k$ is $V(\prod(f - a_i))$ which is Zariski closed by definition.

Given any other topology with this property, $V(f)$ would be closed for it. Therefore $V(S)$ is closed in this other topology for any S. □

More generally, we see that morphisms $F : \mathbb{A}^n \to \mathbb{A}^m$ are continuous when both spaces are given their Zariski topologies.

When $k = \mathbb{C}$, we have two choices of topologies, the Zariski topology and the Euclidean metric space topology that we will call the classical or strong topology. The strong topology on $\mathbb{A}^n_{\mathbb{C}}$ is Hausdorff, noncompact, and coincides with the product topology $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}} \ldots \mathbb{A}^1_{\mathbb{C}}$. All of these fail for Zariski.

Exercise 1.3.5.

1. *Finish the proof of proposition 1.3.1.*
2. *Let* $D(f) = \mathbb{A}^n - V(f)$*. Show that any open set is a union of* $D(f)$*'s, in other words, these sets form a basis for the topology.*
3. *Show that* $\mathbb{A}^n$ *is not Hausdorff if* k *is infinite.*
4. *Prove that any nonempty open subset of* $\mathbb{A}^n$ *is dense.*
5. *Prove that* $\mathbb{A}^n$ *is compact.*
6. *Show that the Zariski topology of* $\mathbb{A}^2$ *is finer than the product topology of* $\mathbb{A}^1 \times \mathbb{A}^1$*.*

1.4 The Cayley-Hamilton theorem

Given a square matrix A, recall that its characteristic polynomial is $\det(tI - A) \in k[t]$. Its roots are precisely the eigenvalues of A.

Theorem 1.4.1 (Cayley-Hamilton)**.** *If* $p_A(t) = \det(tI - A)$*, then* $p_A(A) = 0$*.*

We give a proof which makes use of the Zariski topology.

Lemma 1.4.2. *The subset of matrices* D_n *in* $Mat_{n\times n}$ *with distinct eigenvalues is dense in the Zariski topology.*

Proof. The set of these matrices is certainly nonempty and open since complement is an algebraic set by example 1.1.11. As observed in the exercises of the last section, nonempty open sets are dense. □

We recall the following standard fact from linear algebra.

Theorem 1.4.3. *Any matrix* $A \in D_n$ *can be diagonalized, that this there exist* $T \in Gl_n(k)$ *such that* $T^{-1}AT$ *is diagonal.*

Proof. Take D to be the diagonal matrix of eigenvalues λ_i of A and T a matrix with eigenvectors v_i as columns. Then $AT = TD$. We will be done if we can show that T is invertible. For this it is enough to prove that v_i are linearly independent. Suppose that

$$\sum a_i v_i = 0$$

where not all the $a_i = 0$. We can assume the number of $a_i \neq 0$ is chosen as small as possible and that $i = 1$ is among these indices. Then multiplying the equation by $T - \lambda_1 I$ yields a shorter relation

$$\sum a_i(\lambda_i - \lambda_1)v_i = 0$$

which is a contradiction. □

Proof of Cayley-Hamilton. We have to show that the morphism $\mathbb{A}^{n^2} \to \mathbb{A}$ given by $A \mapsto p_A(A)$ vanishes identically. It suffices to check this for $A \in D_n$ since it is dense. By the previous theorem, any matrix $A \in D_n$ can be diagonalized. Since $p_{T^{-1}AT}(T^{-1}AT) = T^{-1}p_A(A)T$, we can reduce to the case where

$$A = \begin{pmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & 0\dots \\ & \dots & \end{pmatrix}$$

The result is now straightforward calculation using $p_A(t) = \prod(t - \lambda_i)$. $\square$

1.5 Affine Varieties

The Noetherian property of $k[x_1, \dots x_n]$ has the following topological interpretation.

Lemma 1.5.1. *Any descending chain $X_1 \supseteq X_2 \supseteq \dots$ stabilizes ($X_N = X_{N+1} = \dots$ for some N).*

Proof. The chain of ideals $\mathcal{I}(X_1) \subseteq \mathcal{I}(X_2) \subseteq \dots$ has to stabilize by the Noetherian property. $\square$

A space satisfying this lemma is called *Noetherian.* Let $X = V(I) \subseteq \mathbb{A}^n_k$. We give X the induced topology, which means that a subset of X is closed if it is closed as a subset of $\mathbb{A}^n_k$. X is again Noetherian. We say that X is *reducible* if it is a union of two proper closed sets. Otherwise X is called *irreducible.*

Exercise 1.5.2. *Show that $V(f)$ is irreducible if f is an irreducible polynomial.*

The unique factorization property for polynomials can be generalized as follows.

Theorem 1.5.3. *Any Noetherian space X can be expressed as a union of $X = X_1 \cup X_2 \cup \dots X_n$ of irreducible closed sets, where no X_i is contained in an X_j. This is unique up to reordering.*

Proof. If X is irreducible, there is nothing to prove. Suppose X is reducible, then we can write $X = X_{(0)} \cup X_{(1)}$ where $X_{(i)}$ are proper and closed. Repeat this for each $X_{(i)}$, and continue doing this. Let's represent this as a tree:

$$\begin{tikzcd}[arrows=dash]
 & & X \arrow[dl] \arrow[dr] & & \\
 & X_{(0)} \arrow[dl] \arrow[d] & & X_{(1)} \arrow[d] \arrow[dr] & \\
X_{(00)} & X_{(01)} & & X_{(10)} & X_{(11)}
\end{tikzcd}$$

By the Noetherian property, we can't continue this forever. Thus the tree must be finite (we're using König's lemma that an infinite binary tree contains an infinite path). The "leaves", i.e. the lowest elements give the X_i.

Suppose we have another decomposition, $X = X'_1 \cup X'_2 \cup \dots X'_m$. Then

$$X'_i = (X_1 \cap X'_i) \cup (X_2 \cap X'_i) \cup \dots$$

Since the left side is irreducible, we must have $X'_i = X_j \cap X'_i$ for some j. So that $X'_i \subseteq X_j$. By symmetry, $X_j \subseteq X'_\ell$. Therefore $X'_i = X'_\ell$ by assumption, and this forces $X'_i = X_j$. This proves

$$\{X'_1, X'_2, \dots\} \subseteq \{X_1, X_2, \dots\}$$

We get the opposite inclusion by symmetry. $\square$

The X_i in the theorem are called the irreducible components. An irreducible closed subset of some $\mathbb{A}^n_k$ is called an *affine algebraic variety.*

Lemma 1.5.4. *$X \subseteq \mathbb{A}^n_k$ is irreducible iff $\mathcal{I}(X)$ is a prime ideal.*

In terms of ideal theory, the irreducible components of X correspond to the minimal primes of $I = \mathcal{I}(X)$. That is primes ideals p containing I and minimal with respect to this property.

1.6 Hilbert's Nullstellensatz

We haven't completely finished our analysis of V and I. We need to understand what happens if we follow one by the other. One direction is easy.

Lemma 1.6.1. *$V(\mathcal{I}(X))$ is the closure of X (in the Zariski topology).*

Exercise 1.6.2. *Prove this.*

The other direction is harder. The *radical* of an ideal I is defined by

$$\sqrt{I} = \{f \mid \exists n \in \mathbb{N}, f^n \in I\}$$

This is an ideal containing I. We define the localization of a ring R at $f \in R$ by

$$R[1/f] = R[T]/(Tf - 1)$$

The image of T will be denoted by $1/f$. Notice that $R[1/0]$ makes sense, but the whole ring is 0. More generally,

Lemma 1.6.3. *$R[1/f] = 0$ iff f is nilpotent.*

Corollary 1.6.4. *Let $R = k[x_1, \dots x_n]/I$, $f \in k[x_1, \dots x_n]$ and $\bar{f}$ its image. Then $R[1/\bar{f}] = 0$ iff $f \in \sqrt{I}$.*

Theorem 1.6.5 (Hilbert's Nullstellensatz)**.** *If k is algebraically closed, $\mathcal{I}(V(I)) = \sqrt{\mathcal{I}}$.*

Proof. The inclusion $\sqrt{\mathcal{I}} \subseteq \mathcal{I}(V(I))$ is obvious: if f^n vanishes on $V(I)$ then so does f.

