Complex Algebraic Geometry

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Chapter 1

Manifolds

1.1 Topological Manifolds

In imprecise terms, a manifold is a space which looks locally like Euclidean space. Actually there several kinds. Let's start with the most basic.

Definition 1.1.1. *A topological manifold of dimension n (or n-manifold) is a metrizable topological space such point has a nbhd U homeomorphic to an open ball of $\mathbb{R}^n$. We refer to U as a coordinate nbhd, and U with a fixed homeomorphism $\phi : U \to V \subset \mathbb{R}^N$ as a chart.*

Recall that metrizable means that it comes from a metric. The condition ensures that the underlying space is sufficiently nice, but it can be omitted at this point. Note that any connected component of a manifold is also a manifold, so we usually just study the connected ones. And if I forget to say "connected manifold" , you should assume that's what I meant.

Example 1.1.2. *Any open subset of $\mathbb{R}^n$ is clearly an n-manifold.*

Example 1.1.3. *The n-sphere $S^n = \{(x_1, \ldots, x_n) \mid x_1^2 + \ldots + x_n^2 = 1\}$ is manifold. Given $p = (1, 0, \ldots 0)$, the hemisphere $S^n \cap \{x_1 > 0\}$ is a coordinate nbhd, with ϕ given by stereographic projection.*

Example 1.1.4. *If X and Y are manifolds of dimension n and m, then $X \times Y$ is manifold of dimension $n+m$. In particular, the torus $T^n = S^1 \times \ldots S^1$ (n times) is an n-manifold.*

Example 1.1.5. *The cone defined by $z^2 = x^2 + y^2$ in $\mathbb{R}^3$ is* **not** *a manifold. Why not?*

The next example is really important in algebraic geometry. So we study it in some detail.

Example 1.1.6. *Complex projective space $\mathbb{CP}^n = \mathbb{P}^n_{\mathbb{C}}$ is the set of complex lines through the origin (= one dimensional complex subspaces) in $\mathbb{C}^{n+1}$. Alternatively, let $t \in \mathbb{C}^*$ act on $\mathbb{C}^{n+1}$ by multiplications. Then $\mathbb{P}^n_{\mathbb{C}} = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$.*

We give $\mathbb{P}^n_{\mathbb{C}}$ the quotient topology, i.e. U is open iff it's preimage $\mathbb{C}^{n+1} - \{0\}$ is open.

Lemma 1.1.7. *$\mathbb{P}^n_{\mathbb{C}}$ is a manifold of dimension $2n$ (NB: algebraic geometers generally use complex dimension, which would be n.)*

Proof. Given $p = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} - \{0\}$, let $[p]$ denote the corresponding point in projective space. The variables z_i are called homogenous coordinates. Let $U_i = \{[z_0, \ldots, z_n] \mid z_i \neq 0\}$. This forms an open cover. The map

$$[z_0, \ldots, z_n] \mapsto (z_1/z_0, \ldots z_n/z_0)$$

defines a homeomorphism $U_0 \cong \mathbb{C}^n$. A similar construction applies to all U_i. □

The manifold $\mathbb{P}^n_{\mathbb{C}}$ is compact (why?). By the above proof it is a compactification of $\mathbb{C}^n$. If one wants to compactify $\mathbb{C}^n$ to a complex manifold (to be defined later). Then this is a simplest way to do it.

Finally, we describe a few more examples constructed by "cut and paste".

Example 1.1.8. *If one glues the ends of $[0, 1] \times \mathbb{R}$ by identifying $(0, x)$ with $(1, x)$, one gets $S^1 \times \mathbb{R}$. However, identifying $(0, x)$ with $(1, -x)$ results in the Moebius strip.*

Example 1.1.9. *Take two copies of T^2, say T_1, T_2. Choose open coordinate disks $D_i \subset T_i$. Note that boundaries $\partial \bar{D}_i$ are circles. Glue $T_1 - D_1$ to $T_2 - D_2$ along the circles $\partial \bar{D}_i$. This construction is called a connected sum, and denoted by $T^2 \# T^2$. This forms a new 2-manifold called a genus 2 surface. It can be visualized by drawing a 2 holed donut. This can be repeated several times. The manifold $T^2 \# T^2 \# \ldots T^2$ (g times) is a called a genus g surface.*

1.2 C^∞-manifolds

It is possible to do calculus on manifolds, but first we have to refine the definition.

Definition 1.2.1. *A C^∞ manifold of dimension n is a topological manifold X equipped with a collection of charts (called an atlas) $\phi_i : U_i \to V_i \subset \mathbb{R}^n$ such that $\phi_i \circ \phi_j^{-1}$ are C^∞, and of course $X = \bigcup U_i$.*

If we write $\phi_i(p) = (x_1(p), \ldots, x_n(p))$. The functions $x_1, \ldots$ are called local coordinates. We are requiring that new coordinates can be expressed as C^∞ functions of old coordinates, and visa versa. We say that two atlases are equivalent if their union is an atlas. Then to be a bit more pedantic, a C^∞ manifold is given by an equivalence class of atlases. (Alternatively, some authors will tell you pick the maximal one by Zorn's lemma.)

Example 1.2.2. *Open subsets of $\mathbb{R}^n$, S^n, T^n and $\mathbb{P}^n_{\mathbb{C}}$ are all C^∞ manifolds.*

For the last item, note that homogenous coordinates are *not* coordinates in the sense of the previous paragraph, but the ratios $z_0/z_i, z_2/z_i \dots$ on U_i are. On $U_i \cap U_j$, we have two systems of coordinates z_k/z_i and z_k/z_j related by multiplying by z_i/z_j or its inverse.

The Moebius strip and the last example above $T^2\#\dots$ can also be made into C^∞ manifold if the gluing is done with care. For the rest of this section manifold means C^∞ manifold.

We can produce many examples, with the help of the implicit function theorem from calculus. In the simplest form it says that if $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is C^∞ such that $f(0) = 0$ and $\frac{\partial f}{\partial x_{n+1}}(0) \neq 0$, and if

$$f(x_1, \dots, x_{n+1}) = 0$$

then we can "solve for" x_{n+1} in terms of the previous variables, at least near the origin. Here is a more precise and stronger statement.

Theorem 1.2.3. *Suppose that $0 \in U \subseteq \mathbb{R}^{n+m}$ is open, and $f : U \to \mathbb{R}^m$ is a C^∞ function such that $f(0) = 0$ and $(\frac{\partial f_i}{\partial x_{n+i}}(0))_{i=1,\dots m}$ is invertible. Then $f^{-1}(0)$ is the graph of another C^∞ function g near the origin. More precisely, there exists open sets $0 \in V \subseteq \mathbb{R}^n, 0 \in W \subseteq \mathbb{R}^m$ and C^∞ $g : V \to W$ such that $V \times W \subset U$, and*

$$\forall (p, q) \in V \times W, f(p, q) = 0 \Leftrightarrow q = g(p)$$

Corollary 1.2.4. *Suppose that $f : U \to \mathbb{R}^m$ is C^∞ such that $X = f^{-1}(0) \neq \emptyset$ and the Jacobian $(\partial f_i/\partial x_{n+i})$ is invertible along all points of X Then X has the structure of C^∞ n-manifold.*

Sketch. Suppose $p_1 \in X$, which for simplicity we assume is 0. Then the implicit function theorem produces V, W, g as above. Set $U_1 = (V \times W) \cap X$. Then the projection $\phi_1 : U_1 \to V = V_1$ is a homeomorphism, because the inverse is given by $p \mapsto (p, g(p))$. This gives a chart at p_1. One can check that for any other choice $p_2, \dots, \phi_2$, the transition function $\phi_2 \circ \phi_1^{-1}$ is C^∞. ∎

Definition 1.2.5. *A map $f : X \to Y$ between manifolds is C^∞ if*

1. *f is continuous*
2. *$\psi_j \circ f \circ \phi_i^{-1}$ is C^∞ for any charts $\phi_i : U_i \to V_i \subset \mathbb{R}^n$ and $\psi_j : U'_j \to V'_j \subset \mathbb{R}^m$.*

The last condition says that f is C^∞ when expressed in local coordinates. Using standard facts from calculus.

Theorem 1.2.6. *The composition of two C^∞.*

It follows that manifolds and C^∞ maps constitute a category. An isomorphism in this category is called a diffeomorphism. To be more explicit:

Definition 1.2.7. *A diffeomorphism between manifolds is a C^∞ bijection such that the inverse is also C^∞. Two manifolds are called diffeomorphic if a diffeomorphism exists between them.*

1.3 Riemann surfaces

We will postpone the discussion of definition of general complex manifolds, and focus on the important special case of Riemann surfaces (= complex curves) for now.

Definition 1.3.1. *A Riemann surface is a topological manifold X equipped with an altas $\phi_i : U_i \to V_i \subset \mathbb{C}$, such that $\phi_i \circ \phi_j^{-1}$ is holomorphic.*

By definition, a Riemann has complex local coordinate z in every chart. Coordinate changes are required to be holomorphic. A Riemann surface can be viewed as the C^∞ 2-manifold, with (real) coordinates $x = Re\, z, y = Im\, z$.

Example 1.3.2. *Any open subset of $\mathbb{C}$ is a Riemann surface.*

The next example is covered in a standard complex analysis class. It is the simplest example of Riemann surface which is not a subset of $\mathbb{C}$.

Example 1.3.3. *The Riemann sphere is $S^2 = \mathbb{C} \cup \{\infty\}$ has two charts $U_0 = \mathbb{C}$ with the identity $\phi_0 : \mathbb{C} \to \mathbb{C}$ or standard coordinate z, and $U_1 = \mathbb{C} - \{0\} \cup \{\infty\}$ with coordinate $\zeta = 1/z$. Alternatively, the sphere can be described as $\mathbb{P}^1_{\mathbb{C}}$, where $z = z_1/z_0$, and $\zeta = z_1/z_0$ in homogeneous coordinates.*

Example 1.3.4. *Given two $\mathbb{R}$-linearly independent complex numbers $a, b \in \mathbb{C}$, let $L = \mathbb{Z}a \oplus \mathbb{Z}b$ be the lattice generated by them. Let $E = \mathbb{C}/L$ with a projection $\pi : \mathbb{C} \to E$. If D is a disk of radius $r < \min(|a+b|, |a-b|)/2$ (so that D lies in a period parallegram), π will map D homeomorphically to its image $\pi(D)$. We take $\pi(D) \xrightarrow{\pi^{-1}} D$ to be a chart. In this way E becomes a Riemann surface called an* elliptic curve. *As a C^∞ manifold, $E = T^2$, and any two 2-tori are diffeomorphic. However, its structure as a Riemann surface depends in a subtle way on a, b.*

We will discuss many other examples later on. But let us consider a nonexample. As noted already, a Riemann surface is a C^∞ 2-manifold. The converse is not true however. For example, *a Moebius strip cannot be turned into a Riemann surface.* The reason is that the strip is not orientable. Informally, a surface in $\mathbb{R}^3$ is orientable if there is a nowhere nonzero normal vector field on it. With a little calculus, we can turn this into a better definition.

Definition 1.3.5. *A C^∞ 2-manifold is orientable if the Jacobian determinants of the coordinate changes*

$$\det \begin{pmatrix} \partial x'/\partial x & \partial x'/\partial y \\ \partial y'/\partial x & \partial y'/\partial y \end{pmatrix}$$

are either all strictly positive or strictly negative.

Theorem 1.3.6. *A Riemann surface is orientable.*

Proof. Let $z = x + iy, w = u + iv$ be holomorphic coordinates. The Cauchy-Riemann equations imply that

$$\det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \det\begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix} = u_x^2 + v_x^2 > 0$$

(In case, you forget what the Cauchy-Riemann says, remember that it means that the derivative is $\mathbb{C}$-linear, or equivalently that the Jacobian matrix commutes with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.) □

We can give another "proof" using the informal definition. It has the advantage of being calculation free. Suppose a surface $X \subset \mathbb{R}^3$ was a Riemann surface. Choose a unit tangent vector v at $p \in X$. Since the tangent plane can be identified with $\mathbb{C}$, we can multiply by i to get new tangent vector u. The cross product $v \times u$ gives a preferred unit normal. Note that the argument tells us that Riemann surfaces are not just orientable, but naturally oriented.

1.4 Complex manifolds

We start a few more remarks about holomorphic functions in one variable. Let us write

$$z = x + iy$$

as usual, and introduce complex valued differential forms

$$dz = dx + idy,\ d\bar{z} = dx - idy$$

(If you aren't sure what differential forms are, you can just be view them as formal expressions for now.) Therefore

$$dx = \frac{1}{2}(dz + d\bar{z})$$

$$dy = \frac{1}{2i}(dz - d\bar{z})$$

Given a C^∞ function $f : U \to \mathbb{C}$, the total differential

$$df = f_x dx + f_y dy = \frac{1}{2}(f_x - if_y)dz + \frac{1}{2}(f_x + if_y)d\bar{z}$$

We introduce operators

$$\partial f = \frac{1}{2}(f_x - if_y)dz$$

$$\bar{\partial} f = \frac{1}{2}(f_x + if_y)d\bar{z}$$

so that

$$d = \partial + \bar{\partial}$$

If we set $u = Re\, f, v = Im\, f$, then

$$\bar{\partial} f = \frac{1}{2}[(u_x - v_y) + i(u_y + v_x)]d\bar{z}$$

This makes it clear that the condition $\bar{\partial} f = 0$ is precisely the Cauchy-Riemann equations. Therefore

Lemma 1.4.1. *f is holomorphic iff $\bar{\partial} f = 0$.*

Let us now go to several variables.

Definition 1.4.2. *Let $U \subseteq \mathbb{C}^n$ be open. A function $f : U \to \mathbb{C}$ is holomorphic if it is C^∞ (or just C^1) and holomorphic in each variable, i.e. when all but one variable is fixed, $f(z_1, \ldots, z_n)$ is holomorphic in the remaining variable.*

We define the Cauchy-Riemann operator in several variables by

$$\bar{\partial} f = \sum_{j=0}^{n} \frac{1}{2}(f_{x_j} + i f_{y_j}) d\bar{z}_j$$

Theorem 1.4.3. *Given a C^∞ function $f : U \to \mathbb{C}$, the following are equivalent:*

1. *f is holomorphic.*
2. *$\bar{\partial} f = 0$*
3. *f is analytic: Writing $z = (z_1, \ldots, z_n)$, then for each $p \in U$, there exists an expansion*
$$f(z + p) = \sum_{j_1, \ldots, j_n \geq 0} a_{j_1 \ldots j_n} z_1^{j_1} \ldots z_n^{j_n}$$
which converges uniformly in a nbhd of 0.