Suppose that $f \in \sqrt{I}$. Let $R = k[x_1, \dots x_n]/I$. Then $R[1/\bar{f}] \neq 0$. Choose a maximal ideal $m \subset R[1/\bar{f}]$. Then $R[1/\bar{f}]/m = k$ by the Algebraic Nullstellensatz. Thus we have a homomorphism $h : R[1/\bar{f}] \to k$. The composition $R \to R[1/\bar{f}] \to k$ is necessarily of the form ev_a with $a \in V(I)$. The fact that ev_a factors through $R[1/\bar{f}]$ means that $f(a)$ has an inverse i.e. $f(a) \neq 0$. Which shows that $f \notin \mathcal{I}(V(I))$. $\square$

1.7 Nilpotent matrices

Let A be 2×2 matrix over a field k. The Cayley-Hamilton theorem tells us that

$$A^2 - trace(A)A + det(A)I = 0$$

Therefore $det(A) = trace(A) = 0$ implies that A is nilpotent of order 2. Conversely, these vanish for a nilpotent matrix since it has zero eigenvalues. Let's try and understand this using the Nullstellensatz. Let

$$A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

be the generic matrix. The polynomials $det(A), trace(A)$ generate an ideal $I \subset k[x_1, \dots x_4]$. The entries of A^2 generate another ideal J. We need to check that $\sqrt{I} = \sqrt{J}$.

We have already hinted that many of these computations can be carried algorithmically. The name of the game here is Gröbner basis theory, and the book by Cox, Little, O'shea [CLO] gives an introduction to this. These algorithms have been implemented in several computer packages. We are going to use Grayson and Stillman's Macaulay2 program

http://www.math.uiuc.edu/Macaulay2/

which is particularly convenient for algebraic geometry/commutative algebra. We will check this in characteristic 0, however we need to work over a field where the elements and operations can be represented precisely on a machine. We will use the prime field $k = \mathbb{Q}$ even though we are interested in algebraically closed fields containing it. This is justified by the following:

Lemma 1.7.1. *Let $k_1 \subset k_2$ be a field extension. Suppose that $I \subset k_1[x_0, \dots x_n]$ is an ideal, and let $I' \subset k_2[x_0, \dots x_n]$ be the ideal generated by I. Then $I' \cap k_1[x_0, \dots x_n] = I$ and $\sqrt{I'}$ is generated by $\sqrt{I}$.*

Proof. We prove this using tensor products (see Atiyah-MacDonald for a review). We have $k_2[x_0, \dots x_n] = k_2 \otimes_{k_1} k_1[x_0, \dots x_n]$ as algebras. Furthermore $I' = k_2 \otimes_{k_1} I$ and the first statement follows easily.

Let $J = k_2 \otimes \sqrt{I}$. We certainly have $I' \subset J \subseteq \sqrt{I'}$, we just have to check that J is radical. This is equivalent to condition that the ring

$$k_2[x_1, \ldots x_n]/J \cong k_2 \otimes_{k_1} (k_1[x_1, \ldots x_n]/\sqrt{I})$$

has no nonzero nilpotents. This is clear, since $(a \otimes f)^n = a^n \otimes f^n = 0$ forces a or f to be 0. $\square$

Beware that for some questions the field does matter. For example, the ideal $(x^2 - 2)$ is prime over $\mathbb{Q}$ but not over $\mathbb{C}$.

Below is a transcript of a Macaulay 2 session which shows that $\sqrt{I} = \sqrt{J}$ and that $J \subsetneqq I$. It shouldn't be too hard to understand what the commands are doing. The ; is used to suppress output, = is used to assign values, and == for testing equality.

```
i1 : R = QQ[x_1..x_4];

i2 : A = matrix{{x_1,x_2},{x_3,x_4}};

i3 : D = det A;

i4 : T = trace A;

i5 : I =ideal {D,T};

i6 : J = ideal A^2;

i7 : I == J

o7 = false

i8 : isSubset(J,I)

o8 = true

i9 : radical I == radical J

o9 = true
```

We will do a similar calculation for 3×3 matrices over the finite field $k = \mathbb{Z}/101\mathbb{Z}$, since Macaulay 2 is more efficient in finite characteristic. We let I denote the ideal defined by the coefficients of the characteristic polynomial of a generic 3×3 matrix which in addition to the *det* and *trace* includes the trace of the matrix of 2×2 minors generated by the *exteriorPower* command. We show that $V(I)$ is the set of nilpotent matrices of order 3.

```
i1 : R = ZZ/101[x_1..x_9];
```

```
i2 : A = genericMatrix(R,x_1,3,3)

o2 = | x_1 x_4 x_7 |
     | x_2 x_5 x_8 |
     | x_3 x_6 x_9 |

             3       3
o2 : Matrix R  <--- R

i3 : I = ideal { det(A), trace(A), trace(exteriorPower(2,A))};

o3 : Ideal of R

i4 : J = ideal A^3;

o4 : Ideal of R

i5 : radical I == radical J

o5 = true
```

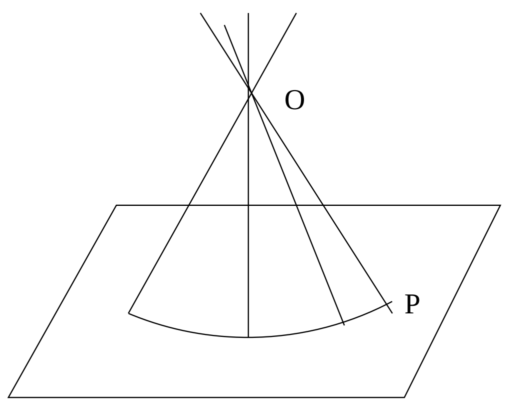
Exercise 1.7.2. *Let K be the ideal defining nilpotent 2×2 matrices of order 3. Show that $\sqrt{J}$ coincides with $\sqrt{K}$. Does $J = K$?*

Chapter 2

Projective Geometry

2.1 Projective space

In Euclidean plane geometry, we need seperate the cases of pairs of lines which meet and parallel lines which don't. Geometry becomes a lot simpler if any two lines met possibly "at infinity". There are various ways of arranging this, the most convenient method is to embed the $\mathbb{A}^2$ into 3 dimensional space as depicted. To each point $P \in \mathbb{A}^2$, we can associate the line OP. The lines parallel to the plane correspond to the points at infinity.



We now make this precise. n dimensional projective space $\mathbb{P}^n_k$ over a (not necessarily algebraically closed) field k consists of the set of lines through 0, or equivalently one dimensional subspaces, in $\mathbb{A}^{n+1}$. There is a map $\pi : \mathbb{A}^{n+1}_k - \{0\} \to \mathbb{P}^n_k$ which sends v to its span. We will usually write $[a_0, \ldots a_n]$ for $\pi((a_0, \ldots a_n))$. We identify $(a_1, \ldots a_n) \in \mathbb{A}^n$ with the point $[1, a_1, \ldots a_n] \in \mathbb{P}^n$. The complement of $\mathbb{A}^n$ is called the hyperplane at infinity. It can be identified with $\mathbb{P}^{n-1}$.

2.2 Projective varieties

We want to do algebraic geometry on projective space. Given $X \subset \mathbb{P}^n$, define the *cone* over X to be $Cone(X) = \pi^{-1}X \cup \{0\} \subseteq \mathbb{A}^{n+1}$. A subset of $\mathbb{A}^{n+1}$ of this form is called a cone. We define $X \subseteq \mathbb{P}^n$ to be *algebraic* iff $Cone(X)$ is algebraic in $\mathbb{A}^{n+1}$.

Lemma 2.2.1. *The collection of algebraic subsets are the closed for a Noetherian topology on $\mathbb{P}^n$ also called the Zariski topology. $\mathbb{A}^n \subset \mathbb{P}^n$ is an open subset.*

Proof. Exercise! □

There are natural embeddings $\mathbb{A}^n \to \mathbb{P}^n$ given by

$$(a_1, \dots a_n) \mapsto [a_0, \dots a_{i-1}, 1, a_i \dots a_n]$$

This identifies the image with the open set $U_i = \{x_i \neq 0\}$. This gives an open cover of $\mathbb{P}^n$ which allows many problems to be reduced to affine geometry.

We can define the strong topology of $\mathbb{P}^n_{\mathbb{C}}$ in the same way as the Zariski topology.