We just give the briefest sketch. See Voisin or other references for more. The equivalence of 1 and 2 should be clear from what we said above. Also $3 \Rightarrow 1$ is clear. Let's consider the converse when $p = 0$. This requires a version of Cauchy's formula in several variables, which can be proved by induction:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_1| = r_1} \cdots \int_{|\zeta_n| = r_n} \frac{f(\zeta)}{\prod(\zeta_j - z_j)} d\zeta_1 \wedge \ldots d\zeta_n$$

(ignore the $\wedge$ if you aren't sure what it means). The integrand can be expanded into a product of geometric series. Since this is uniformly convergent for small $|z|$, we can integrate term by term to obtain a power series expansion for $f(z)$.

We record another consequence of Cauchy's formula, which should be familiar from one variable.

Theorem 1.4.4 (Maximum principle)**.** *If f holomorphic function on a closed polydisk (= product of disks), then either $|f(z)|$ takes a maximum on the boundary or f is constant.*

In addition to analogues of results from one complex variable, there are also some new phenomena:

Theorem 1.4.5 (Hartogs theorem)**.** *Suppose that $n > 1$ and that $p \in U \subseteq \mathbb{C}^n$ is open. A holomorphic function on $U - \{p\}$ extends to U.*

A proof, along with stronger versions, can be found in any book on several complex variables. We are now ready to give the definition.

Definition 1.4.6. *A complex manifold of dimension n is topological manifold X equipped with charts $\phi_i : U_i \to V_i^* \subset \mathbb{C}^n$ such that $\phi_i \circ \phi_j^{-1}$ is holomorphic in the above sense.*

Note n above is the complex dimension. The real dimension would be $2n$. The axioms say about each point, we can find holomorphic (or analytic) coordinates $z_1, \ldots, z_n$. The real and imaginary parts $x_1 = Re\, z_1, y_1 = Im\, z_1, \ldots$ give $2n$ coordinates as the C^∞ manifold.

We already have many examples, such Riemann surfaces (where the complex dim =1), products of Riemann surfaces, and $\mathbb{P}^n_{\mathbb{C}}$. Here are a few examples, which come from algebraic geometry.

Example 1.4.7. *Let $f(z_1, \ldots, z_n)$ be a holomorphic function, e.g. a polynomial, such that the gradient $(\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$ is nonzero along $X = f^{-1}(0) \subset \mathbb{C}^n$. Then X is a complex manifold of dimension $n-1$ called a nonsingular (algebraic or analytic) hypersurface of $\mathbb{C}^n$. This requires the holomorphic version of the implicit function theorem. We will say more about that later.*

Before explaining the projective analogue, note that a polynomial $f \in \mathbb{C}[z_0, \ldots, z_n]$ in the homogenous coordinates does not define a function on $\mathbb{P}^n_{\mathbb{C}}$. However, if f is homogenous, then its zero set

$$V(f) = \{[a] \mid f(a) = 0\}$$

is well defined, because $f(a) = 0 \Rightarrow f(b) = 0$ for all $b \in [a]$. Also

$$f(z_0/z_i, \ldots, 1, \ldots z_n/z_n) \in \mathbb{C}[z_/z_0, \ldots, z_n/z_0]$$

gives a well defined non homogeneous polynomial in the true coordinates of U_i. The zero set of the latter, can be indentified with $V(f) \cap U_i$. Similar remarks apply to a set of homogenous polynomials.

Example 1.4.8. *Suppose that $f \in \mathbb{C}[z_0, \ldots, z_n]$ is a homogenous polynomial of degree d, such that the intersection of*

$$X = V(f)$$

with any U_i is a nonsingular hypersurface in $\mathbb{C}^n$, then X is a complex manifold in $\mathbb{P}^n_{\mathbb{C}}$. It is called a (projective algebraic) hypersurface of degree d.

In the last two examples, when X is zero set of a polynomial in $\mathbb{C}^2$ or $\mathbb{P}^2_{\mathbb{C}}$, it is called a nonsingular affine or projective algebraic plane curve. A degree one curve is isomorphic in an obvious to $\mathbb{P}^1$. The same is true when the degree is 2, but this is less obvious. For degree 3, we get something different, namely an elliptic curve. The proof uses the theory of elliptic functions.

In spite of the similarities between C^∞ manifolds and complex manifolds, there are also big differences. These stem from the fact that C^∞ functions are very flexible, while holomorphic functions are somewhat rigid. The following function on $\mathbb{R}^n$

$$f(x_1, \dots, x_n) = g(x_1) \dots g(x_n)$$

$$g(x) = \begin{cases} e^{1/(x-1)^2} e^{1/(x+1)^2} & \text{if } x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$$

is supported on the cube $[-1, 1]^n$. Viewing this as a function on a coordinate nbhd, and then extending by zero, shiows that any C^∞ manifold has many nonconstant global C^∞ functions. By contrast:

Theorem 1.4.9. *A holomorphic function on a connected compact complex manifold is constant.*

Proof. Let $f(z)$ be a holomorphic function on a compact manifold X. By compactness, $|f(z)|$ takes a maximum value at a point p_0. Then $S = \{p \in X \mid f(p) = f(p_0)\}$ is closed and nonempty. Given $p \in S$, and let U be coordinate nbhd of p equivalent to a polydisk. Then $f|_U$ is constant by the maximum principle. Therefore S is also open. Consequently $X = S$ by connectedness. □

Chapter 2

Sheaves of functions

2.1 Sheaves

It is convenient at this point to introduce the language of sheaves, although in the limited way. We say that such a collection of functions is a presheaf if it is closed under restriction. Given sets X and T, let $Map_T(X)$ denote the set of maps from X to T. Here is the precise definition of a presheaf.

Definition 2.1.1. *Suppose that X is a topological space and T a nonempty set. A presheaf of T-valued functions on X is a collection of subsets $\mathcal{P}(U) \subseteq Map_T(U)$, for each nonempty open $U \subseteq X$, such that the restriction $f|_V \in \mathcal{P}(V)$ whenever $f \in \mathcal{P}(U)$ and $V \subset U$.*

The collection of all functions $Map_T(U)$ is of course a presheaf. Less trivially:

Example 2.1.2. *Let T be a topological space, then the set of continuous functions $C_{X,T}(U)$ from $U \subseteq X$ to T is a presheaf.*

Example 2.1.3. *Let X be a topological space and T be a set. The set $T^{pre}(U)$ of constant functions from U to T is a presheaf called the constant presheaf.*

Upon comparing these two examples, we see an essential difference. Continuity is a local condition, which means that it can be checked in a neighbourhood of a point. Constancy is, however, not local. A presheaf is called a sheaf if the defining condition is local as in the first example. More precisely:

Definition 2.1.4. *A presheaf of functions $\mathcal{P}$ is called a sheaf if given any open set U with an open cover $\{U_i\}$, a function f on U lies in $\mathcal{P}(U)$ if $f|_{U_i} \in \mathcal{P}(U_i)$ for all i.*

The first example $C_{X,T}(U)$ is certainly a sheaf, while the second is not in general. Suppose that T has at least two elements t_1, t_2, and that X contains a disconnected open set U. Then we can write $U = U_1 \cup U_2$ as a union of two disjoint open sets. The function τ taking the value of t_i on U_i is not in $T^{pre}(U)$, but $\tau|_{U_i} \in T^{pre}(U_i)$. Therefore T^{pre} is not sheaf.

However, there is a simple remedy.

Example 2.1.5. *A function is locally constant if it is constant in a neighbourhood of a point. For instance, the function τ constructed above is locally constant but not constant. The set of locally constant functions, denoted by $T(U)$ or $T_X(U)$, is a now sheaf, precisely because the condition can be checked locally. A sheaf of this form is called a constant sheaf.*

There are a number of further examples that will come up frequently.

Example 2.1.6. *Let $X = \mathbb{R}^n$ or a C^∞ manifold, the sets $C^\infty(U)$ of C^∞ real valued functions form a sheaf.*

Example 2.1.7. *Let $X = \mathbb{C}^n$ or a complex manifold, the sets $\mathcal{O}(U)$ of holomorphic functions on U form a sheaf.*

Example 2.1.8. *Let L be a linear differential operator on $\mathbb{R}^n$ with C^∞ coefficients (e. g. $\sum \partial^2/\partial x_i^2$). Let $S(U)$ denote the space of C^∞ solutions in U. This is a sheaf.*

Example 2.1.9. *Let $X = \mathbb{R}^n$, the sets $L^1(U)$ of L^1 or summable functions forms a presheaf which is not a sheaf, because summability is a global condition and not a local one.*

We can always create a sheaf from a presheaf by the following construction.

Example 2.1.10. *Given a presheaf $\mathcal{P}$ of functions from X to T. Define the*

$$\mathcal{P}^s(U) = \{f : U \to T \mid \forall x \in U, \exists \text{ a neighbourhood } U_x \text{ of } x, \text{ such that } f|_{U_x} \in \mathcal{P}(U_x)\}$$

This is a sheaf called the sheafification of $\mathcal{P}$.

When $\mathcal{P}$ is a presheaf of constant functions, $\mathcal{P}^s$ is exactly the sheaf of locally constant functions. When this construction is applied to the presheaf L^1, we obtain the sheaf of locally L^1 functions.

2.2 Manifolds again

We can now reinterpret the notion of a C^∞ or complex manifold. We start with a metrizable space X and a sheaf of C^∞ or holomorphic functions. We require that each point has a nbhd homeomorphic to a ball in $\mathbb{R}^n$ or $\mathbb{C}^n$ so that the homeomorphism preserves this special class of functions. To be more precise, let us fix a field k such as $k = \mathbb{R}$ or $\mathbb{C}$. Then $Map_k(X)$ is a commutative k-algebra with pointwise addition and multplication.

Definition 2.2.1. *Let $\mathcal{R}$ be a sheaf of k-valued functions on X. We say that $\mathcal{R}$ is a sheaf of algebras if each $R(U) \subseteq Map_k(U)$ is a subalgebra. Call the pair $(X, \mathcal{R})$ a concrete ringed space over k simply a k-space. We will sometimes refer to elements of $\mathcal{R}(U)$ as distinguished functions.*

The sheaf $\mathcal{R}$ is called the structure sheaf of X. Basic examples of $\mathbb{R}$-spaces are $(\mathbb{R}^n, C_{\mathbb{R}^n,\mathbb{R}})$, $(\mathbb{R}^n, C^\infty)$, and $(\mathbb{C}^n, \mathcal{O})$ is an example of a $\mathbb{C}$-space.

Definition 2.2.2. *A morphism of k-spaces $(X, \mathcal{R}) \to (Y, \mathcal{S})$ is a continuous map $F : X \to Y$ such that the pullback of distinguished functions are distinguished. More precisely, the condition is that if $f \in \mathcal{S}(U)$ then $F^*f \in \mathcal{R}(F^{-1}U)$, where $F^*f = f \circ F|_{F^{-1}U}$.*

For example, a C^∞ map $F : \mathbb{R}^n \to \mathbb{R}^m$ induces a morphism $(\mathbb{R}^n, C^\infty) \to (\mathbb{R}^m, C^\infty)$ of $\mathbb{R}$-spaces, and a holomorphic map $F : \mathbb{C}^n \to \mathbb{C}^m$ induces a morphism of $\mathbb{C}$-spaces. The converse is also true, and will be left for the exercises.

We note that the collection of k-spaces and morphisms forms a category. In any category, we have a notion of isomorphism. We will spell this out for k-spaces.

Definition 2.2.3. *An isomorphism of k-spaces $(X, \mathcal{R}) \cong (Y, \mathcal{S})$ is a homeomorphism $F : X \to Y$ such that $f \in \mathcal{S}(U)$ if and only if $F^*f \in \mathcal{R}(F^{-1}U)$.*

Given a sheaf S on X and open set $U \subset X$, let $S|_U$ denote the sheaf on U defined by $V \mapsto S(V)$ for each $V \subseteq U$. The following gives an alternative approach to manifolds.

Proposition 2.2.4. *An n-dimensional C^∞ manifold (resp. complex manifold) is the same thing as an $\mathbb{R}$-space (X, C^∞_X) (resp. $\mathbb{C}$-space $(X, \mathcal{O}_X)$) such that*

1. *X is metrizable*
2. *X admits an open cover $\{U_i\}$ such that each $(U_i, C^\infty_X|_{U_i})$ is isomorphic to $(B_i, C^\infty_{B_i})$ for open balls $B_i \subset \mathbb{R}^n$ (resp. ... isomorphic to $(B_i, \mathcal{O}_{B_i})$ for balls in $B_i \subset \mathbb{C}^n$).*

A C^∞ (resp. holomorphic) map between manifolds is the same thing as a morphism of the corresponding $\mathbb{R}$-spaces. (resp. $\mathbb{C}$-spaces).

We omit the proof which is not hard. We give one further example of a complex manifold with the help of this characterization.

Example 2.2.5. *The complex Grassmanian $G = Gr(2, n)$ is the set of 2 dimensional subspaces of $\mathbb{C}^n$. Let $M \subset \mathbb{C}^{2n}$ be the open set of $2 \times n$ matrices of rank 2. Let $\pi : M \to G$ be the surjective map which sends a matrix to the span of its rows. Give G the quotient topology induced from M, and define $f \in \mathcal{O}_G(U)$ if and only if $\pi \circ f \in \mathcal{O}_M(\pi^{-1}U)$. For $i \neq j$, let $U_{ij} \subset M$ be the set of matrices with $(1, 0)^t$ and $(0, 1)^t$ for the ith and jth columns. One can check that*

$$\mathbb{C}^{2n-4} \cong U_{ij} \cong \pi(U_{ij})$$

as ringed spaces. Since the images $\pi(U_{ij})$ cover G, we conclude that G is a $2n - 4$ dimensional complex manifold.

We note that the sheaf theory approach to manifolds is not commonly discussed in most references, but it has some advantages in algebraic or complex geometry where we consider more general kinds of spaces.

Example 2.2.6. *Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be a nonconstant polynomial, and let $X = f^{-1}(0)$. Since we didn't impose a condition on the gradient, X need not be a topological manifold. However, we can still introduce a sheaf of holomorphic functions, where $\mathcal{O}_X(U)$ consists of restrictions of holomorphic functions from an open subset of $\mathbb{C}^n$ containing U. (This is more accurately the sheaf of holomorphic functions on the reduced space corresponding to X.)*

2.3 Stalks

Given two functions defined in possibly different neighbourhoods of a point $x \in X$, we say they have the same *germ* at x if their restrictions to some common neighbourhood agree. This is is an equivalence relation. The germ at x of a function f defined near X is the equivalence class containing f. We denote this by f_x.

Definition 2.3.1. *Given a presheaf of functions $\mathcal{P}$, its stalk $\mathcal{P}_x$ at x is the set of germs of functions contained in some $\mathcal{P}(U)$ with $x \in U$.*

It will be useful to give a more abstract characterization of the stalk using *direct limits* (which are also called inductive limits, or filtered colimits). We explain direct limits in the present context. Suppose that a set L is equipped with a family of maps $\mathcal{P}(U) \to L$, where U ranges over open neighbourhoods of x. We will say that the family is a compatible family if $\mathcal{P}(U) \to L$ factors through $\mathcal{P}(V)$, whenever $V \subset U$. The maps $\mathcal{P}(U) \to \mathcal{P}_x$ given by $f \mapsto f_x$ forms a compatible family. A set L equipped with a compatible family of maps is called a direct limit of $\mathcal{P}(U)$ if and only if for any M with a compatible family $\mathcal{P}(U) \to M$, there is a unique map $L \to M$ making the obvious diagrams commute. This property characterizes L up to isomorphism, so we may speak of *the* direct limit

$$\varinjlim_{x \in U} \mathcal{P}(U).$$

Lemma 2.3.2. $\mathcal{P}_x = \varinjlim_{x\in U} \mathcal{P}(U)$.

Proof. Suppose that $\phi : \mathcal{P}(U) \to M$ is a compatible family. Then $\phi(f) = \phi(f|_V)$ whenever $f \in \mathcal{P}(U)$ and $x \in V \subset U$. Therefore $\phi(f)$ depends only on the germ f_x. Thus ϕ induces a map $\mathcal{P}_x \to M$ as required. □

All the examples of k-spaces encountered so far satisfy the following additional property.

Definition 2.3.3. *We will say that a concrete k-space $(X, \mathcal{R})$ is* locally ringed *if $1/f \in \mathcal{R}(U)$ when $f \in \mathcal{R}(U)$ is nowhere zero.*

Recall that a ring R is *local* if it has a unique maximal ideal, say m. The quotient R/m is called the residue field.

Lemma 2.3.4. *Suppose that $k = \mathbb{R}$ or $\mathbb{C}$, and $(X, \mathcal{R})$ is a ringed space, such that $\mathcal{R}(U)$ consists of continuous functions and $1/f \in \mathcal{R}(U)$ when $f \in \mathcal{R}(U)$ is nowhere zero. Then for any $x \in X$ $\mathcal{R}_x$ is a local ring with residue field isomorphic to k. In particular, this applies to C^∞ and complex manifolds.*

Proof. Let m_x be the set of germs of functions vanishing at x. For $\mathcal{R}_x$ to be local with maximal ideal m_x, it is necessary and sufficient that each $f \in \mathcal{R}_x \setminus m_x$ is invertible. This is clear since $1/f|_U \in R(U)$ for some $x \in U$.

To see that $\mathcal{R}_x/m_x = k$, it is enough to observe that the ideal m_x is the kernel of the evaluation map $ev : \mathcal{R}_x \to k$ given by $ev(f) = f(x)$, and the map is surjective, because $ev(a) = a$ when $a \in k$. □

Proposition 2.3.5. *When $(X, \mathcal{O}_X)$ is an n-dimensional complex manifold, the local ring $\mathcal{O}_{X,x}$ can be identified with ring of convergent power series in n variables.*

Proof. We can replace (X, x) by $(\mathbb{C}^n, 0)$. Then the germ of a holomorphic function at 0 is completely determined by its Taylor series, which converges in a nbhd of 0. □

We write

$$\mathbb{C}\{z_1, \ldots, z_n\} \subset \mathbb{C}[z_1, \ldots, z_n]$$

for the rings of convergent and formal power series in the above variables. Both rings are local with maximal ideal $m = (z_1, \ldots, z_n)$. Also both rings are known to be noetherian [see for example Zariski-Samuel Vol II], so standard results from commutative algebra can be applied. By contrast, when X is a C^∞-manifold, the stalks are non-noetherian local rings. This is because $\cap_n m^n$ contains nonzero functions such as

$$\begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

when $X = \mathbb{R}$, so it violates Krull's theorem [Atiyah-Macdonald, pp 110-111]. Nevertheless, the maximal ideals are finitely generated.

Proposition 2.3.6. *If R is the ring of germs at 0 of C^∞ functions on $\mathbb{R}^n$, then its maximal ideal m is generated by the coordinate functions $x_1, \ldots x_n$.*

Proof. One checks that if $f \in m$, then

$$f = \sum_i x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots tx_n)dt$$

□

If R is a local ring with maximal ideal m then $k = R/m$ is a field called the *residue field.* We will often convey all this by referring to the triple (R, m, k) as a local ring. For stalks of C^∞ and complex manifolds, the residue fields are respectively, $\mathbb{R}$ and $\mathbb{C}$. We note the following properties hold in these cases.

1. There is an inclusion $k \subset R$ which gives a splitting of the natural map $R \to k$.

2. The ideal m is finitely generated.

2.4 Tangent spaces

The tangent space to a manifold at a point is the best linear approximation to it. For a hypersurface $X = f^{-1}(0) \subset \mathbb{R}^n$ with $\nabla f|_X \neq 0$. We could use the definition from calculus: the tangent space at $p \in X$ is

$$T_p = \{v \in \mathbb{R}^n \mid v \cdot \nabla(f)(p) = 0\}$$

But this depends on the embedding. It is better to use a more intrinsic approach. The idea is that to each tangent vector $v \in T_p$ we can associate a directional derivative δ_v which operates on germs of functions at p. Since $v \mapsto \delta_v$ involves no loss of information, we may as well identify them. Thus we arrive the following abstract definition.

Definition 2.4.1. *Let (R, m, k) be a local ring of a C^∞ or complex manifold X at a point p, or more generally a ring satisfying the conditions at the end of the last section. Define the tangent space $T_p = T_R$ to be the set of k-linear derivations $Der_k(R, k)$ i.e. linear maps $\delta : R \to k$ satisfying $\delta(fg) = f(p)\delta g + g(p)\delta f$.*

When (R, m, k) satisfies the above conditions, R/m^2 splits canonically as $k \oplus m/m^2$. The second factor m/m^2 is finite dimensional. Let us focus on $R = C_p^\infty$ for now. The decomposition is given by $f \mapsto (f(p), f - f(p))$. Set $df = f - f(p)$. To get a better sense of what this means, expand f using Taylor's formula

$$f(x_1, \ldots, x_n) = f(p) + \sum \left.\frac{\partial f}{\partial x_i}\right|_p x_i + r(x_1, \ldots, x_n)$$

where the remainder r lies in m^2. We thus

$$df = \sum \left.\frac{\partial f}{\partial x_i}\right|_p \tag{2.1}$$

as the notation suggests.

Lemma 2.4.2. *$d : R \to m/m^2$ is a $\mathbb{R}$-linear derivation. There is an isomorphism*

$$T_p = Hom(m/m^2, \mathbb{R})$$

given by $\delta \mapsto \delta|_{m/m^2}$.

Proof. The first statement is clear from the formula (2.1). Given $\delta' \in Hom(m/m^2, k)$, let $\delta = \delta' \circ d$. This lies in T_p, and the map $\delta' \mapsto \delta$ gives the inverse to the map above. □

Corollary 2.4.3. *$T_p^* = T_R^* \cong m/m^2$ This is called the cotangent space for obvious reasons.*

Lemma 2.4.4. *When (R, m, k) is the ring of germs at 0 of C^∞ functions on $\mathbb{R}^n$. Then a basis for $Der_k(R, k)$ is given*

$$D_i = \left.\frac{\partial}{\partial x_i}\right|_0 \quad i = 1, \dots n$$

The lemma is straight forward using previous facts. A homomorphism $F : S \to R$ of local rings is called local if it takes the maximal ideal of S to the maximal ideal of R. Under these conditions, we get map of cotangent spaces $T_S^* \to T_R^*$ called the codifferential of F. When residue fields coincide, we can dualize this to get a map $dF : T_R \to T_S$ called the derivative or differential. To see the name is justified, suppose $f : U \to V$ is a C^∞ map, where $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ are open. Given $p \in U$, and $q = f(p)$, we get a homomorphism $f^* : S \to R$ between rings of germs of functions on V and U by $f^*g = g \circ f$. The following is straight forward.

Lemma 2.4.5. *Using standard bases, df^* is represented by the Jacobian matrix*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots \frac{\partial f_1}{\partial x_n} \\ & \dots \\ \frac{\partial f_m}{\partial x_1} & \dots \frac{\partial f_m}{\partial x_n} \end{pmatrix} (p)$$

where f_i are the components of f.

Proof. Writing $y_i = f_i(x_1, \dots, x_n)$. The chain rule gives

$$\frac{\partial}{\partial x_j} = \sum_i \frac{\partial f_i}{\partial x_j} \frac{\partial}{\partial y_i}$$

□

Let X be a complex manifold with $p \in X$, let $\mathcal{O}_{X,p} = \mathcal{O}_p$ be the ring of germs of holomorphic functions, and let $C^\infty_{X,p}$ be the ring of germs of C^∞ functions. Let us define the holomorphic tangent space as $T_p^h = T_{\mathcal{O}_p}$, and the real tangent space $T_p = T_{C^\infty_p}$. We have local homomorphism $\mathcal{O}_{X,p} \to C^\infty_{X,p}$, which induces a map $T_p^h \to T_p$ of real vector spaces. If $z_1, \dots, z_n$ are local analytic coordinates, the previous map is given by

$$\frac{\partial}{\partial z_j} \mapsto \frac{\partial}{\partial x_j}$$

Usually, we identify

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$$

So the map above is essentially the real part. From this description, we see that

$$T_p^h \cong T_p$$

as $\mathbb{R}$-vector spaces. This isomorphism imparts on T_p the structure of a complex vector space.

2.5 Vector fields and the tangent bundle

Let X be a C^∞ manifold. A C^∞ vector field on X is $\mathbb{R}$-linear derivation $Der_{\mathbb{R}}(C^\infty(X), C^\infty(X))$. Let $Vect(X)$ denote the set of these. It is clearly an abelian group. Given $f \in C^\infty(X)$ and $D \in Vect(X)$, $fD = g \mapsto f(x)D(g(x))$ is another vector field. This structure makes $Vect(X)$ into a module over $C^\infty(X)$. The following is not hard.

Proposition 2.5.1. *If $U \subset X$ is a coordinate nbhd of X with coordinates $x_1, \ldots, x_n$, then $\frac{\partial}{\partial x_i}$ gives a basis for $Vect(U)$ as $C^\infty(U)$-module. In particular, $Vect(U)$ is a free module of rank n.*

There is an alternative way to understand what a vector field is; it is simply a "C^∞ family" of vectors $v_p \in T_p$ for each p. The C^∞ requirement can be made precise by choosing coordinates as above. We describe a coordinate free approach in the case of a hypersurface $X = f^{-1}(0) \subset \mathbb{R}^n$. Define the manifold

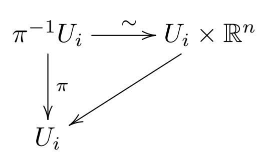
$$T_X = \{(p, v) \in \mathbb{R}^n \times \mathbb{R}^n \mid \nabla f(p) \cdot v = 0\}$$

This has a projection $\pi : T_X \to X$ given by $\pi(p, v) = p$. The space T_X together with π is called the *tangent bundle* of X. The fibres are $\pi^{-1}(p) = T_p$ (using the calculus definition). Then a vector field is simply a C^∞ map $\sigma : X \to T_X$ such that $\pi \circ \sigma = 1$, because we want $\sigma(p) \in T_p$.

In general:

Theorem 2.5.2. *Given a C^∞ n-manifold X, there exists a $2n$-manifold T_X with a C^∞ map $\pi : T_X \to X$ such that*

1. *Each fibre $\pi^{-1}(p) \cong T_p$*
2. *There exists an open cover $\{U_i\}$ and isomorphisms*



which are linear on the fibres.

3. *$Vect(X)$ is isomorphic to the set of C^∞ maps $\sigma : X \to T_X$ (called sections) such that $\pi \circ \sigma = 1$.*

The data in item 2 is called a "local trivialization". One can a choose collection of charts for the cover $\{U_i\}$. A detailed construction can be found in any basic book on manifolds. Here we describe it when we have two charts U_1 and U_2 with coordinates x_i and x_i' respectively. We extend these to coordinates $x_1, \ldots, x_{2n}$ on $U_1 \times \mathbb{R}^n$, and $x_1', \ldots, x_{2n}'$ on $U_2 \times \mathbb{R}^n$. We want to think of

$x_{n+i} = \partial/\partial x_i$ etc. On the intersection $U_1 \cap U_2$ we are given functions expressing $x'_i = \phi_i(x_1, \ldots, x_n)$ and visa versa. Differentiating gives the transformation rule

$$x'_{n+i} = \frac{\partial}{\partial x'_i} = \sum_j \frac{\partial x_j}{\partial x'_i} \frac{\partial}{\partial x_j} = \sum_j \frac{\partial x_j}{\partial x'_i} x_{n+j}$$

for the remaining coordinates. This allows us to glue $U_1 \times \mathbb{R}^n$ to $U_2 \times \mathbb{R}^n$.

The tangent bundle is called trivial, or X is called parallelizable, if we can take $\{U_i\} = \{X\}$, i.e. there exists an isomorphism $T_X \cong X \times \mathbb{R}^n$ as above. In algebraic terms, this equivalent to $Vect(X)$ to being a free module of rank n. Most manifolds are not parallelizable. The simplest counter example is the sphere S^2.