Lemma 2.2.2. *$\mathbb{P}^n_{\mathbb{C}}$ is a compact Hausdorff space which contains $\mathbb{A}^n_{\mathbb{C}}$ as a dense open set. In other words, it is a compactification of affine space.*

Proof. $\mathbb{P}^n_{\mathbb{C}}$ is compact Hausdorff since it is the image of the unit sphere $\{z \in \mathbb{C}^{n+1} \mid |z| = 1\}$. □

Let's make the notion of algebraic set more explicit. We will use variables $x_0, \dots x_n$. Thus X is algebraic iff $Cone(X) = V(S)$ for some set of polynomials in $k[x_0, \dots x_n]$. Let's characterize the corresponding ideals. Given a polynomial f, we can write it as a sum of homogeneous polynomials $f = f_0 + f_1 + \dots$. The f_i will be called the homogeneous components of f.

Lemma 2.2.3. *$I \subset k[x_0, \dots x_n]$ is generated by homogeneous polynomials iff I contains all the homogeneous components of its members.*

Proof. Exercise! □

An $I \subset k[x_0, \dots x_n]$ is called *homogeneous* if it satisfies the above conditions

Lemma 2.2.4. *If I is homogeneous then $V(I)$ is a cone. If X is a cone, then $\mathcal{I}(X)$ is homogeneous .*

Proof. We will only prove the second statement. Suppose that X is a cone. Suppose that $f \in \mathcal{I}(X)$, and let f_n be its homogenous components. Then for $a \in X$,

$$\sum t^n f_n(a) = f(ta) = 0$$

which implies $f_n \in \mathcal{I}(X)$. □

We let $\mathbb{P}V(I)$ denote the image of $V(I) - \{0\}$ in $\mathbb{P}^n$. Once again, we will revert to assuming k is algebraically closed. Then as a corollary of the weak Nullstellensatz, we obtain

Theorem 2.2.5. *If I is homogeneous , then $\mathbb{P}V(I) = \emptyset$ iff $(x_0, \ldots x_n) \subseteq \sqrt{I}$.*

A *projective variety* is an irreducible algebraic subset of some $\mathbb{P}^n$.

2.3 Projective closure

Given a closed subset $X \subset \mathbb{A}^n$, we let $\overline{X} \subset \mathbb{P}^n$ denote its closure. Let us describe this algebraically. Given a polynomial $f \in k[x_1, \ldots x_n]$, it homogenization (with respect to x_0) is

$$f^H = x_0^{\deg f} f(x_1/x_0, \ldots x_n/x_0)$$

The inverse operation is $f^D = f(1, x_1, \ldots x_n)$. The second operation is a homomorphism of rings, but the first isn't. We have $(fg)^H = f^H g^H$ for any f, g, but $(f + g)^H = f^H + g^H$ only holds if f, g have the same degree.

Lemma 2.3.1. *$\mathbb{P}V(f^H)$ is the closure of $V(f)$.*

Proof. Obviously, $f(a) = 0$ implies $f^H([1, a]) = 0$. Thus $\mathbb{P}V(f^H)$ contains $V(f)$ and hence its closure. Conversely, it's enough to check that

$$\mathcal{I}_P(\mathbb{P}V(f^H)) \subseteq \mathcal{I}_P(\overline{V(f)})$$

For simplicity assume that f is irreducible. Then the left hand ideal is (f^H). Suppose that $g \in \mathcal{I}_P(\overline{V(f)})$, then $g^D \in \mathcal{I}(V(f))$. This implies $f|g^D$ which shows that $f^H|g$. □

We extend this to ideals

$$I^H = \{f^H \mid f \in k[x_1, \ldots x_n]\}$$

$$I^D = \{f^D \mid f \in k[x_0, \ldots x_n]\}$$

Lemma 2.3.2. *I^H is a homogenous ideal such that $(I^H)^D = I$.*

Theorem 2.3.3. *$\overline{V(I)} = \mathbb{P}V(I^H)$.*

Proof.

$$\overline{V(I)} = \overline{\bigcap_{f \in I} V(f)} \subseteq \bigcap \overline{V(f)}$$

Conversely, proceed as above. Let $g \in \mathcal{I}_P(\overline{V(I)})$ then $g^D \in \mathcal{I}(V(I)) = \sqrt{I}$. Thus

$$(g^D)^N = (g^D)^N \in I$$

for some N. Therefore $g^N \in I^H$. So that $g \in \mathcal{I}_P(\mathbb{P}V(I^H)) = \sqrt{I^H}$.

□

For principal ideals, homogenization is very simple. $(f)^H = (f^H)$. In general, homogenizing the generators need **not** give the right answer. For example, the affine twisted cubic is the curve in $\mathbb{A}^3$ defined by the ideal $I = (x_2 - x_1^2, x_3 - x_1^3)$. Then

$$(x_2x_0 - x_1^2, x_3x_0^2 - x_1^3) \subsetneqq I$$

We will check on the computer for $k = \mathbb{Q}$. But first, let us give the correct answer,.

Lemma 2.3.4. *Let $I = (f_1, \dots f_N)$ and $J = (f_1^H, \dots f_N^H)$. Then*

$$I^H = \{f \in k[x_0, \dots x_n] \mid \exists m, x_0^m f \in J\}$$

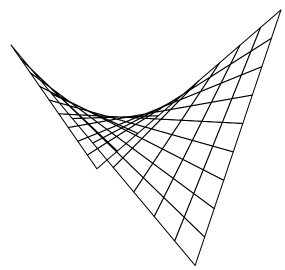
The process of going from J to I^H above is called saturation with respect to x_0. It can be computed with the *saturate* command in Macaulay2.

```
i1 : R = QQ[x_0..x_3];


i2 : I = ideal {x_2-x_1^2, x_3 -x_1^3};

o2 : Ideal of R

i3 : J = homogenize(I, x_0)

                2             3    2
o3 = ideal (- x  + x x , - x  + x x )
                1    0 2     1    0 3

o3 : Ideal of R

i4 : IH = saturate(J, x_0)

              2                        2
o4 = ideal (x  - x x , x x  - x x , x  - x x )
              2    1 3   1 2    0 3   1    0 2
```

Exercise 2.3.5. *Prove that I^H is generated by the polynomials given in the computer calculation, and conclude $I^H \neq J$ (for arbitary k).*

2.4 Miscellaneous examples

Consider the quadric Q given by $x_0x_3 - x_1x_2 = 0$ in $\mathbb{P}^3$.



This is a doubly ruled surface which means that it has two families of lines. This can be see explicitly by setting up a bijection $\mathbb{P}^1 \times \mathbb{P}^1 \cong Q$ by

$$([s_0, s_1], [t_0, t_1]) \mapsto [s_0t_0, s_0t_1, s_1t_0, s_1t_1]$$

More generally, the *Segre embedding* of

$$\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{(m+1)(n+1)-1}$$

is given by sending $([s_i], [t_j])$ to $[s_it_j]$ ordered appropriately.

Exercise 2.4.1. *Check that the image of the Segre embedding is a projective variety.*

The rational normal curve of degree n is the image of $\mathbb{P}^1$ in $\mathbb{P}^n$ under

$$[x_0, x_1] \mapsto [x_0^n, x_0^{n-1}x_1, \ldots x_1^n]$$

Exercise 2.4.2. *Check that the rational normal curve is a projective variety.*

2.5 Grassmanians

Let's turn to a fairly important but subtle example of a projective variety. Let $r \leq n$. As a set the Grassmanian $Gr(r, n)$ is the set of r dimensional subspaces of k^n. For example, $Gr(1, n) = \mathbb{P}^{n-1}$. Let $Mat = Mat_{r\times n} \cong \mathbb{A}^{nr}$ denote the space of $r \times n$ matrices, and let $R(r, n) \subset Mat$ denote the subset of matrices of rank r. This is a dense open subset of Mat. Choose the standard basis of k^n, and represent the elements by row vectors. Then we have a surjective map $R(r, n) \to Gr(r, n)$ which sends A to the span of its rows. This is not a very good parameterization since it is very far from one to one. In fact, A and A' represent the same subspace iff $A' = MA$ for some invertible $r \times r$ matrix. Let $N = \binom{n}{r}$ and label the minors of A by integers $1, \ldots N$. Let us consider the map $pl : Mat \to \mathbb{A}^N$, which sends A to its vector of $r \times r$ minors. We call $pl(A)$ the Plücker vector of A. Note that $pl^{-1}\mathbb{A}^N - \{0\} = R(r, n)$. If A and A' define the same point, so that $A' = AM$, then $pl(A') = \det(M)pl(A)$. Therefore, we have proven

Lemma 2.5.1. *The map $A \mapsto [pl(A)]$ is an injection from $Gr(r,n) \to \mathbb{P}^{N-1}$.*

We are going to prove that

Theorem 2.5.2. *The image of $Gr(r,n)$ in $\mathbb{P}^{N-1}$ is a projective variety.*

Let's start by trying to discover the equations for $Gr(2,4)$. Identify $(x_1, \dots x_8) \in \mathbb{A}^8$ with

$$\begin{pmatrix} x_1 & x_3 & x_5 & x_7 \\ x_2 & x_4 & x_6 & x_8 \end{pmatrix}$$

Order the minors by

$$1 \times 2, 1 \times 3, 2 \times 3, 1, 2 \times 4, 3 \times 4$$

Then pl is associated to the map of polynomials rings:

$$k[y_1, \dots y_6] \to k[x_1, \dots x_8]$$

sending y_i to the ith minor. We can discover the relations among the minors by looking at the kernel of this map. We do this using Macaulay 2.

```
i1 : R = QQ[x_1..x_8];


i2 : S = QQ[y_1..y_6]


i3 : A = genericMatrix(R,x_1,2,4)

o3 = | x_1 x_3 x_5 x_7 |
     | x_2 x_4 x_6 x_8 |

              2       4
o3 : Matrix R  <--- R


i4 : M2 = exteriorPower(2,A);

              1       6
o4 : Matrix R  <--- R;

i5 : pl = map(R,S,M2);


i6 : ker pl

o6 = ideal(y y  - y y  + y y )
            3 4    2 5    1 6
```

Having discovered the basic equation

$$y_3y_4 - y_2y_5 + y_1y_6 = 0 \tag{2.1}$$

by machine over $\mathbb{Q}$, let's check this by hand for any k. Let $R(2,4)_i$ be the set of matrices where the ith minor is nonzero. This gives an open covering of $R(2,4)$. The matrices in $R(2,4)$ are of the form

$$\begin{pmatrix} x_1 & x_3 & 1 & 0 \\ x_2 & x_4 & 0 & 1 \end{pmatrix}$$

times a nonsingular 2×2 matrix. Thus it's Plücker vector is nonzero multiple of

$$(x_1x_4 - x_2x_3, -x_2, -x_4, x_1, x_3, 1)$$

It follows easily that (2.1) holds for this. Moreover, the process is reversable, any vector satisfying (2.1) with $y_6 = 1$ is the Plücker vector of

$$\begin{pmatrix} y_4 & y_5 & 1 & 0 \\ -y_2 & -y_3 & 0 & 1 \end{pmatrix}$$

By a similar analysis for the other $R(2,4)_i$, we see that $Gr(2,4)$ is determined by (2.1).

In order to analyze the general case, it is convenient to introduce the appropriate tool. Given a vector space V, the exterior algebra $\wedge^* V$ is the free associative (noncommutative) algebra generated by V modulo the relation $v \wedge v = 0$ for all $v \in V$. This relation forces anticommutivity:

$$(v + w) \wedge (v + w) = v \wedge w + w \wedge v = 0$$

If $v_1, \ldots v_n$ is a basis for V, then

$$\{v_{i_1} \wedge \ldots v_{i_n} \mid i_1 < \ldots i_n\}$$

forms a basis for $\wedge^* V$. $\wedge^r V$ is the space of $\wedge^* V$ spanned by r-fold products of vectors. $\dim \wedge^r V = \binom{n}{r}$.

Let $V = k^n$, with v_i the standard basis. The exterior algebra has a close connection with determinants.

Lemma 2.5.3. *Let A be an $r \times n$ matrix and let $w_i = \sum a_{ij} v_j$ be the ith row, then*

$$w_1 \wedge \ldots w_r = \sum (i_1 \ldots i_r th\ minor\ ) v_{i_1} \wedge \ldots v_{i_r}$$

This says that the $w_1 \wedge \ldots w_r$ is just the Plücker vector $pl(A)$. The condition for an element of $\wedge^r V$ to be of the form $pl(A)$ is that it is decomposable, this means that it is a wedge product of r elements of V.

Lemma 2.5.4. *Given $\omega \in \wedge^s V$ and linearly independent vectors $v_1, \ldots v_r \in V$, $\forall i, \omega \wedge v_i = 0$ iff $\omega = \omega' \wedge v_1 \wedge \ldots v_r$.*

Proof. One direction is clear. For the other, we can assume that v_i is part of the basis. Writing

$$\omega = \sum_{i_1<\ldots<i_s} a_{i_1\ldots i_s} v_{i_1} \wedge \ldots v_{i_s}$$

Then

$$\omega \wedge v_1 = \sum_{i_1<\ldots<i_s; i\neq i_1,\ldots i\neq i_s} a_{i_1\ldots i_s} v_{i_1} \wedge \ldots v_{i_s} \wedge v_1 = 0$$

precisely if 1 is included in the set of indices for which $a_{i_1\ldots i_s} \neq 0$. This means that v_1 can be factored out of ω. Continuing this way gives the whole lemma. $\square$

Corollary 2.5.5. *A nonzero ω is decomposable iff the dimension of the kernel of $v \mapsto \omega \wedge v$ on V is at least r, or equivalently if the rank of this map is at most $n-r$.*

Corollary 2.5.6. *The image of pl is algebraic.*

Proof. The conditions on the rank of $v \mapsto \omega \wedge v$ are expressible as the vanishing of $(n-r+1) \times (n-r+1)$ minors of its matrix. $\square$

2.6 Elimination theory

Although $\mathbb{A}^n$ is compact with its Zariski topology, there is a sense in which it isn't. To motivate the precise statement, we start with a result in point set topology. Recall that a map of topological spaces is *closed* if it takes closed sets to closed sets.

Theorem 2.6.1. *If X is a compact metric space then for any metric space Y, the projection $p : X \times Y \to Y$ is closed*

Sketch. Given a closed set $Z \subset X \times Y$ and a convergent sequence $y_i \in p(Z)$, we have to show that the limit y lies in $p(Z)$. By assumption, we have a sequence $x_i \in X$ such that $(x_i, y_i) \in Z$. Since X is compact, we can assume that x_i converges to say $x \in X$ after passing to a subsequence. Then we see that (x, y) is the limit of (x_i, y_i) so it must lie in Z because it is closed. Therefore $y \in p(Z)$. $\square$

The analogous property for algebraic varieties is called completeness or properness. To state it precisely, we need to define products. For affine varieties $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$, their set theoretic product $X \times Y \subset \mathbb{A}^{n\times m}$ is again algebraic and irreducible.

Exercise 2.6.2. *Prove this.*

Given a polynomial $f \in k[x_0, \ldots x_n, y_1, \ldots y_m]$ which is homogeneous in the x_i, we can defines its zeros in $\mathbb{P}^n \times \mathbb{A}^m$ as before. Similarly if f is homogeneous seperately in the x's and y's or bihomogeneous, then we can define its zeros in $\mathbb{P}^n \times \mathbb{P}^{m-1}$. A subset of either product is called algebraic if it is the intersection of the zero sets of such polynomials. We can define a Zariski topology whose closed sets are exactly the algebraic sets.

Exercise 2.6.3. *Show that the Zariksi topology on $\mathbb{P}^n \times \mathbb{P}^m$ coincides with induced topology under the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^{(n+1)(m+1)-1}$.*

With this it is easy to see that:

Lemma 2.6.4. *The projections $\mathbb{P}^n \times \mathbb{A}^m \to \mathbb{P}^n$ etcetera are continuous.*

We say that a variety X is *complete* if the projections $X \times \mathbb{A}^m \to \mathbb{A}^m$ are all closed. Then $\mathbb{A}^n$ is not complete if $n > 0$. To see this, observe that the projection $\mathbb{A}^n \times \mathbb{A}^1 \to \mathbb{A}^1$ is not closed since $V(x_1 y - 1)$ maps onto $\mathbb{A}^1 - \{0\}$. It is possible to describe the closure of the images explicitly. Given coordinate $x_1, \ldots x_n$ on $\mathbb{A}^n$ and $y_1, \ldots y_m$ on $\mathbb{A}^m$. A subvariety of the product is given by an ideal $I \subset k[x_1, \ldots x_n, y_1, \ldots y_m]$. The intersection $I' = I \cap k[y_1, \ldots y_m]$ is called the elimination ideal (with respect to the chosen variables). Then it known $V(I') = \overline{V(I)}$ [CLO, chap 3].