Finally, let us return to sheaf viewpoint. Given C^∞ n-manifold X, if we view $v \in Vect(U)$ as a section $U \to T_X|_U = T_U$, we can restrict it to any subset $V \subset U$. It should be clear that the assignment $Vect_X : U \mapsto Vect(U)$ forms a sheaf of abelian groups. In fact, restrictions are compatible with the module structure in the following sense. Given $f \in C^\infty(U)$ and $v \in Vect(U)$, $fv|_V = f|_V v|_V$. We say that $U \mapsto Vect(U)$ is a sheaf of modules over the sheaf C^∞_X. Finally, if U is a coordinate nbhd, proposition 2.5.1 implies that

$$Vect_X|_U \cong (C^\infty_U)^n$$

A sheaf of modules with this property is locally free of rank n. In summary:

Proposition 2.5.3. *$Vect_X$ is a locally free sheaf of modules over C^∞_X of rank n.*

There is a parallel story in the holomorphic case.

Theorem 2.5.4. *Given a complex n-manifold X, there exists a complex $2n$-manifold T^h_X with a holomorphic map $\pi : T^h_X \to X$ such that*

1. *Each fibre $\pi^{-1}(p) \cong T^h_p$*

2. *There exists an open cover $\{U_i\}$ and isomorphisms*

$$\begin{array}{ccc} \pi^{-1}U_i & \xrightarrow{\sim} & U_i \times \mathbb{C}^n \\ \downarrow{\scriptstyle\pi} & \swarrow & \\ U_i & & \end{array}$$

   which are linear on the fibres.

3. *As C^∞ manifolds $T^h_X = T_X$.*

Thanks to 3, we will usually drop the h in the future. We can define a holomorphic vector field as holomphic section $\sigma : X \to T_X$. We can form a sheaf of holomorphic vector fields as above.

Chapter 3

Differential forms

3.1 Introduction

Let us assume that $U \subset \mathbb{R}^n$ is a open with coordinates x_i. Then a differential form of degree 1, or simply a 1-form, on U is an expression

$$\sum f_i dx_i$$

where $f_i \in C^\infty(U)$. The (exterior) derivative of a C^∞ function is

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

So this plays the role of the gradient. A 1-form can be integrated along a path $\gamma : [0,1] \to U$ by the usual rules of calculus. In order to define surface integrals etc., we need differential forms of higher degree. A 2-form is an expression

$$\sum f_{ij} dx_i \wedge dx_j$$

The symbol $\wedge$ is a distributive and anticommutative product. Anticommutiivity means $dx_i \wedge dx_j = -dx_j \wedge dx_i$. This might seem like a strange rule to impose, but it is extremely useful to do so. Given a 1-form

$$\alpha = \sum f_j dx_j$$

define its exterior derivative

$$d\alpha = \sum_j df_j \wedge dx_j = \sum_i \sum_j \frac{\partial f_j}{\partial x_i} dx_i \wedge dx_j$$

When $n = 3$, a basis of 2-forms consists of $dx_1 \wedge dx_2, dx_1 \wedge dx_3, dx_2 \wedge dx_3$ by anticommutivity. Simplifying $d\alpha$ shows that it is essentially the curl. So a single operation d replaces and generalizes the standard operations of vector calculus.

3.2 1-forms on manifolds

Let X be C^∞ n-manifold. A 1-form at $x \in X$ is simply a cotangent vector in T_x^*. A 1-form on X is a collection of cotangent vectors

$$\alpha \in \prod_{x \in X} T_x^*$$

which varies in C^∞ fashion. Here is the precise condition.

Definition 3.2.1. *A 1-form on X is a collection α as above such that*

$$\forall v \in Vect(U), \langle v, \alpha \rangle \in C^\infty(U)$$

where $\langle , \rangle$ is the obvious pairing between tangent and contangent vectors. Let $\mathcal{E}^1(X)$ denote the set of 1-forms.

Recall that we defined vector fields as simply derivations. So given $v \in Vect(X)$ and $f \in C^\infty(X)$, let $v(f) \in C^\infty(X)$ be the result of applying v as a derivation.

Definition 3.2.2. *Given $f \in C^\infty(X)$, the exterior derivative is the 1-form defined by*

$$\langle v, df \rangle = v(f)$$

Thus we have now given meaning to the symbols dx_i used before. It should be clear that

Lemma 3.2.3. *$d : C^\infty(X) \to \mathcal{E}^1(X)$ is an $\mathbb{R}$-linear derivation.*

Since we basically saying that

$$\mathcal{E}(X) \subset Map_{\prod T_x^*}(X)$$

we can ask whether $U \mapsto \mathcal{E}^1(U)$ is a sheaf of functions, and it is.

Proposition 3.2.4. *The sheaf $\mathcal{E}^1_X$ defined by $U \mapsto \mathcal{E}^1(U)$ is a sheaf of functions.*

Proof. It should be clear that the restriction of a 1-form is 1-form by the way we defined it. So $\mathcal{E}^1$ is a presheaf. Suppose $\{U_i\}$ is an open cover of U, and the restriction of $\alpha \in \prod_{x \in U} T_x^*$ to each U_i is a 1-form. Suppose that $v \in Vect(V)$, then $\langle v|_{U_i \cap V}, \alpha|_{U_i \cap V} \rangle \in C^\infty(U_i \cap V)$. Therefore $\langle v|, \alpha \rangle \in C^\infty(V)$. □

In fact, more is true

Theorem 3.2.5. *$\mathcal{E}^1_X$ is a locally free sheaf of modules over C^∞ of rank n.*

Sketch. The fact that $\mathcal{E}^1_X$ is a sheaf of modules should be clear. If U is a coordinate nbhd, one checks that $\mathcal{E}^1_U$ is has basis given by dx_i. □

1-forms can also be understood as sections of the contangent bundle. The construction is similar to the tangent bundle. Details can be found elsewhere.

3.3 p-forms

Before discussing differential forms of higher degree, we need to pause for some algebra. Let R be a commutative ring, and M an R-module. Define

$$T^*(M) = R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \dots$$

This an R-module, with a product given by tensor product. This makes $T^*(M)$ into a *noncommutative* associative R-algebra called the *tensor algebra* of M. Note that it is also a graded algebra with the nth graded piece

$$T^n(M) = M \otimes \dots \otimes M \ (n \text{ times})$$

We define the *exterior algebra* (sometimes called the Grassman algebra)

$$\wedge^* M = T^*(M)/I$$

where I is the 2-sided ideal generated by $\{m \otimes m \mid m \in M\}$. The operation $\otimes$ induces a product denoted by $\wedge$, which makes $\wedge^* M$ into a graded algebra again. Notice that we are forcing

$$m \wedge m = 0$$

which implies

$$(m_1 + m_2) \wedge (m_1 + m_2) = m_1 \wedge m_2 + m_2 \wedge m_1 = 0$$

for $m_1, m_2 \in M$ Therefore $\wedge$ is anticommutative. (The technically correct term is graded commutative.)

Lemma 3.3.1. *When M is a free module with basis $m_1, \dots, m_n$, then $\wedge^d M$ is free with a basis $m_{i_1} \wedge \dots m_{i_d}$ with $i_1 < \dots i_d$. In particular, $\wedge^d M = 0$ when $d > n$.*

This has been pretty abstract, so let's specialize to the case of a finite dimensional vector space M over a field k. Essentially by the definition of tensor product,

$$Hom_k(T^d M, k)$$

is the space of multilinear functions

$$f : M \times \dots M \to k$$

in d-variables. The space

$$Hom_k(\wedge^d M, k)$$

consists of multilinear functions which are antisymmetric in the sense that switching two arguments results in a sign change. Many basic books start with this point view, but the drawback is the wedge product looks really complicated and unatural when described this way.

Definition 3.3.2. *Given a C^∞ manifold X, a 0-form is simply a C^∞-function. When $p > 0$, a p-form is an element of*

$$\alpha \in \prod_{x\in X} Hom(\wedge^p T_x, \mathbb{R})$$

such that given vector fields $v_1, \ldots, v_p \in Vect(U)$,

$$\alpha(v_1, \ldots, v_p) \in C^\infty(U)$$

Let $\mathcal{E}^p(X)$ be the set of p-forms

The following can be proved as we did before.

Theorem 3.3.3. *The assignment $\mathcal{E}^p_X : U \to \mathcal{E}^p(U)$ forms a locally free sheaf of modules.*

The sheaf property allows us to describe a differential form locally in coordinates, if we need to. One case where this useful is in describing the exterior derivative.

Theorem 3.3.4. *There exists a linear operation $d : \mathcal{E}^p(X) \to \mathcal{E}^{p+1}(X)$ compatible with restriction and satisfying the Leibnitz rule*

$$d(f\alpha) = df \wedge \alpha + f d\alpha, \quad f \in C^\infty(X), \alpha \in \mathcal{E}^p(X)$$

and

$$d^2 f = 0$$

Proof. Let $\{U_j\}$ be a covering by charts. The above rules imply that in any coordinate system on U_j, d must be computed by

$$d(\sum_{i_1<\ldots i_p} f_{i_1\ldots i_p} dx_{i_1} \wedge \ldots dx_{i_p}) = (\sum\sum_j \frac{\partial f_{i_1\ldots i_p}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots dx_{i_p})$$

Given this formula, linearity and Leibnitz is clear. We check that

$$d^2 f = d(\sum_i f_i dx_i) = \sum_{i,j} f_{ij} dx_j \wedge dx_i = \sum_{i<j} (f_{ij} - f_{ji}) dx_i \wedge dx_j = 0$$

where $f_i = \partial f/\partial x_i$ etc. It follows that there is exactly one such operation $d : \mathcal{E}^p(U_j) \to \mathcal{E}^{p+1}(U_j)$. It follows that these local expressions $d(\alpha|_{U_j})$ must patch to yield a well defined $p+1$ form $d\alpha$. □

3.4 de Rham cohomology

Lemma 3.4.1. *Let X be a manifold. The operation $d^2 : \mathcal{E}^p(X) \to \mathcal{E}^{p+2}(X)$ is zero.*

Proof. This is easily checked in coordinates. □

A differential form is called *exact* if it lies in the image of d, and *closed* if its derivative is zero.

Corollary 3.4.2. *Exact forms are closed.*

Conversely, we can ask whether closed forms are exact? This is important both in pure math and in various applications. The answer, however, is in general no.

Example 3.4.3. *Let $S^1 = \mathbb{R}/\mathbb{Z}$. We can identify both $C^\infty(S^1)$ and $\mathcal{E}^1(S^1)$ with periodic functions on $\mathbb{R}$, and d with the derivative. Then $1 \in C^\infty(S^1)$, but it is not the derivative of a periodic function.*

We can try to measure the failure of the last question as follows.

Definition 3.4.4. *The nth (real) de Rham cohomology group of a manifold X is*

$$H^n_{dR}(X) = \frac{\{\text{closed } n\text{-forms}\}}{\{\text{exact } n\text{-forms}\}}$$

We will see that this is very computable. A key fact which helps with computations is the following.

Theorem 3.4.5 (Poincaré's lemma)**.** *For all $i > 0$, $H^i_{dR}(\mathbb{R}^n) = 0$*

Proof. Assume, by induction, that the theorem holds for $n-1$. Consider the maps $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $\iota : \mathbb{R}^{n-1} \to \mathbb{R}^n$ defined by $p(x_1, x_2, \ldots x_n) = (x_2, \ldots x_n)$ and $\iota(x_2, \ldots x_n) = (0, x_2, \ldots x_n)$. Let I be the identity transformation and let $R = (\iota \circ p)^*$. More explicitly, $R : \mathcal{E}^k(\mathbb{R}^n) \to \mathcal{E}^k(\mathbb{R}^n)$ is the $\mathbb{R}$-linear operator defined by

$$R(f(x_1, \ldots x_n)dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = \begin{cases} f(0, x_2, \ldots x_n)dx_{i_1} \wedge \ldots \wedge dx_{i_k} & \text{if } 1 \notin \{i_1, i_2 \ldots\} \\ 0 & \text{otherwise} \end{cases}$$

where we always choose $i_1 < i_2 < \ldots$. The image of R can be identified with $p^*\mathcal{E}^k(\mathbb{R}^{n-1})$. Note that R commutes with d. Therefore if $\alpha \in \mathcal{E}^k(\mathbb{R}^n)$ is closed, $dR\alpha = Rd\alpha = 0$. By the induction assumption, $R\alpha$ is exact.

For each k, define a linear map $h : \mathcal{E}^k(\mathbb{R}^n) \to \mathcal{E}^{k-1}(\mathbb{R}^n)$ by

$$h(f(x_1, \ldots x_n)dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = \begin{cases} (\int_0^{x_1} f dx_1)dx_{i_2} \wedge \ldots \wedge dx_{i_k} & \text{if } i_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then the fundamental theorem of calculus shows that $dh + hd = I - R$. (In other words, h is homotopy from I to R.) Given $\alpha \in \mathcal{E}^k(\mathbb{R}^n)$ satisfying $d\alpha = 0$. We have $\alpha = dh\alpha + R\alpha$, which is exact. □

3.5 Complex manifolds

Let X be a complex manifold of dimension n. We can treat this as a C^∞-manifold of dimension $2n$. It will be convenient to work with complex valued C^∞ forms. We define

$$\mathcal{E}^k_{\mathbb{C}}(X) = \mathcal{E}^k(X) \otimes_{\mathbb{R}} \mathbb{C}$$

When the context makes it clear, we may drop the subscript $\mathbb{C}$. If $z_1, \ldots, z_n$ are local analytic coordinates in U, we write

$$dz_j = dx_j + idy_j$$

$$d\bar{z}_j = dx_j - idy_j$$

where $x_j = Re\, z_j, y_j = Im\, z_j$. Note that this gives a basis of $\mathcal{E}^1_{\mathbb{C}}(U)$ which is more convenient to work with. We can extend this to a basis of $\mathcal{E}^k_{\mathbb{C}}$ by taking wedge products.

Definition 3.5.1. *A differential form is of type (p, q) if it expressible in coordinates as a linear combination of*

$$dz_{i_1} \wedge \ldots dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots d\bar{z}_{j_q}$$

We let $\mathcal{E}^{pq}(X)$ denote the space of these.

Since the notation above gets quite cumbersome, let's abbreviate it as

$$dz_I \wedge d\bar{z}_J$$

for ordered sets $I = i_1 < i_2 \ldots$, $J = j_1 < j_2 \ldots$.

Given a function f, define

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j})$$

$$\partial f = \sum_{j=0}^{n} \frac{\partial f}{\partial z_j} dz_j$$

$$\frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j})$$

$$\bar{\partial} f = \sum_{j=0}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j$$

Recall that the last operator is the Cauchy-Riemann operator. We extend these to higher degree forms as follows

$$\partial(\sum f_{IJ} dz_I \wedge d\bar{z}_J) = \sum (\partial f_I) \wedge dz_I \wedge d\bar{z}_J$$

$$\bar{\partial}(\sum f_{IJ} dz_I \wedge d\bar{z}_J) = \sum (\bar{\partial} f_I) \wedge dz_I \wedge d\bar{z}_J$$

Although we define these using coordinates:

Theorem 3.5.2. *The operators ∂ and $\bar{\partial}$ are well defined.*

Proof. We just do this for the original operators on functions. Rather than proving this by calculation, we give an intrinsic description. Recall that we have an isomorphism of real vector space $T_x^h \cong T_x$, where the first space is the holomorphic tangent space. Let $J : T_x \to T_x$ denote the linear map induced by multiplication by i on T_x^h. This induces maps $T_x^* \to T_x^*$ and on the complexification that we also denote by J. In coordinates $J(dx_k) = dy_k$ and $J(dy_k) = -dx_k$. It follows that $Jdz_k = -idz_k$ and $Jd\bar{z}_k = id\bar{z}_k$. So these are eigenvectors for J. To define ∂f or $\bar{\partial} f$, we can project df onto the $-i$ or $+i$ eigenspaces □

Lemma 3.5.3. *The following identities hold*

$$\partial^2 = \bar{\partial}^2 = \bar{\partial}\partial + \partial\bar{\partial} = 0$$

$$d = \partial + \bar{\partial}$$

Proof. The lemma follows from the identities

$$\frac{\partial^2 f}{\partial z_j \partial z_k} = \frac{\partial^2 f}{\partial z_k \partial z_j}$$

$$\frac{\partial^2 f}{\partial \bar{z}_j \partial \bar{z}_k} = \frac{\partial^2 f}{\partial \bar{z}_k \partial \bar{z}_j}$$

$$\frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} = \frac{\partial^2 f}{\partial \bar{z}_k \partial z_j}$$

which are easily checked. □

An analogue of de Rham cohomology is provided by:

Definition 3.5.4. *The (p,q)th Dolbeault cohomology is*

$$H^{pq}_{Dol}(X) = \frac{\ker \mathcal{E}^{pq}(X) \xrightarrow{\bar{\partial}} \mathcal{E}^{p,q+1}(X)}{\operatorname{im} \mathcal{E}^{p,q-1}(X) \xrightarrow{\bar{\partial}} \mathcal{E}^{p,q}(X)}$$

For people comfortable with homological algebra, the previous lemma says that $\mathcal{E}^{\bullet\bullet}(X)$ forms a double complex. This gives rise to a spectral sequence, which we may study later. Dolbeault cohomology gives first page of the spectral sequence.

For now, we want to understand the simplest case.

Definition 3.5.5. *A p-form $\alpha \in \mathcal{E}^p_{\mathbb{C}}(X)$ is called holomorphic if it can be expressed in local coordinates as*

$$\alpha = \sum f_I dz_I$$

with f_I holomorphic. Let $\Omega^p_X(X) = \Omega^p(X)$ be the space of these.

Lemma 3.5.6. $H^{p,0}(X) = \Omega^p(X)$

Proof. $H^{p,0}(X) = \{\alpha \in \mathcal{E}^p(X) \mid \bar{\partial}\alpha = 0\} = \Omega^p(X)$ □

Similar to the C^∞ case, we have

Theorem 3.5.7. *The assigment $\Omega^p_X : U \mapsto \Omega^p(U)$ is a sheaf, and in fact a locally free $\mathcal{O}_X$-module.*

We will see later that the other Dolbeault groups can be understood as higher cohomology of these sheaves.

Chapter 4

Sheaf Cohomology

4.1 Presheaves and Sheaves

It will be convenient to define presheaves of things other than functions. For instance, one might consider sheaves of equivalence classes of functions, distributions and so on. For this more general notion of presheaf, the restrictions maps have to be included as part the datum:

Definition 4.1.1. *A presheaf $\mathcal{P}$ of sets (respectively groups or rings) on a topological space X consists of a set (respectively group or ring) $\mathcal{P}(U)$ for each open set U, and maps (respectively homomorphisms) $\rho_{UV} : \mathcal{P}(U) \to \mathcal{P}(V)$ for each inclusion $V \subseteq U$ such that:*

1. $\rho_{UU} = id_{\mathcal{P}(U)}$
2. $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$ *if* $W \subseteq V \subseteq U$.

We will usually write $f|_V = \rho_{UV}(f)$.

Definition 4.1.2. *A sheaf $\mathcal{P}$ is a presheaf such that for any open cover $\{U_i\}$ of U and $f_i \in \mathcal{P}(U_i)$ satisfying $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, there exists a unique $f \in \mathcal{P}(U)$ with $f|_{U_i} = f_i$.*

In English, this says that a collection of local sections can be patched together provided they agree on the intersections. Here is an example constructed abstractly.

Example 4.1.3. *Given a space X, a point $p \in X$ and a group A, the skyscraper sheaf*

$$A_p(U) = \begin{cases} A & \text{if } p \in U \\ 0 & \text{otherwise} \end{cases}$$

The restriction $\rho_{UV} = id$ if both sets contain p, otherwise it's 0.

Definition 4.1.4. *Given presheaves of sets (respectively groups) $\mathcal{P}, \mathcal{P}'$ on the same topological space X, a morphism $f : \mathcal{P} \to \mathcal{P}'$ is collection of maps (respectively homomorphisms) $f_U : \mathcal{P}(U) \to \mathcal{P}'(U)$ which commute with the restrictions. Given morphisms $f : \mathcal{P} \to \mathcal{P}'$ and $g : \mathcal{P}' \to \mathcal{P}''$, the compositions $g_U \circ f_U$ determine a morphism from $\mathcal{P} \to \mathcal{P}''$. The collection of presheaves of Abelian groups and morphisms with this notion of composition constitutes a category $PAb(X)$.*

Definition 4.1.5. *The category $Ab(X)$ is the full subcategory of $PAb(X)$ generated by sheaves of Abelian groups on X. In other words, objects of $Ab(X)$ are sheaves, and morphisms are defined in the same way as for presheaves.*

Example 4.1.6. *The exterior derivative $d : \mathcal{E}^p_X \to \mathcal{E}^{p+1}_X$ is a morphism of sheaves.*

A special case of a morphism is the notion of a *subsheaf* of a sheaf. This is a morphism of sheaves $f : \mathcal{P} \to \mathcal{P}'$ where each $f_U : \mathcal{P}(U) \subseteq \mathcal{P}'(U)$ is an inclusion.

Example 4.1.7. *The sheaf of C^∞-functions on $\mathbb{R}^n$ is a sub sheaf of the sheaf of continuous functions.*

Example 4.1.8. *Let Y be a closed subset of a complex manifold $(X, \mathcal{O}_X)$, the ideal sheaf associated to Y,*

$$\mathcal{I}_Y(U) = \{f \in \mathcal{O}_X(U) \mid f|_Y = 0\},$$

is a subsheaf of $\mathcal{O}_X$

Example 4.1.9. *Given a sheaf of rings of functions $\mathcal{R}$ over X, and $f \in \mathcal{R}(X)$, the collection of maps $\mathcal{R}(U) \to \mathcal{R}(U)$ given by multiplication by $f|_U$ is a morphism.*

Example 4.1.10. *Let X be a C^∞ manifold, then $d : C^\infty_X \to \mathcal{E}^1_X$ is a morphism of sheaves.*

Fix a space X and a point $x \in X$. We define the *stalk* $\mathcal{P}_x$, of a presheaf $\mathcal{P}$ at x, as the direct limit $\varinjlim \mathcal{P}(U)$ over neighbourhoods of x. The elements can be represented by germs of sections of $\mathcal{P}$ in concrete cases considered earlier. Given a morphism $\phi : \mathcal{P} \to \mathcal{P}'$, we get an induced map $\mathcal{P}_x \to \mathcal{P}'_x$ taking the germ of f to the germ of $\phi(f)$. This gives a functor from $PAb(X) \to Ab$.

Theorem 4.1.11. *There is a functor $\mathcal{P} \mapsto \mathcal{P}^+$ from $PAb(X) \to Ab(X)$ called sheafification, with the following properties:*

(a) There is a canonical morphism $\mathcal{P} \to \mathcal{P}^+$.

(b) If $\mathcal{P}$ is a sheaf then the morphism $\mathcal{P} \to \mathcal{P}^+$ is an isomorphism.

(c) Any morphism from $\mathcal{P}$ to a sheaf factors uniquely through $\mathcal{P} \to \mathcal{P}^+$

(d) The map $\mathcal{P} \to \mathcal{P}^+$ induces an isomorphism on stalks.

We just explain a construction when $\mathcal{P}$ is a presheaf of functions. We can define

$$\mathcal{P}^+(U) = \{f : U \to T \mid \forall x \in U, \exists \text{ a nbhd } U_x \text{ of } x, \text{ such that } f|_{U_x} \in \mathcal{P}(U_x)\}$$

In other words, we add enough to make it into a sheaf. Properties (a) and (b) should be clear, and (d) can also be checked with a bit of thought.

4.2 Exact Sequences

The categories $PAb(X)$ and $Ab(X)$ are *additive* which means among other things that $Hom(A, B)$ has an Abelian group structure such that composition is bilinear. Actually, more is true. These categories are *Abelian* (see Weibel or any other modern book on homological algebra for the definition). This implies that they possess many of the basic constructions and properties of the category of Abelian groups. In particular, there are intrinsic notions of exactness in these categories. We give a non intrinsic, but equivalent, formulation of this notion for $Ab(X)$.

Definition 4.2.1. *A sequence of sheaves on X*

$$\ldots \mathcal{A} \to \mathcal{B} \to \mathcal{C} \ldots$$

is called exact in the if and only if the sequence of stalks

$$\ldots \mathcal{A}_x \to \mathcal{B}_x \to \mathcal{C}_x \ldots$$

is exact for every $x \in X$.

The key point is that no matter how complicated X and the sheaves are globally, exactness is a local issue, and this what gives the notion its power. We will let the symbols $\mathcal{A}, \mathcal{B}, \mathcal{C}$ stand for sheaves for the remainder of this section unless stated otherwise. We will also say that morphism $\mathcal{A} \to \mathcal{B}$ is a *monomorphism* (respectively *epimorphism*) if $\mathcal{A}_x \to \mathcal{B}_x$ is injective (respectively surjective) for all $x \in X$

Lemma 4.2.2. *Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$, then $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is exact if and only if for any open $U \subseteq X$*

(1) $g_U \circ f_U = 0$.

(2) Given $b \in \mathcal{B}(U)$ with $g(b) = 0$, there exists an open cover $\{U_i\}$ of U and $a_i \in \mathcal{A}(U_i)$ such that $f_{U_i}(a_i) = b|_{U_i}$.

Proof. We will prove one direction, leaving the other as an exercise. Suppose that $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is exact. To simply notation, we write suppress the subscript U. Given $a \in \mathcal{A}(U)$, $g(f(a)) = 0$, since $g(f(a))_x = g(f(a_x)) = 0$ for all $x \in U$. This shows (1).

Given $b \in \mathcal{B}(U)$ with $g(b) = 0$, then for each $x \in U$, b_x is the image of a germ in $\mathcal{A}_x$. Choose a representative a_i for this germ in some $\mathcal{A}(U_i)$ where U_i is a neighbourhood of x. After shrinking U_i if necessary, we have $f(a_i) = b|_{U_i}$. As x varies, we get an open cover $\{U_i\}$, and sections $a_i \in \mathcal{A}(U_i)$ as required. □

Corollary 4.2.3. *If $\mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U)$ is exact for every open set U, then $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is exact.*

The converse is false, but we do have:

Lemma 4.2.4. *If*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is an exact sequence of sheaves, then

$$0 \to \mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U)$$

is exact for every open set U.

Proof. Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ denote the maps. By lemma 4.2.2, $g_u \circ f_U = 0$. Suppose $a \in \mathcal{A}(U)$ maps to 0 under f, then $f(a_x) = f(a)_x = 0$ for each $x \in U$ (we are suppressing the subscript U once again). Therefore $a_x = 0$ for each $x \in U$, and this implies that $a = 0$.

Suppose $b \in \mathcal{B}(U)$ satisfies $g(b) = 0$. Then by lemma 4.2.2, there exists an open cover $\{U_i\}$ of U and $a_i \in \mathcal{A}(U_i)$ such that $f(a_i) = b|_{U_i}$. Then $f(a_i|_{U_i \cap U_j} - a_j|_{U_i \cap U_j}) = 0$, which implies $a_i|_{U_i \cap U_j} - a_j|_{U_i \cap U_j} = 0$ by the first paragraph. Therefore $\{a_i\}$ patch together to yield an element of $a \in \mathcal{A}(U)$ such that $f(a) = b$.

□

We give some *natural* examples to show that $\mathcal{B}(X) \to \mathcal{C}(X)$ is not usually surjective.

Example 4.2.5. *Consider the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Let $\mathbb{R}_{S^1}$ denote the sheaf of locally constant $\mathbb{R}$-valued function. Then*

$$0 \to \mathbb{R}_{S^1} \to C^\infty_{S^1} \xrightarrow{d} \mathcal{E}^1_{S^1} \to 0$$

is exact. However $C^\infty(S^1) \to \mathcal{E}^1(S^1)$ is not surjective.

To see the first statement, let $U \subset S^1$ be an open set diffeomorphic to an open interval. Then the sequence

$$0 \to \mathbb{R} \to C^\infty(U) \xrightarrow{f \to f'} C^\infty(U)dx \to 0$$

is exact by calculus. Thus we get exactness on stalks. The 1-form dx gives a global section of $\mathcal{E}^1(S^1)$ since it is translation invariant. However, it is not the differential a periodic function. Therefore $C^\infty(S^1) \to \mathcal{E}^1(S^1)$ is not surjective.

Example 4.2.6. *Let $(X, \mathcal{O}_X)$ be a complex manifold and $Y \subset X$ a submanifold. Let*

$$\mathcal{I}_Y(U) = \{f \in \mathcal{O}_X(U) \mid f|_Y = 0\}$$

then this is a sheaf called the ideal sheaf of Y, and

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

is exact. The map $\mathcal{O}_X(X) \to \mathcal{O}_Y(X)$ need not be surjective. For example, let $X = \mathbb{P}^1_{\mathbb{C}}$ with $\mathcal{O}_X$ the sheaf of holomorphic functions. Let $Y = \{p_1, p_2\} \subset \mathbb{P}^1$ be a set of distinct points. Then the function $f \in \mathcal{O}_Y(X)$ which takes the value 1 on p_1 and 0 on p_2 cannot be extended to a global holomorphic function on $\mathbb{P}^1$ since all such functions are constant by Liouville's theorem.