Exercise 2.6.5. *Prove that $V(I') \supseteq \overline{V(I)}$.*

Theorem 2.6.6 (Main theorem of elimination theory)**.** *$\mathbb{P}^n$ is complete. That is given a collection of polynomials $f_i(\vec{x}, \vec{y})$ homogeneous in x, there exists polynomials $g_j(\vec{y})$ such that*

$$\exists \vec{x} \neq 0, \forall i f_i(\vec{x}, \vec{y}) = 0 \Leftrightarrow \forall j g_j(\vec{y}) = 0$$

Proofs can be found in the references at the end. Note that the images can be computed explicitly by using elimination ideals.

Corollary 2.6.7. *The projections $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ are closed.*

Proof. Let $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ be closed. $\mathbb{P}^m$ has a finite open cover U_i defined earlier. Since $\mathcal{U}_i$ can be identified with $\mathbb{A}^m$, it follows that $Z \cap U_i$ is closed in U_i. Therefore $\mathbb{P}^m = \cup(U_i - Z)$ is open. ☐

2.7 Simultaneous eigenvectors

Given two $n \times n$ matrices A, B, a vector v is a simultaneous eigenvector if it is an eigenvector for both A and B with possibly different eigenvalues. It should be clear that the pairs admiting simultaneous eigenvectors are special. The question is how special.

Proposition 2.7.1. *The set S of $(A, B) \in (Mat_{n\times n}(k))^2 = \mathbb{A}^{n^2} \times \mathbb{A}^{n^2}$ such that A and B admit a simultaneous eigenvector is a proper Zariski closed set.*

Proof. We first introduce an auxillary set

$$E_2 = \{(A, B, v) \in \mathbb{A}^{n^2} \times \mathbb{A}^{n^2} \times k^n \mid v \text{ is an eigenvector for } A\&B\}$$

This is defined by the conditions $rank(Av, v) \leq 1$ and $rank(Bv, v) \leq 1$ which is expressable as the vanishing of the 2×2 minors of these matrices. These are multihomogeneous equations, therefore they define a closed set in $\mathbb{P}^{n^2-1} \times \mathbb{P}^{n^2-1} \times \mathbb{P}^{n-1}$. Let PS be the image of this set in $\mathbb{P}^{n^2-1} \times \mathbb{P}^{n^2-1}$ under projection. This is closed by the elimination theorem and therefore the preimage of PS in $(\mathbb{A}^{n^2} - \{0\})^2$ is also closed. It is now easy to see that our set S, which is the union of the preimage with the axes $\{0\} \times \mathbb{A}^{n^2} \cup \mathbb{A}^{n^2} \times \{0\}$, is also closed. $\square$

The proposition implies that the set S above is defined by polynomials. Let's work out the explicit equations when $n = 2$ and $k = \mathbb{Q}$. We write our matrices as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

Now we work out the ideal I defining E_2:

```
i1 : R = QQ[x,y,a,b,c,d,e,f,g,h];

i2 :  Avv = matrix {{a*x+ b*y, x},{c*x+d*y, y}}

o2 = | xa+yb x |
     | xc+yd y |

               2        2
o2 : Matrix R  <--- R

i3 : Bvv = matrix {{e*x+f*y,x}, {g*x+h*y, y}}

o3 = | xe+yf x |
     | xg+yh y |

               2        2
o3 : Matrix R  <--- R

i4 : I= ideal( determinant Avv, determinant Bvv)

                       2     2                          2     2
o4 = ideal (x*y*a + y b - x c - x*y*d, x*y*e + y f - x g - x*y*h)
```

Now try to eliminate x, y:

```
i5 : eliminate(I, {x,y})

o5 = 0
```

And we get 0? The problem is that affine algebraic set $V(I)$ has two irreducible components which we can see by calculating minimal primes

```
i6 : minimalPrimes I


o6 = {ideal (- x*y*a - ... ),ideal (y, x)}
```

(we are suppressing part of the output). The uninteresting component $x = y = 0$ corresponding to the second ideal is throwing off our answer. Setting J to the first ideal, and then eliminating gives us the right answer.

```
i7 : J= o6#0;

o7 : Ideal of R

i8 : eliminate(J,{x,y})
```

After cleaning this up, we see that the locus of pairs of matrices with a simultaneous eigenvector is defined by

$$bce^2 - acef + cdef - c^2f^2 - abeg + bdeg + a^2fg + 2bcfg - 2adfg + d^2fg - b^2g^2 - 2bceh$$
$$+acfh - cdfh + abgh - bdgh + bch^2 = 0$$

The corresponding computation for $n = 3$ seems painfully slow (at least on my computer). So we ask a more qualitative question: can S be defined by a single equation for $n > 2$? We will see that the answer is no in the next chapter.

Chapter 3

The category of varieties

3.1 Rational functions

Given an affine variety $X \subset \mathbb{A}^n$, its *coordinate ring*

$$\mathcal{O}(X) = k[x_1, \ldots x_n]/\mathcal{I}(X)$$

An $f \in \mathcal{O}(X)$ is an equivalence class of polynomials. Any two polynomial representatives $\tilde{f}_i$ of f give the same value $\tilde{f}_1(a) = \tilde{f}_2(a)$ for $a \in X$. Thus the elements can be identified with functions $X \to k$ called regular functions. Since $\mathcal{I}(X)$ is a prime ideal, it follows that $\mathcal{O}(X)$ is an integral domain. Its field of fractions is called the *function field* $k(X)$ of X. An element $f/g \in k(X)$ is called a rational function. It is regular on the open set $D(g) = \{g(x) \neq 0\}$. Note that $D(g)$ can be realized as the affine variety

$$\{(a, b) \in \mathbb{A}^{n+1} \mid g(a)b = 1\}$$

Its coordinate ring is the localization $\mathcal{O}(X)[1/g]$. This has the same function field as X. Thanks to this, we can define the function field of a projective variety $X \subset \mathbb{P}^n$ as follows. Choose i so that U_i has a nonempty intersection with X, then set $k(X) = k(X \cap U_i)$. If $X \cap U_j \neq \emptyset$ then

$$k(X \cap U_i) = k(X \cap U_i \cap U_j) = k(X \cap U_j)$$

so this is well defined.

Example 3.1.1. $k(\mathbb{P}^n) = k(\mathbb{A}^n) = k(x_1 \ldots x_n)$.

It is convenient to enlarge the class of varieties to a class where affine and projective varieties can be treated together. A *quasi-projective* variety is an open subset of a projective variety. This includes both affine and projective varieties and some examples which are neither such as $\mathbb{P}^n \times \mathbb{A}^m$. The local study of quasi-projective varieties can be reduced to affine varieties because of

Lemma 3.1.2. *Any quasi-projective variety X has a finite open cover by affine varieties.*

Proof. Suppose that X is an open subset of a projective variety $Y \subset \mathbb{P}^n$. Then $X \cap U_i$ is open in $Y \cap U_i$. Let f_{ij} be a finite set of generators of the ideal of $(Y - X) \cap U_i$ Then $Y \cap U_i - V(f_{ij})$ is our desired open cover. These sets are affine since they can be embedded as closed sets into $\mathbb{A}^{n+1}$. $\square$

When X is quasi-projective with a cover as above, we define $k(X) = k(U_i)$ for some (and therefore any) i.

3.2 Quasi-projective varieties

We have defined morphisms of affine varieties. These are simply maps defined by polynomials. We can compose morphisms to get new morphisms. Thus the collection of affine varieties and morphisms between them forms a category. The definition of categories and functors can be found, for example, in [L].