4.3 Sheaf cohomology

Let Ab denote the category of abelian groups. Here is the basic result

Theorem 4.3.1. *Given a space X, there exists a sequence of functors*

$$H^i(X, -) : Ab(X) \to Ab, i \in \mathbb{N}$$

with the following properties:

1. *$H^0(X, \mathcal{A}) \cong \mathcal{A}(X)$*
2. *Given a short exact sequence*

   $$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

   of sheaves, there is a long exact sequence

   $$0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to H^0(X, \mathcal{C}) \to H^1(X, \mathcal{A}) \to H^1(X, \mathcal{B}) \to \dots$$

   extending what we found in lemma 4.2.4.
3. *There is a class of sheaves called injective sheaves, such that*
   (a) *any sheaf can be embedded (via a monomorphism) into an injective sheaf,*
   (b) *an injective sheaf $\mathcal{B}$ is acyclic, which means $H^i(X, \mathcal{B}) = 0$ for all $i > 0$.*

We omit the proof, or for that matter the definition of an injective object, since it is better left for a class in homological algebra. As an abstract statement it certainly does have content. However, to be truly useful, we need to better understand what these higher cohomology groups are, and how to compute them in good cases. In particular, we would like to know when $H^1(X, \mathcal{A}) = 0$, as this would tell us that $\mathcal{B}(X) \to \mathcal{C}(X)$ is surjective. Of course, injective sheaves have this property, but it is hard to find examples "in nature". Fortunately, there is more accessible class of sheaves with this property.

Definition 4.3.2. *A sheaf $\mathcal{A}$ is flasque (or flabby) if $\mathcal{A}(X) \to \mathcal{A}(U)$ is surjective for all open U.*

We note that the class of flasque is wider:

Proposition 4.3.3. *Injective sheaves are flasque.*

Proof. Hartshorne, p 207. □

Here are a couple of natural examples.

Example 4.3.4. *Skyscraper sheaves are flasque.*

Example 4.3.5. *Let X be an irreducible algebraic variety some field with its* Zariski topology *(closed sets are finite unions of subvarieties). A preseaf of locally constant functions (with any target) is a flasque sheaf.*

Lemma 4.3.6. *If $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is an exact sequence of sheaves with $\mathcal{A}$ flasque, then $\mathcal{B}(X) \to \mathcal{C}(X)$ is surjective.*

Proof. Let's assume for simplicity that X has a countable basis. (Although this isn't necessary, it holds in all the cases that we care about.) Let $\gamma \in \mathcal{C}(X)$. By assumption, there is an open cover $\{U_i\}_{i\in\mathbb{N}}$, such that $\gamma|_{U_i}$ lifts to a section $\beta_i \in \mathcal{B}(U_i)$. We will define

$$\sigma_i \in \mathcal{B}(\bigcup_{j<i} U_j)$$

inductively, so that it maps to the restriction of γ. Set $\sigma_1 = \beta_0$. Now suppose that σ_i exists. Let $U = U_i \cap (\cup_{j<i} U_j)$. Then $\beta_i|_U - \sigma_i|_U$ is the image of a section $\alpha_i' \in \mathcal{A}(U)$. By hypothesis α_i' extends to a global section $\alpha_i \in \mathcal{A}(X)$. Then set

$$\sigma_{i+1} = \begin{cases} \sigma_i \text{ on } \bigcup_{j<i} U_j \\ \beta_i - \alpha_i|_{U_i} \text{ on } U_i \end{cases}$$

□

Corollary 4.3.7. *If $\mathcal{A}$ is flasque, then $H^1(X, \mathcal{A}) = 0$.*

Proof. Embed $\mathcal{A}$ into an injective sheaf $\mathcal{B}$, we can complete this to an exact sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

by taking $\mathcal{C}$ to be the sheafification of $U \mapsto \mathcal{B}(U)/\mathcal{A}(U)$. Then we have

$$\mathcal{B}(X) \to \mathcal{C}(X) \to H^1(X, \mathcal{A}) \to 0$$

Since the first map is surjective, the corollary follows. □

More is true.

Theorem 4.3.8. *Flasque sheaves are acyclic.*

Proof. We show that $H^i(\mathcal{A}) = 0$ for $i > 0$ and $\mathcal{A}$ flasque by induction. The case of $i = 1$ was just proved. Now suppose that it holds for i. Let $\mathcal{A}$ be flasque. Embed it into an injective sheaf $\mathcal{B}$, and complete this to a short exact sequence

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

as above. By the previous lemma $\mathcal{B}(U) \to \mathcal{C}(U)$ is surjective. Since $\mathcal{B}$ is flasque, $\mathcal{B}(X) \to \mathcal{C}(X)$ is surjective. This forces $\mathcal{C}(X) \to \mathcal{C}(U)$ to be surjective. Therefore $\mathcal{C}$ is flasque. So by induction

$$0 = H^i(X, \mathcal{C}) \to H^{i+1}(X, \mathcal{A}) \to H^{i+1}(X, \mathcal{B}) = 0$$

□

For applications to number theory, Weil proposed in 1949 that there should exist a cohomology theory for algebraic varieties over finite fields, which behaves like singular cohomology, i.e. the kind of cohomology one learns in a topology class. Such theories (étale and crystalline cohomology) were constructed by Grothendieck and his collaborators in the early 60's. However, sheaf cohomology with the Zariski topology won't work, because you'll just zero by the previous theorem.

4.4 Soft sheaves

While flasque sheaves are better than injective sheaves in terms of finding examples, most sheaves won't be flasque. However, if we replace open by closed sets in the definition, this situation improves dramatically.

Definition 4.4.1. *A sheaf $\mathcal{A}$ is soft if for any closed set S, and section $\alpha \in \mathcal{A}(U)$, where $S \subset U$, there exists $\tilde{\alpha} \in \mathcal{A}(X)$ such that $\tilde{\alpha}|_{U'} = \alpha|_{U'}$ for some $S \subset U' \subseteq U$.*

The definition can be restated as saying the germ of α along S can be extended to an element of $\mathcal{A}(X)$. Here the set of germs $\mathcal{A}(S)$ can be defined like we did for stalks by taking a direct limit of $\mathcal{A}(U)$ as $U \supset S$ shrinks down to S.

Lemma 4.4.2. *Suppose that X is metric space (or more generally a paracompact Hausdorff space). If $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is an exact sequence of sheaves with $\mathcal{A}$ soft, then $\mathcal{B}(X) \to \mathcal{C}(X)$ is surjective.*

We should probably recall that a topological space is paracompact if any open cover has a locally finite subcover. Metric spaces are paracompact [Stone, Paracompactness and product spaces, Bull AMS, 1948]. Finally, we note that the conditions on X are essential, and this limits the utility of softness for the sorts of spaces that come in up in abstract algebraic geometry (schemes). This is why books like Hartshorne don't even discuss this concept.

Sketch. The argument is similar to the flasque case, so we just outline it. Let $\gamma \in \mathcal{C}(X)$. By the assumptions, there exists a locally finite open cover $\{U_i\}_{i\in I}$, such that $\gamma|_{U_i}$ lifts to $\beta_i \in \mathcal{B}(U_i)$. Choose another open cover $\{V_i\}$ such that $S_i = \overline{V}_i \subset U_i$. The differences $(\beta_i - \beta_j)|_{S_i \cap S_j}$ define local sections of $\mathcal{A}$. Softness of $\mathcal{A}$ allows us to choose lifts of these sections to global sections. This allows to correct β_i so they patch. □

Theorem 4.4.3. *On a paracompact Hausdorff space, soft sheaves are acyclic.*

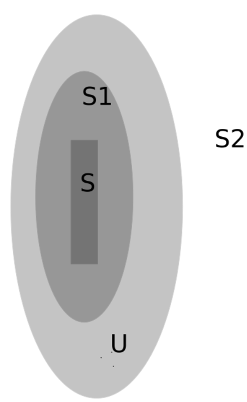
We omit the proof because it is very similar to the corresponding theorem for flasque sheaves.

Now let us consider examples.

Theorem 4.4.4. *The sheaf C_X of continuous real valued functions on a metric space X is soft. The sheaf of C_X^∞ of C^∞ functions on a manifold is soft. More generally, a sheaf of modules $\mathcal{M}$ over C_X or C_X^∞ is soft.*

Proof. The basic strategy for the proof of all these statements is the construction of a continuous or C^∞ "cutoff" function ρ which is 0 outside a given neighbourhood U of a closed set S, and 1 close to S. Suppose we have ρ. Then given a section $f \in \mathcal{M}(U)$ for a module over one of these rings, ρf can be extended by 0 to all of X. Since f and ρf have the same germ along S, this would prove the surjectivity of $\mathcal{M}(X) \to \mathcal{M}(S)$ as required

We spell out the construction of ρ in the continuous case. Let $S_1 \subset U$ be a closed set containing S in its interior. This can be constructed by expressing U as a union of open balls, and taking the union of closed balls of half the radii. Let $S_2 = X - U$. Then Urysohn's lemma from point set topology guarantees the existence of a continuous ρ, taking a value of 1 on S_1 and 0 on S_2.



□

We are ready to do a serious computation, which gives a special case of (a version of) de Rham's theorem.

Example 4.4.5. *From the exact sequence*

$$0 \to \mathbb{R}_{S^1} \to C^\infty_{S^1} \xrightarrow{d} \mathcal{E}^1_{S^1} \to 0$$

we deduce

$$C^\infty(S^1) \to \mathcal{E}^1(S^1) \to H^1(S^1, \mathbb{R}_{S^1}) \to 0$$

and

$$H^i(S^1, \mathbb{R}_{S^1}) = 0, \; i > 1$$

In other words, sheaf cohomology with these coefficients equals de Rham.

$$H^i(S^1, \mathbb{R}_{S^1}) \cong H^i_{DR}(S^1)$$

4.5 de Rham's theorem

We want to generalize the calculation in the last example. But first we need a method for computing all sheaf cohomology in one go.

Definition 4.5.1. *An acyclic resolution of a sheaf $\mathcal{F}$ is an exact sequence*

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \dots$$

of sheaves such that each $\mathcal{F}^i$ is acyclic (e.g. flasque or soft).

Let us write

$$\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$$

This a functor. Given an exact sequence of sheaves $\mathcal{F}^\bullet$, the induced sequence

$$\Gamma(X, \mathcal{F}^0) \to \Gamma(X, \mathcal{F}^1) \to \dots$$

may fail to be exact (as we've seen). But it is necessarily a complex of Abelian groups in the sense that the composition of any two consecutive maps is 0. Define the cohomology of this (or any complex) by

$$\mathcal{H}^i(\Gamma(X, \mathcal{F}^\bullet)) = \frac{\Gamma(X, \mathcal{F}^i) \to \Gamma(X, \mathcal{F}^{i+1})}{\Gamma(X, \mathcal{F}^{i-1}) \to \Gamma(X, \mathcal{F}^i)}$$

This measure the failure of exactness.

Theorem 4.5.2 (Theorem on acyclic resolutions)**.** *Given an acyclic resolution $\mathcal{F}^\bullet$ of $\mathcal{F}$,*

$$H^i(X, \mathcal{F}) \cong \mathcal{H}^i(\Gamma(X, \mathcal{F}^\bullet))$$

Proof. Let $\mathcal{K}^{-1} = \mathcal{F}$ and $\mathcal{K}^i = \ker(\mathcal{F}^{i+1} \to \mathcal{F}^{i+2})$ for $i \geq 0$. Then there are exact sequences

$$0 \to \mathcal{K}^{i-1} \to \mathcal{F}^i \to \mathcal{K}^i \to 0$$

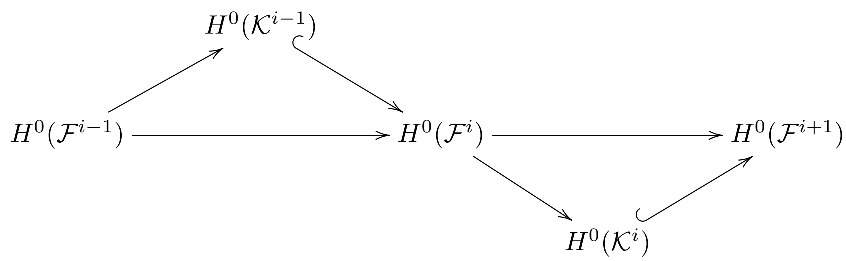
for $i \geq 0$. Since each $\mathcal{F}^i$ is acyclic, Theorem 4.3.1 implies that

$$0 \to H^0(\mathcal{K}^{i-1}) \to H^0(\mathcal{F}^i) \to H^0(\mathcal{K}^i) \to H^1(\mathcal{K}^{i-1}) \to 0 \qquad (4.1)$$

is exact, and

$$H^j(\mathcal{K}^i) \cong H^{j+1}(\mathcal{K}^{i-1}) \tag{4.2}$$

for $j > 0$. We have a diagram



which is commutative since the morphism $\mathcal{F}^{i-1} \to \mathcal{F}^i$ factors through $\mathcal{K}^{i-1}$ and so on. The oblique line in the diagram is part of (4.1), so it is exact. In particular, the first hooked arrow is injective. The injectivity of the second hooked arrow follows for similar reasons. Thus

$$\text{im}[H^0(\mathcal{F}^i) \to H^0(\mathcal{K}^i)] = \text{im}[H^0(\mathcal{F}^i) \to H^0(\mathcal{F}^{i+1})] \tag{4.3}$$

Suppose that $\alpha \in H^0(\mathcal{F}^i)$ maps to 0 in $H^0(\mathcal{F}^{i+1})$, then it maps to 0 in $H^0(\mathcal{K}^i)$. Therefore α lies in the image of $H^0(\mathcal{K}^{i-1})$. Thus

$$H^0(\mathcal{K}^{i-1}) = \ker[H^0(\mathcal{F}^i) \to H^0(\mathcal{F}^{i+1})] \tag{4.4}$$

This already implies the theorem when $i = 0$. Replacing i by $i+1$ in (4.4), and combining it with (4.1) and (4.3) shows that

$$H^1(\mathcal{K}^{i-1}) \cong \frac{H^0(\mathcal{K}^i)}{\text{im}[H^0(\mathcal{F}^i) \to H^0(\mathcal{K}^i)]} = \frac{\ker[H^0(\mathcal{F}^{i+1}) \to H^0(\mathcal{F}^{i+2})]}{\text{im}[H^0(\mathcal{F}^i) \to H^0(\mathcal{F}^{i+1})]}$$

Combining this with the isomorphisms

$$H^{i+1}(\mathcal{F}) = H^{i+1}(\mathcal{K}^{-1}) \cong H^i(\mathcal{K}^0) \cong \ldots H^1(\mathcal{K}^{i-1})$$

of (4.2) proves the theorem for positive exponents. □

Let $\mathbb{R}_X$ denote the of locally constant real valued functions on a space X. But ee often just write it as $\mathbb{R}$ when the context makes it clear.