A morphism $F : X \to Y$ of affine varieties given by

$$(x_1, \dots x_n) \mapsto (F_1(x_1 \dots x_n), \dots F_m(x_1, \dots x_n))$$

induces an algebra homomorphism $F^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ called pull back, given by

$$F^*(p(y_1, \dots y_m)) = p(F_1(\vec{x}), \dots F_m(\vec{x}))$$

This can be identified with the composition $p \mapsto p \circ F$ of functions. If $F : X \to Y$ and $G : Y \to Z$, then we have $(G \circ F)^* = F^* \circ G^*$. Therefore the assignment $X \mapsto \mathcal{O}(X)$, $F \mapsto F^*$ is a contravariant functor from the category of affine varieties to the category of finitely generated k-algebra which happen to be domains. In fact:

Theorem 3.2.1. *The category of affine varieties is antiequivalent to this category of algebras.*

Exercise 3.2.2. *The theorem amounts to the following two assertions.*

1. *Show that any algebra of the above type is isomorphic to $\mathcal{O}(X)$ for some X.*

2. *Show that there is a one to one correspondence between the set of morphisms from X to Y and k-algebra homomorphisms $\mathcal{O}(Y) \to \mathcal{O}(X)$.*

We want to have a category of quasi-projective varieties. The definition of morphism is a little trickier than before, since we can allow rational expressions provided that the singularities are "removable". We start with a weaker notion. A *rational map* $F : X \dashrightarrow Y$ of projective varieties is given by a morphism from an affine open set of X to an affine open set of Y. Two such morphisms define the same rational map if they agree on the intersection of their domains. So a rational map is really an equivalence class. A *morphism* $F : X \to Y$ of quasi-projective varieties is a rational map such that for an affine open cover $\{U_i\}$ $F : X \cap U_i \dashrightarrow Y$ extends to a morphism for all i.

Here are two examples.

Example 3.2.3. *The map $\mathbb{P}^1 \to \mathbb{P}^3$ given by $[x_0, x_1] \mapsto [x_0^3, x_0^2x_1, x_0x_1^2, x_1^3]$ is a morphism.*

Example 3.2.4. *Let E be $\mathbb{P}V(zy^2 - x(x-z)(x-2z)) \subset \mathbb{P}^2$. Then $E \to \mathbb{P}^1$ given by $[x, y, z] \mapsto [x, z]$ is a morphism.*

The first should be obvious, but the second isn't. The second map appears to be undefined at $[0, 1, 0] \in E$. To check, we can work on $U_2 = \{y \neq 0\}$. Normalizing $y = 1$ and substituting into the defining equation gives

$$[x, y, z] \mapsto [x, z] = [x, x(x-z)(x-2z)] = [1, (x-z)(x-2z)]$$

The last expression is well defined even on $[0, 1, 0]$.

A regular function on a quasi-projective variety is morphism from it to $\mathbb{A}^1$. For affine varieties, this coincides with the definition given earlier. The following exercise gives a characterization of morphism which is taken as the definition in more advanced presentations. (Of course one would need to define regular functions first.)

Exercise 3.2.5. *Show that a continuous map $F : X \to Y$ of quasi-projective varieties is a morphism if and only if $f \circ F :^{-1} U \to k$ is regular whenever $f : U \to k$ is regular.*

3.3 Graphs

Proposition 3.3.1. *If $f : X \to Y$ is a morphism of quasi-projective varieties, then the graph $\Gamma_f = \{(x, f(x)) \mid x \in X\}$ is closed.*

Proof. We first treat the case of the identity map $id : X \to X$. The graph is just the diagonal $\Delta_X = \{(x, x) \mid x \in X\}$. This is closed when $X = \mathbb{P}^n$ since it is defined by the equations $x_i - y_i = 0$. For $X \subset \mathbb{P}^n$, $\Delta_X = \Delta_{\mathbb{P}^n} \cap X \times X$ is therefore also closed. Define the morphism $f \times id : X \times Y \to Y \times Y$ by $(x, y) \mapsto (f(x), y)$, then $\Gamma_f = (f \times id)^{-1}\Delta_Y$. So it is closed. $\square$

Exercise 3.3.2. *Let $C = V(y^2 - x^3)$. Show that $f : C \to \mathbb{A}^1$ given by $(x, y) \mapsto y/x$ if $x \neq 0$, and $f(0, 0) = 0$ has closed graph but is not a morphism.*

Proposition 3.3.3. *If X is projective and Y quasi-projective, then any morphism $f : X \to Y$ is closed*

Proof. Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projections. q is closed by theorem 2.6.6. If $Z \subset X$ is closed then $f(Z) = q(p^{-1}Z \cap \Gamma_f)$ is also closed. $\square$

The next theorem can be viewed as an analogue of Lioville's theorem in complex analysis, that bounded entire functions are constant.

Theorem 3.3.4. *Any regular function on a projective variety is constant.*

Proof. Let X be a projective variety. A regular function $f : X \to \mathbb{A}^1$ is closed, so $f(X)$ is either a point or $\mathbb{A}^1$. The second case is impossible since f composed with the inclusion $\mathbb{A}^1 \subset \mathbb{P}^1$ is also closed. $\square$

Exercise 3.3.5. *Show that a point is the only variety which is both projective and affine.*

Chapter 4

Dimension theory

4.1 Dimension

We define the *dimension* $\dim X$ of an affine or projective variety to be the trancendence degree of $k(X)$ over k. This is the maximum number of algebraically independent elements of $k(X)$. A point is the only zero dimensional variety. Varieties of dimension $1, 2, 3\ldots$ are called curves, surfaces, threefolds...

Example 4.1.1. *The function field $k(\mathbb{A}^n) = k(\mathbb{P}^n) = k(x_1, \ldots x_n)$. Therefore $\dim \mathbb{A}^n = \dim \mathbb{P}^n = n$ as we would expect.*

A morphism $F : X \to Y$ is called *dominant* if $F(X)$ is dense. If F is dominant then the domain of any rational function on Y meets $F(X)$, so it can be pulled back to a rational function on X. Thus we get a nonzero homomorphism $k(Y) \to K(X)$ which is necessarily injective. A rational map is called dominant if it can be realized by a dominant morphism. Putting all of this together yields:

Lemma 4.1.2. *If $F : X \dashrightarrow Y$ is a dominant rational map, then we have an inclusion $k(Y) \subseteq k(X)$. Therefore $\dim X \geq \dim Y$.*

A dominant morphism $F : X \dashrightarrow Y$ is called *generically finite* if $k(Y) \subseteq k(X)$ is a finite field extension.

Lemma 4.1.3. *If $F : X \dashrightarrow Y$ is a generically finite rational map, then $\dim X = \dim Y$.*

Proof. The extension $k(Y) \subseteq k(X)$ is algebraic, so the transcendence degree is unchanged. $\square$

Example 4.1.4. *Suppose that $f(x_1, \ldots x_n) = \sum f_i(x_1, \ldots x_{n-1})x_n^i$ is a polynomial nonconstant in the last variable. Then projection $V(f) \to \mathbb{A}^{n-1}$ onto the first $n-1$ coordinates is generically finite.*

Lemma 4.1.5. *If f is a nonconstant (homogeneous) polynomial in n (respectively $n+1$) variables, then $\dim V(f) = n-1$ (respectively $\dim \mathbb{P}V(f) = n-1$).*

Proof. By reordering the variables, we can assume that the conditions of the previous example hold. Then $\dim V(f) = \dim \mathbb{A}^{n-1}$. The projective version follows from this. $\square$

In fact, much more is true:

Theorem 4.1.6 (Krull's principal ideal theorem)**.** *If $X \subset \mathbb{A}^n$ is a closed set and $f \in k[x_1, \ldots x_n]$ gives a nonzero divisor in $k[x_1, \ldots x_n]/\mathcal{I}(X)$, then the dimension of any irreducible component of $V(f) \cap X$ is $\dim X - 1$.*

Although we have stated it geometrically, it is a standard result in commutative algebra [AM, E]. By induction, we get:

Corollary 4.1.7. *Suppose that $f_1, f_2, \ldots f_r$ gives a regular sequence in $k[x_1, \ldots x_n]/\mathcal{I}(X)$ which means f_1 is a nonzero divisor in $k[x_1, \ldots x_n]/\mathcal{I}(X)$, f_2 a nonzero divisor in $k[x_1, \ldots x_n]/\mathcal{I}(X) + (f_1)$ and so on. Then any component of $V(f_1, \ldots f_r) \cap X$ has dimension $\dim X - r$.*

Regularity can be understood as a kind of nondegeneracy condition. Without it, $\dim X - r$ only gives a lower bound for the dimension of the components.