Theorem 4.5.3 (de Rham's theorem). *If X is a C^∞-manifold,*

$$H^i(X, \mathbb{R}_X) \cong H^i_{DR}(X)$$

Proof. We have a sequence of sheaves

$$\mathbb{R}_X \to \mathcal{E}^0_X \xrightarrow{d} \mathcal{E}^1_X \xrightarrow{d} \ldots$$

where the first map is simply inclusion. We claim that the sequence

$$0 \to \mathbb{R}_X \to \mathcal{E}^0_X \xrightarrow{d} \mathcal{E}^1_X \xrightarrow{d} \dots \tag{4.5}$$

is exact. Exactness at the first step is clear, since the map is inclusion. For the rest of the sequence, we check exactness at stalks. Then we can replace X by a ball. Since this is diffeomorphic to $\mathbb{R}^n$, exactness follows from the the Poincaré lemma. Since $\mathcal{E}^i_X$ are soft, tells us that (4.5) is an acyclic resolution. Therefore this theorem follows from the previous theorem. □

One might complain that de Rham's theorem is supposed to say that de Rham cohomology is the same as *singular* cohomology with real coefficients. It is easy to deduce this too by showing singular cohomology equals sheaf cohomology. This can be proved by another acyclic resolution. Finally, we note that we can define complex valued de Rham cohomology $H^*_{dR}(X, \mathbb{C})$ by using complex valued forms. The argument as above shows

$$H^i(X, \mathbb{C}_X) \cong H^i_{dR}(X, \mathbb{C})$$

4.6 Dolbeault's theorem

In this section, X will be a complex manifold and $\mathcal{E}^\bullet(X)$ will stand for the space of complex valued forms. Recall that Dolbeault cohomology is the cohomology of the complex

$$\mathcal{E}^{p0}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p1}(X) \xrightarrow{\bar\partial} \dots$$

The key fact that we need is an analogue of Poincaré's lemma for the Cauchy-Riemann operator.

Theorem 4.6.1. *If Δ is a polydisk, then given $q > 0$ and $\alpha \in \mathcal{E}^{pq}(\bar\Delta)$ satisfying $\bar\partial\alpha = 0$, there exists $\beta \in \mathcal{E}^{p,q-1}(\Delta)$ such that $\alpha = \bar\partial\beta$.*

A proof can be found on p 25 of Griffiths-Harris for example. We just indicate how it goes in one variable. Let $\Delta \subset \mathbb{C}$ be an open disk. Given $f \in C^\infty(\bar\Delta)$, we need to find a function $g \in C^\infty(\Delta)$ such that $\frac{\partial g}{\partial \bar z} = f$. A version of Cauchy's formula shows that

$$g(\zeta) = \frac{1}{2\pi i} \iint_\Delta \frac{f(z)}{z - \zeta}\, dz \wedge d\bar z$$

gives the desired solution.

Theorem 4.6.2 (Dolbeault's theorem)**.**

$$H^q(X, \Omega^p_X) \cong H^{pq}_{Dol}(X)$$

Proof. Let $\mathcal{E}^{pq}_X$ denote the sheaf of C^∞ (p, q) forms. This is soft. We have sequence of sheaves

$$0 \to \Omega^p_X \to \mathcal{E}^{p0}_X \xrightarrow{\bar\partial} \mathcal{E}^{p1}_X(X) \xrightarrow{\bar\partial} \dots$$

where the first morphism is inclusion. We claim this sequence is exact. We check this at stalks. For this, we can replace X by a polydisk and apply the previous theorem. Therefore we have an acyclic resolution, and the theorem follows. □

4.7 Poincaré duality

Let X be a C^∞ manifold. Let $\mathcal{E}_c^k(X)$ denote the set of C^∞ k-forms with compact support. Since $d\mathcal{E}_c^k(X) \subset \mathcal{E}_c^{k+1}(X)$, these form a complex.

Definition 4.7.1. *Compactly supported de Rham cohomology is defined by* $H^k_{cdR}(X) = \mathcal{H}^k(\mathcal{E}_c^\bullet(X))$.

Lemma 4.7.2. *For all* n*,*

$$H^k_{cdR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} \text{ if } k = n \\ 0 \text{ otherwise} \end{cases}$$

Proof. See Spivak. □

This computation suggests that these groups are roughly opposite to the usual de Rham groups. There is another piece of evidence, which is that H_{cdR} behaves covariantly in certain cases. For example, given an open set $U \subset X$, a form in $\mathcal{E}_c^k(U)$ can be extended by zero to $\mathcal{E}_c^k(X)$. This induces a map $H^k_{cdR}(U) \to H^k_{cdR}(X)$.

The precise statement of duality requires the notion of orientation. An orientation on an n dimensional real vector space V is a connected component of $\wedge^n V - \{0\}$ (there are two). An ordered basis $v_1, \ldots v_n$ is positively oriented if $v_1 \wedge \ldots \wedge v_n$ lies in the given component. If V were to vary, there is no guarantee that we could choose a orientations consistently. So we make a definition:

Definition 4.7.3. *An* n *dimensional manifold* X *is called orientable if* $\wedge^n T_X$ *minus its zero section has two components. If this is the case, an orientation is a choice of one of these components.*

This is equivalent to the definition we gave earlier, but more convenient. The following test is immediate.

Lemma 4.7.4. *An n-manifold is orientable if it has a nonwhere zero* C^∞ *n-form.*

Theorem 4.7.5 (Poincaré duality, version I)**.** *Let* X *be a connected oriented n-dimensional manifold. Then*

$$H^k_{cdR}(X) \cong H^{n-k}(X, \mathbb{R})^*$$

There is a standard proof of this using currents, which are to forms what distributions are to functions. However, we can get by with something much weaker. We define the space of *pseudocurrents* of degree k on an open set $U \subset X$ to be

$$\mathcal{C}^k(U) = \mathcal{E}_c^{n-k}(U)^* :\, = Hom(\mathcal{E}_c^{n-k}(U), \mathbb{R}).$$

This is "pseudo" because we are using the ordinary (as opposed to topological) dual. We make this into a presheaf as follows. Given $V \subseteq U$, $\alpha \in \mathcal{C}_X^k(U)$, $\beta \in \mathcal{E}_c^{n-k}(V)$, define $\alpha|_V(\beta) = \alpha(\tilde{\beta})$ where $\tilde{\beta}$ is the extension of β by 0.

Lemma 4.7.6. *$\mathcal{C}_X^k$ is a sheaf.*

Proof. Let $\{U_i\}$ be an open cover of U, which we may assume is locally finite. Suppose that $\alpha_i \in \mathcal{C}_X^k(U_i)$ is a collection of sections such that $\alpha_i|_{U_i \cap U_j} = \alpha_j|_{U_i \cap U_j}$. This means that $\alpha_i(\beta) = \alpha_j(\beta)$ if β has support in $U_i \cap U_j$. Let $\{\rho_i\}$ be a C^∞ partition of unity subordinate to $\{U_i\}$. Then define $\alpha \in \mathcal{C}_X^k(U)$ by

$$\alpha(\beta) = \sum_i \alpha_i(\rho_i \beta|_{U_i})$$

We have to show that $\alpha(\tilde{\beta}) = \alpha_j(\beta)$ for any $\beta \in \mathcal{E}_c^{n-k}(U_j)$ with $\tilde{\beta}$ its extension to U by 0. The support of $\rho_i \tilde{\beta}$ lies in $U_i \cap supp(\beta) \subset U_i \cap U_j$, so only finitely many of these are nonzero. Therefore

$$\alpha(\tilde{\beta}) = \sum_i \alpha_i(\rho_i \tilde{\beta}) = \sum_i \alpha_j(\rho_i \tilde{\beta}) = \alpha_j(\beta)$$

as required. We leave it to the reader to check that α is the unique current with this property. □

Define a map $\delta : \mathcal{C}_X^k(U) \to \mathcal{C}_X^{k+1}(U)$ by $\delta(\alpha)(\beta) = (-1)^{k+1}\alpha(d\beta)$. One automatically has $\delta^2 = 0$. Thus one has a complex of sheaves.

Let X be an oriented n-dimensional manifold. Then we will recall [Spivak,...] that one can define an integral $\int_X \alpha$ for any n-form $\alpha \in \mathcal{E}_c^n(X)$. Using a partition of unity, the definition can reduced to the case where α is supported in a coordinate neighbourhood U. Then we can write $\alpha = f(x_1, \ldots x_n) dx_1 \wedge \ldots \wedge dx_n$, where the order of the coordinates are chosen so that $\partial/\partial x_1, \ldots, \partial/\partial x_n$ gives a positive orientation of T_X. Then

$$\int_X \alpha = \int_{\mathbb{R}^n} f(x_1, \ldots x_n) dx_1 \ldots dx_n$$

The functional $\int_X$ defines a canonical global section of $\mathcal{C}_X^0$.

Theorem 4.7.7 (Stokes' theorem)**.** *Let X be an oriented n-dimensional manifold, then $\int_X d\beta = 0$.*

Proof. See Spivak or almost any book on manifolds. □

Corollary 4.7.8. $\int_X \in ker[\delta]$.

We define a map $\mathbb{R}_X \to \mathcal{C}_X^0$ by sending $r \to r\int_X$. The key lemma to establish Theorem 4.7.5 is:

Lemma 4.7.9.
$$0 \to \mathbb{R}_X \to \mathcal{C}_X^0 \to \mathcal{C}_X^1 \to \dots$$
is an acyclic resolution.

Proof. Lemma 4.7.2 implies that this complex is exact. Given $f \in C^\infty(U)$ and $\alpha \in \mathcal{C}^k(U)$ define
$$f\alpha(\beta) = \alpha(f\beta)$$
This makes $\mathcal{C}^k$ into a C^∞-module, and it follows that it is soft and therefore acyclic. □

Proof of theorem 4.7.5. We can now use the complex $\mathcal{C}_X^\bullet$ to compute the cohomology of $\mathbb{R}_X$ to obtain
$$H^i(X,\mathbb{R}) \cong \mathcal{H}^i(\mathcal{C}_X^\bullet(X)) = \mathcal{H}^i(\mathcal{E}_c^{n-\bullet}(X)^*).$$
The right hand space is isomorphic to $H_{cdR}^i(X,\mathbb{R})^*$. This completes the proof of the theorem. □

Corollary 4.7.10. *If X is a compact oriented n-dimensional manifold. Then*
$$H^k(X,\mathbb{R}) \cong H^{n-k}(X,\mathbb{R})^*$$

The following is really a corollary of the proof.

Corollary 4.7.11. *If X is a connected oriented n-dimensional manifold. Then the map $\alpha \mapsto \int_X \alpha$ induces an isomorphism*
$$\int_X : H_{cdR}^n(X,\mathbb{R}) \cong \mathbb{R}$$

We can make the Poincaré duality isomorphism more explicit:

Theorem 4.7.12 (Poincaré duality, version II)**.** *If $f \in H_{cdR}^{n-k}(X)^*$, then there exists a closed form $\alpha \in \mathcal{E}^k(X)$ such that $f([\beta]) = \int_X \alpha\wedge\beta$. Moreover the class $[\alpha] \in H_{dR}^k(X)$ is unique.*

If $\alpha \in \mathcal{E}^i(X)$ and $\beta \in \mathcal{E}_c^j(X)$ are closed forms, then $\alpha\wedge\beta$ is also closed by the Leibnitz rule. The *cup product* of the associated cohomology classes is defined by $[\alpha]\cup[\beta] = [\alpha\wedge\beta]$. This is a well defined operation which makes de Rham cohomology into a graded ring when X is compact. The theorem tells us that:

Corollary 4.7.13. *The cup product followed by integration gives a nondegenerate pairing*
$$H_{dR}^k(X) \times H_{cdR}^{n-k}(X) \to H_{cdR}^n(X) \cong \mathbb{R}$$

Here is a simple example to illustrate of this.

Example 4.7.14. *Consider the torus $T = \mathbb{R}^n/\mathbb{Z}^n$. We will show later that: Every de Rham cohomology class on T contains a unique form with constant coefficients. This will imply that there is an algebra isomorphism $H^*(T, \mathbb{R}) \cong \wedge^*\mathbb{R}^n$. Poincaré duality becomes the standard isomorphism*

$$\wedge^k\mathbb{R}^n \cong \wedge^{n-k}\mathbb{R}^n.$$

4.8 Cech interpretation for H^1

There is another approach to sheaf cohomology which is quite explicit, and this makes it useful for many computations. Let us start with a sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

on X. Given a section $\gamma \in \mathcal{C}(X)$, let us try to directly understand when it lifts to a section $\beta \in \mathcal{B}(X)$. We can find an open cover $\mathcal{U} = \{U_i\}$ and sections $\beta_i \in \mathcal{B}(U_i)$ which map to $\gamma|_{U_i}$. Let $U_{ij} = U_i \cap U_j$, then

$$\alpha_{ij} = \beta_i|_{U_{ij}} - \beta_j|_{U_{ij}}$$

can viewed as a collection of sections of $\mathcal{A}(U_{ij})$. These satisfy the 1-cocycle identies

$$\alpha_{ii} = 0$$

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 0$$

We say that it is 1-coboundary if we can find a collection $\alpha_i \in \mathcal{A}(U_i)$ such that

$$\alpha_{ij} = \alpha_i|_{U_{ij}} - \alpha_j|_{U_{ij}}$$

We now come to be key observation:

Lemma 4.8.1. *If α_{ij} is a coboundary, then the sections $\beta_i - \alpha_i$ will patch to form a global section of $\mathcal{B}$ lifting γ.*

We can put of all this together Let $Z^1(\mathcal{U}, \mathcal{A})$ denote the set of 1-cocyles. It is naturally an abelian group. Let $B^1(\mathcal{U}, \mathcal{A})$ denote subgroup of 1-coboundaries.