4.2 Dimension of fibres

We want to state a generalization of the "rank-nullity" theorem from elementary linear algebra. The role of the kernel of a linear map is played by the fibre. Given a morphism $f : X \to Y$ of quasi-projective varieties. The fibre $f^{-1}(y) \subset X$ is a closed set. So it is a subvariety if it is irreducible.

Theorem 4.2.1. *Let $f : X \to Y$ be a dominant morphism of quasi-projective varieties and let $r = \dim X - \dim Y$. Then for every $y \in f(X)$ the irreducible components of $f^{-1}(y)$ have dimension at least r. There exists a nonempty open set $U \subseteq f(X)$ such that for $y \in U$ the components of $f^{-1}(y)$ have dimension exactly r.*

A complete proof can be found in [M, I§8]. We indicate a proof of the first statement. We can reduce to the case where X and Y are affine, and then to the case $Y = \mathbb{A}^m$ using Noether's normalization theorem. Let y_i be the coordinates on $\mathbb{A}^m$. Then the fibre over $a \in f(X)$ is defined by m equations $y_1 - a_1 = 0, \ldots y_m - a_m = 0$. So the inequality follows from the remark at the end of the last section. For the second part, it be enough to find an open set U such that $y_1 - a_1, \ldots y_m - a_m$ gives a regular sequence whenever $a \in U$.

Corollary 4.2.2. *With the same notation as above, $\dim X = \dim Y$ if and only if there is nonempty open set $U \subseteq f(X)$ such that $f^{-1}(y)$ is finite for all $y \in U$.*

Corollary 4.2.3. $\dim X \times Y = \dim X + \dim Y$.

The dimensions of the fibres can indeed jump.

Example 4.2.4. *Let $B = V(y - xz) \subset \mathbb{A}^3$. Consider the projection to the xy-plane. The fibre over $(0,0)$ is the z-axis, while the fibre over any other point in the range is $(x, y, y/x)$.*

We can apply this theorem to calculate the dimension of the Grassmanian $Gr(r,n)$. Recall there is a surjective map π from the space $R(r,n)$ of $r \times n$ matrices of rank r to $Gr(r,n)$. This is a morphism of quasi-projective varieties. An element $W \in Gr(r,n)$ is just an r dimensional subspace of k^n. The fibre $\pi^{-1}(W)$ is the set of all matrices $A \in R(r,n)$ whose columns span W. Fixing $A_0 \in \pi^{-1}(W)$. Any other element of the fibre is given by MA_0 for a unique matrix Gl_r. Thus we can identify $\pi^{-1}(W)$ with Gl_r. Since $Gl_r \subset Mat_{r\times r}$ and $R(r,n) \subset Mat_{r\times n}$ are open, they have dimension r^2 and rn respectively. Therefore

$$\dim Gr(r,n) = rn - r^2 = r(n-r)$$

4.3 Simultaneous eigenvectors (continued)

Our goal is to estimate the dimension of the set S of pairs $(A,B) \in Mat^2_{n\times n}$ having a simultaneous eigenvector. As a first step we compute the dimension of

$$E_1 = \{(A,v) \in Mat_{n\times n} \times k^n \mid v \text{ is an eigenvector of } A\}$$

We have a projection to $p_2 : E_1 \to k^n = \mathbb{A}^n$ which is obviously surjective. The fibre over 0 is all of $Mat_{n\times n}$ which has dimension n^2. This however is the exception and will not help us. If $M \in Gl_n$, then $(A,v) \mapsto (MAM^{-1}, Mv)$ defines an *automorphism* of E_1, that is a morphism from E_1 to itself whose inverse exists and is also a morphism. Suppose $v \in k^n$ is nonzero. Then there is an invertible matrix M such that $Mv = (1, 0 \ldots 0)^T$. The automorphism just defined takes the fibre over v to the fibre over $(1, 0 \ldots 0)^T$, so they have the same dimension. $p_2^{-1}((1,0\ldots 0)^T)$ is the set of matrices $A = (a_{ij})$ satisfying $a_{21} = a_{31} = \ldots a_{n1} = 0$. This is an affine space of dimension $n^2 - (n-1)$. Since this is the dimension of the fibres over an open set. Therefore

$$\dim E_1 = n^2 - n + 1 + n = n^2 + 1$$

This can be seen in another way. The first projection $p_1 : E_1 \to Mat_{n\times n}$ is also surjective, and the fibre over A is the union of its eigenspaces. These should be one dimensional for a generic matrix.

Consider

$$E_2 = \{(A, B, v) \mid v \text{ is a simultaneous eigenvector}\}$$

This is an example of fibre product. The key point is that fibre of $(A,B,v) \mapsto v$ over v is the product of the fibres $p_2^{-1}(v) \times p_2^{-1}(v)$ considered above. So this has dimension $2(n^2 - n + 1)$. Therefore

$$\dim E_2 = 2(n^2 - n + 1) + n = 2n^2 - n + 2$$

We have a surjective morphism $E_2 \to S$ given by projection. The fibre is the union of simultaneous eigenspaces, which is at least one dimensional. Therefore

$$\dim S \leq 2n^2 - n + 1$$

To put this another way, the codimension of S in $Mat^2_{n\times n}$ is at least $n-1$. So it cannot be defined by single equation if $n > 2$.

Chapter 5

Differential calculus

5.1 Tangent spaces

In calculus, one can define the tangent plane to a surface by taking the span of tangent vectors to arcs lying on it. In translating this into algebraic geometry, we are going to use "infinitesimally short" arcs. Such a notion can be made precise using scheme theory, but this goes beyond what we want to cover. Instead we take the dual point of view. To give a map of affine varieties $Y \to X$ is equivalent to giving a map of their coordinate algebras $\mathcal{O}(X) \to \mathcal{O}(Y)$. In particular, the points of X correspond to homomorphisms $\mathcal{O}(X) \to k$ (cor. 1.2.6). Our infinitesimal arc should have coordinate ring $k[\epsilon]/(\epsilon^2)$. The intuition is that the parameter ϵ is chosen so small so that $\epsilon^2 = 0$. We define a *tangent vector* to be an algebra homomorphism

$$v : \mathcal{O}(X) \to k[\epsilon]/(\epsilon^2)$$

We can compose this with the map $k[\epsilon]/(\epsilon^2) \to k$ ($a + b\epsilon \mapsto 0$) to get homomorphism to k, and hence a point of X. This will be called the base of the vector. Define the tangent space T_aX to X at $a \in X$ to be the set of all tangent vectors with base a. A tangent vector on $\mathbb{A}^n$ amounts to an assignment $x_i \mapsto a_i + b_i\epsilon$, where a_i are coordinates of the base. Therefore $T_a\mathbb{A}^n \cong k^n$.

Exercise 5.1.1. *Show that the $f \in k[x_1, \dots x_n]$ goes to*

$$f(a_1 + b_1\epsilon, \dots) = f(a) + \sum b_i \frac{\partial f}{\partial x_i}(a)\epsilon \in k[\epsilon]/(\epsilon^2)$$

In general, a vector $v \in T_aX$ is determined by the coefficient of ϵ. These coefficients can be added, and multiplied by scalars. Thus T_aX is a k-vector space.

Theorem 5.1.2. *Given a variety $X \subset \mathbb{A}^n$ and $a \in X$. Let $\mathcal{I}(X) = (f_1, \dots f_N)$.*

Then T_aX is isomorphic to the kernel of the Jacobian matrix

$$J(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \dots \frac{\partial f_1}{\partial x_n}(a) \\ & \dots \\ \frac{\partial f_N}{\partial x_1}(a) & \dots \end{pmatrix}$$

Proof. The homomorphism $x_i \mapsto a_i + b_i\epsilon$ factors through $\mathcal{O}(X)$ if and only if each f_j maps to 0. From the exercise this holds precisely when $J(a)b = 0$. □

Example 5.1.3. *The locus of noninvertible matrices $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ is given by $xt - yz = 0$. The Jacobian $J = (t, -z, -y, x)$ so the tangent space at the zero matrix is 4 dimensional while the tangent space at any other point is 3 dimensional.*

Exercise 5.1.4. *Show that $det(I + B\epsilon) = 1 + trace(B)\epsilon$, and conclude that $T_ISl_n(k) = \{B \in Mat_{n\times n} \mid trace(B) = 0\}$.*

Suppose that $U = D(f) \subset X$ is a basic open set. This is also affine. Suppose that $a \in U$.

Lemma 5.1.5. *$T_aU \cong T_aX$ canonically.*

Proof. Any tangent vector $v : \mathcal{O}(U) = \mathcal{O}(X)[1/f] \to k[\epsilon]/(\epsilon^2)$ with base a induces a tangent vector on X. Conversely any vector $v : \mathcal{O}(X) \to k[\epsilon]/(\epsilon^2)$ with base a must factor uniquely through $\mathcal{O}(X)[1/f]$ since $v(f) = f(a) + b\epsilon$ is a unit. □

Thanks to this, we can define the tangent space of a quasi-projective variety X as T_aU where U is an affine neighbourhood of a.

5.2 Singular points

A point a of a variety X is called *nonsingular* if $\dim T_aX = \dim X$, otherwise it is called *singular*. For example, the hypersurface $xt-yz = 0$ has dimension equal to 3. Therefore the origin is the unique singular point. X is called nonsingular if it has no singular points. For example $\mathbb{A}^n$ and $\mathbb{P}^n$ are nonsingular. Over $\mathbb{C}$ nonsingular varieties are manifolds.

Theorem 5.2.1. *The set of nonsingular points of a quasi-projective variety forms a nonempty open set.*

Details can be found in [Ht, II 8.16] or [M, III.4]. Let us at least explains why the set of singular points is closed. It is enough to do this for affine varieties.

Lemma 5.2.2. *Let $m_a \subset \mathcal{O}(X)$ be the maximal ideal of regular functions vanishing at $a \in X$. Then $T_a \cong (m_a/m_a^2)^*$ (* denotes the k-vector space dual).*

Exercise 5.2.3. *Given $h \in (m_a/m_a^2)^*$, we can interpret it has linear map $m_a \to k$ killing m_a^2. Show that $v(f) = f(a) + h(f - f(a))\epsilon$ defines a tangent vector. Show that this gives the isomorphism $T_a \cong (m_a/m_a^2)^*$.*

We need a couple of facts from commutative algebra [AM, E]

- The dimension of X as defined earlier coincide with the Krull dimension of both $\mathcal{O}(X)$ and its localization $\mathcal{O}(X)_{m_a}$.
- $$\dim m_a/m_a^2 = \dim m_a\mathcal{O}(X)_{m_a}/(m_a\mathcal{O}(X)_{m_a})^2 \geq \dim \mathcal{O}(X)_{m_a}$$

It follows that if $X \subset \mathbb{A}^n$, then

$$rankJ(a) \leq n - \dim X$$

for all $a \in X$. The set of singular points of is the set where this inequality is strict. This set is defined by the vanishing of the $(n - \dim X)^2$ minors of J. Therefore it is closed.

A variety is called homogenous if its group of automorphisms acts transitively. That is for any two points there is an automorphism which one to the other.

Corollary 5.2.4. *A homogenous quasi-projective variety is nonsingular.*

Exercise 5.2.5.

1. *Show that the space of matrices in $Mat_{n\times m}$ of rank r is homogeneous.*
2. *Show that any Grassmanian is homogeneous.*

5.3 Singularities of nilpotent matrices

In this section, we again return to the computer, and work out the singularities of the variety $Nilp$ of nilpotent 3×3 matrices. Since these computations are a bit slow, we work over a field of finite (but not too small) characteristic. We will see that in this case the singular locus of $Nilp$ coincides with the set of matrices $\{A \mid A^2 = 0\}$ which is about as nice as one could hope for. Certainly this calls for proper theoretical explanation and generalization (exercise**)!

Using Cayley-Hamilton, we can see that the ideal I of this variety is generated by the coefficients of the characteristic polynomial (see §1.7). We compute the dimension of the variety, and find that it's 6. (Actually we are computing the Krull dimension of the quotient ring $k[x_1 \dots x_9]/I$ which would be the same.)

```
 R = ZZ/101[x_1..x_9];

i2 :  A = genericMatrix(R, x_1, 3,3);


               3       3
o2 : Matrix R  <--- R
```

```
i3 : I = ideal {det(A), trace(A), trace(exteriorPower(2,A))};

o3 : Ideal of R


i4 : Nilp = R/I;

i5 : dim Nilp

o5 = 6
```

Next we do a bunch of things. We define the ideal K of matrices satisfying $A^2 = 0$, and them compute its radical. On the other side we compute the ideal $Sing$ of the singular locus by running through the procedure of the previous section. $Sing$ is the sum of I with the 3×3 minors of the Jacobian of the generators of I. The final step is to see that $\sqrt{K} = \sqrt{Sing}$.

```
i6 :  K = ideal A^2;

o6 : Ideal of R

i7 :  rK = radical K;

o7 : Ideal of R

i8 : Jac = jacobian I;

             9       3
o8 : Matrix R  <--- R

i9 : J =  minors(3, Jac);

o9 : Ideal of R

i10 : Sing = J + I;

o10 : Ideal of R

i11 : rSing = radical Sing;

o11 : Ideal of R

i12 : rK == rSing

o12 = true
```

5.4 Bertini-Sard theorem

Sard's theorem states that the set of critical points of a C^∞ map is small in the sense of Baire category or measure theory. We state an analogue in algebraic geometry whose sister theorem due to Bertini is actually much older. Given a morphism of affine varieties $f : X \to Y$ and $a \in X$, we get a linear map $df_a : T_a X \to T_{f(a)} Y$ defined by $v \mapsto v \circ f^*$.

Exercise 5.4.1. *Show that for $f : \mathbb{A}^n \to \mathbb{A}^m$, df_a can be represented by the Jacobian of the components evaluated at a.*

Theorem 5.4.2. *Let $f : X \to Y$ be a dominant morphism of nonsingular varieties defined over a field of characteristic 0, then there exists a nonempty open set $U \subset Y$ such that for any $y \in U$, $f^{-1}(y)$ is nonsingular and for any $x \in f^{-1}(y)$, df_x is surjective.*

A proof can be found in [Ht, III 10.7]. This is the first time that we have encountered the characteristic zero assumption. The assumption is necessary:

Example 5.4.3. *Suppose k has characteristic $p > 0$. The Frobenius morphism $F : \mathbb{A}^1 \to \mathbb{A}^1$ is given by $x \mapsto x^p$. Then $dF = px^{p-1} = 0$ everywhere.*

A hyperplane of $\mathbb{P}^n$ is a subvariety of the form $H = \mathbb{P}V(f)$ where f is linear polynomial. It is clear that H depends only on f up to a nonzero scalar multiple. Thus the set of hyperplanes forms an n dimensional projective space in its own right called the dual projective space $\check{\mathbb{P}}^n$.

Lemma 5.4.4. *If $X \subset \mathbb{P}^n$ is a subvariety of positive dimension then $X \cap H \neq \emptyset$ for any hyperplane.*

Proof. If $X \cap H = \emptyset$ then X would be contained in the affine space $\mathbb{P}^n - H$. $\square$

Theorem 5.4.5 (Bertini)**.** *If $X \subset \mathbb{P}^n$ is a nonsingular variety, there exists a nonempty open set $U \subset \check{\mathbb{P}}^n$ such that $X \cap H$ is nonsingular for any $H \in U$.*

Although this theorem is valid in any characteristic, the proof we give only works in characteristic 0.

Proof. Let

$$I = \{(x, H) \in X \times \check{\mathbb{P}}^n \mid x \in H\}$$

be the so called incidence variety with projections $p : I \to X$ and $q : I \to \check{\mathbb{P}}^n$. In more explicit, terms

$$I = \{([x_0, \ldots x_n], [y_0, \ldots y - n]) \mid \sum x_i y_i = 0\}$$

So the preimage

$$p^{-1}(\{x_0 \neq 0\} \cap X) = \{([1, x_1, \ldots x_n], [-x_1y_1 - x_2y_2 - \ldots, y_1, \ldots y_n])\} \cong \mathbb{A}^n \times \mathbb{P}^{n-1}$$

A similar isomorphism holds for $p^{-1}(\{x_i \neq 0\} \cap X)$ for any i. Thus I is nonsingular. The map q is surjective by the previous lemma. Therefore by theorem 5.4.2, there exists a nonempty open $U \subset \check{\mathbb{P}}^n$ such that $X \cap H = q^{-1}(H)$ is nonsingular for every $H \in U$.

$\square$

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