Definition 4.8.2. *The first Cech cohomology groups*

$$\check{H}^1(\mathcal{U}, \mathcal{A}) = Z^1(\mathcal{U}, \mathcal{A})/B^1(\mathcal{U}, \mathcal{A})$$

$$\check{H}^1(X, \mathcal{A}) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{A})$$

where covers $\mathcal{U}$ are ordered by refinement in the direct limit.

Lemma 4.8.3. *Sending γ above to the class of $\{\alpha_{ij}\}$ yields a map*

$$H^0(X, \mathcal{C}) \xrightarrow{\partial} \check{H}^1(X, \mathcal{A})$$

such that $\partial(\gamma) = 0$ iff γ lifts to $H^0(X, \mathcal{B})$.

Corollary 4.8.4. *There is an exact sequence*

$$0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to H^0(X, \mathcal{C}) \to \check{H}^1(X, \mathcal{A})$$

This says the first Cech group is doing the same job as the first sheaf cohomology. In fact:

Theorem 4.8.5. *There is an isomorphism $H^1(X, \mathcal{A}) \cong \check{H}^1(X, \mathcal{A})$ compatible with the connecting maps ∂.*

There are also higher Cech cohomology groups, where cocycles for a fixed cover are collections $\alpha_{i_0,\ldots,i_n} \in \mathcal{A}(U_{i_0} \cap \ldots U_{i_n})$ satisfying appropriate conditions. But the story gets more complicated. Cech groups agree with sheaf cohomology groups under mild assumptions, but need not agree in all cases.

Chapter 5

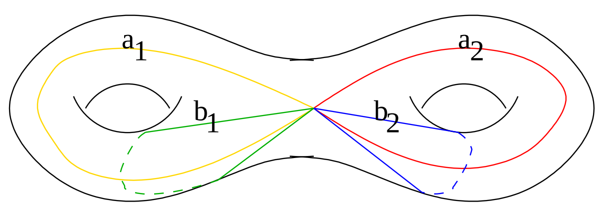
Riemann surfaces

5.1 Topology

Let X be a connected compact Riemann surface. We showed long ago that X is orientable. As a topological space, X is completely understood. The following fact is classical.

Theorem 5.1.1. *A compact orientable* 2*-manifold is homeomorphic to either* S^2 *or a connected sum of a finite number of* 2*-tori.*

See for example [Donaldson, Riemann surfaces] for an explanation of how to prove it. We define the genus of X to be 0 if it is S^2, or g if it is a connected sum of g tori. In more informal terms, X is a g-holed donut.



The first homology group $H_1(X, \mathbb{Z})$ of a space X is defined precisely in any basic book in algebraic topology, such as Hatcher. Very roughly, it is the free abelian group generated by closed paths $\gamma : [0, 1] \to X$ modulo the boundaries of embedded "surfaces" in X. See Hatcher for the key computation:

Theorem 5.1.2. *If* X *is a genus* g *Riemann surface, then*

$$H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}$$

See above picture for a basis of H_1 in the genus 2 case.

De Rham's theorem takes an explicit form. We write $H^i_{dR}(X, \mathbb{R})$ (resp. $H^i_{DR}(X, \mathbb{C})$) for de Rham cohomology using real (resp. complex) valued forms.

Theorem 5.1.3. *If X is a manifold, then*

$$H^1_{dR}(X,\mathbb{R}) \cong Hom(H_1(X,\mathbb{Z}),\mathbb{R})$$

$$H^1_{dR}(X,\mathbb{C}) \cong Hom(H_1(X,\mathbb{Z}),\mathbb{C})$$

where the isomorphisms send a closed form α to

$$\gamma \mapsto \int_\gamma \alpha$$

Corollary 5.1.4. *If X is a genus g Riemann surface,*

$$H^1_{dR}(X,\mathbb{C}) \cong \mathbb{C}^{2g}$$

Finally, we note that $H^1(X,\mathbb{C})$ has a bilinear form given by

$$(\alpha,\beta) = \int_X \alpha \wedge \beta$$

This is skew symmetric. Poincaré duality tells that this is nondegenerate, i.e. any matrix representing it is nonsingular. Under the de Rham isomorphism above, the above pairing is compatible with intersection pairing on homology.

5.2 Sheaf cohomology

Let X be a compact Riemann surface of genus g. Then

$$H^i(X,\mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } i = 0,2 \\ \mathbb{C}^{2g} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

The case $i = 0$ is elementary, $i = 1$ was explained earlier, and the remaining cases follow (for example) from Poincaré duality. In this case, Dolbeault's theorem amounts to the following statements

$$H^1(X,\mathcal{O}_X) = \frac{\mathcal{E}^{(0,1)}(X)}{\bar{\partial}C^\infty(X)}$$

$$H^1(X,\Omega^1_X) = \frac{\mathcal{E}^{(1,1)}(X)}{\bar{\partial}\mathcal{E}^{(1,0)}_X(X)}$$

and

$$H^i(X,\mathcal{O}_X) = H^i(X,\Omega^1_X) = 0$$

if $i > 1$.

Next, we give a holomorphic analogue of the de Rham complex.

Proposition 5.2.1. *There is an exact sequence of sheaves*

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \to 0$$

Proof. The only nontrivial part of the assertion is that $\mathcal{O}_X \to \Omega^1_X$ is an epimorphism. We can check this by replacing X by a disk D. A holomorphic 1-form α on D is automatically closed, therefore $\alpha = df$ by the usual Poincaré lemma. Since df is holomorphic, $\bar{\partial} f = 0$. Therefore f is holomorphic. □

Corollary 5.2.2. *There is a long exact sequence*

$$0 \to H^0(X, \mathbb{C}) \to H^0(X, \mathcal{O}_X) \to H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C}) \ldots$$

Holomorphic 1-forms are closed, and

$$H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C})$$

is the map which sends a holomorphic form to its class in (complex valued) de Rham cohomology.

Lemma 5.2.3. *This map is an injection.*

Proof. Since global holomorphic functions on X are constant

$$H^0(X, \mathbb{C}) \to H^0(X, \mathcal{O}_X)$$

is surjective. □

It follows that $\dim H^0(X, \Omega^1) \leq 2g$. In fact, we can we will show later that

Theorem 5.2.4. *If X is a compact Riemann surface of genus g, then*

$$\dim H^0(X, \Omega) = H^1(X, \mathcal{O}_X) = g$$

So this will give another interpretation of genus. For now we prove a weaker statement.

Lemma 5.2.5. $\dim H^0(X, \Omega^1) \leq g$

Proof. If $\alpha, \beta \in H^0(X, \Omega^1)$, then $\alpha \wedge \beta = 0$ because it would be a $(2, 0)$ form on a 1 dimensional complex manifold. Therefore $(\alpha, \beta) = 0$. This says that $H^0(X, \Omega^1)$ is an *isotropic* subspace. The bound follows from the following fact from linear algebra:

Theorem 5.2.6. *A finite dimensional vector space V with a nondegenerate skew symmetric form is even dimensional. An isotropic subspace has at most half the dimension of V.*

□

5.3 Harmonic forms

We will prove theorem 5.2.4. We start with a seemingly unrelated problem. Recall that de Rham cohomology

$$H^1_{dR}(X, \mathbb{C}) = \frac{\{\alpha \in \mathcal{E}^1(X) \mid d\alpha = 0\}}{\{df \mid f \in C^\infty(X)\}}$$

where $\mathcal{E}^1 = \mathcal{E}^1_{\mathbb{C}}$. So an element of it is really an equivalence class. *Does such a class have a distinguished representative?* The answer will turn out to be yes. To describe it, let us introduce a $\mathbb{C}$-linear operation called the Hodge star given locally by $*dx = dy$, $*dy = -dx$. This is amounts to multiplication by i in the cotangent plane, so it is globally well defined operation. It follows that

Lemma 5.3.1.

$$\langle \alpha, \beta \rangle = (\alpha, *\bar{\beta}) = \int_X \alpha \wedge *\bar{\beta}$$

defines an inner product on $\mathcal{E}^1(X)$.

Proof. One can see that

$$(fdx + gdy) \wedge *\overline{(fdx + gdy)} = (|f|^2 + |g|^2) dx \wedge dy$$

This implies positive definiteness. The other properties are routine. □

Definition 5.3.2. *We define a* 1*-form* α *to be co-closed if* $d(*\alpha) = 0$. *It is harmonic if it is both closed and co-closed, i.e.* $d\alpha = d(*\alpha) = 0$. *A form is called co-exact if it equals* $*df$.

We will explain why these are called harmonic later. The basic properties are given by:

Proposition 5.3.3.

(a) A harmonic 1*-form is a sum of a* $(1,0)$ *harmonic form and* $(0,1)$ *harmonic form.*

(b) A $(1,0)$*-form is holomorphic iff it is closed iff is harmonic.*

(c) A $(0,1)$*-form is harmonic iff it is antiholomorphic i.e. its complex conjugate is holomorphic.*

(d) The space of co-closed (closed) forms is orthogonal to the space of exact (co-exact) forms. In particular, the space of harmonic forms is orthogonal to both spaces.

Proof. If α is a harmonic 1-form, then $\alpha = \alpha' + \alpha''$, where $\alpha' = \frac{1}{2}(\alpha + i * \alpha)$ is a harmonic $(1,0)$-form and $\alpha'' = \frac{1}{2}(\alpha - i * \alpha)$ is a harmonic $(0,1)$-form.

If α is $(1,0)$, then $d\alpha = \bar{\partial}\alpha$. This implies the first half (b). For the second half, use the identity

$$*dz = *(dx + idy) = dy - idx = -idz$$

Finally, note that the harmoncity condition is invariant under conjugation, so the (c) follows from (b).

For (d), suppose α is co-closed. Then so is $\bar{\alpha}$. Integration by parts (essentially Stokes' theorem) implies

$$\langle df, \alpha \rangle = \int_X df \wedge *\bar{\alpha} = \int d(f * \bar{\alpha}) - \int_X f d * \bar{\alpha} = 0$$

Similarly, for α closed,

$$\langle \alpha, *df \rangle = \int_X \alpha \wedge * * d\bar{f} = -\int_X d(\bar{f}\alpha) + \int_X \bar{f} d\alpha = 0$$

□

Here is the key fact. We will say more about this in later on.

Theorem 5.3.4 (Hodge theorem). *Any form in $\mathcal{E}^1(X)$ can be decomposed into a sum $\beta + df + *dg$, where β is harmonic and f, g are C^∞ functions.*

Corollary 5.3.5. *Every de Rham cohomology class has a unique harmonic representative.*

Proof. Suppose α is closed. Write $\alpha = \beta + df + *dg$ as above. Part d of the last proposition implies

$$|| * dg||^2 = \langle \alpha, *dg \rangle = 0$$

So α and the harmonic form β lie in the same cohomology class. Suppose $\alpha = \beta' + dg$ with β' harmonic. Then $\beta - \beta'$ is harmonic and exact. Applying part d again shows that it's zero.

□

Proposition 5.3.6. *$H^1(X, \mathcal{O}_X)$ is isomorphic to the space of antiholomorphic forms.*

Proof. Let $H \subset \mathcal{E}^{01}(X)$ denote the space of antiholomorphic forms. We will show that

$$\pi : H \to \mathcal{E}^{01}(X)/\operatorname{im}\bar{\partial}$$

is an isomorphism. Suppose that $\alpha \in \mathcal{E}^{01}(X)$. By theorem 5.3.4, we may choose a harmonic form β such that $\beta = \alpha + df + *dg$ for some $f, g \in C^\infty(X)$. Then the $(0,1)$ part of β gives an element $\beta' \in H$ such that $\beta' = \alpha + \bar{\partial}(f + ig)$. This shows that π is surjective.

Suppose that $\alpha \in \ker \pi$. Then $\alpha = \bar{\partial} f$ for some f. Therefore $\alpha + \bar{\alpha} = df$. Consequently $\alpha + \bar{\alpha}$ is exact and harmonic, so $\alpha + \bar{\alpha} = 0$. This implies $\alpha = 0$. □

Corollary 5.3.7. *$\alpha \mapsto \bar{\alpha}$ gives a conjugate linear isomorphism*

$$H^0(X, \Omega^1_X) \cong H^1(X, \mathcal{O}_X)$$

The last result is a special case of Serre duality.

Proof of theorem 5.2.4. Combining previous results shows that

$$\dim H^0(X, \Omega^1_X) = \dim H^1(X, \mathcal{O}_X) \leq g$$

We have an exact sequence

$$H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X)$$

which forces

$$2g \leq 2 \dim H^0(X, \Omega^1_X)$$

□

5.4 Riemann-Roch

Let X be Riemann surface of genus g.. A classical problem, sometimes called the Riemann-Roch problem, is construct a meromorphic function with prescribed zeros and poles.This data is a choice of finitely many points $p_1, \ldots, p_k$ with multiplicities $n_1, \ldots, n_k$, which we write as a formal sum $D = \sum n_i p_i$. D is called a *divisor*. The degree $\deg D = \sum n_i$. Write $M(U)$ for the field of meromorphic functions on $U \subseteq X$, and define the sheaf

$$\mathcal{O}_X(D)(U) = \{f \in M(U) \mid ord_{p_i} f \geq -n_i, \forall p_i \in U\}$$

A more precise form of the Riemann-Roch problem is to try calculate the dimension of the global sections of the above sheaf. The key result is as follows:

Theorem 5.4.1 (Riemann-Roch)**.** *The dimensions $h^i(\mathcal{O}_X(D)) = \dim H^i(X, \mathcal{O}_X(D))$ are finite for $i = 0, 1$ and zero for $i \geq 2$. We have*

$$\chi(\mathcal{O}_X(D)) := h^0(\mathcal{O}_X(D)) - h^1(\mathcal{O}_X(D)) = \deg D + 1 - g$$

Proof. Let $m(D) = \sum |n_i|$ denote the "mass" of D. The proof proceeds by induction on $m(D)$. The base case $m(D) = 0$ follows from the computations

$$h^0(\mathcal{O}_X) = 1, \quad h^1(\mathcal{O}_X) = g, \quad h^i(\mathcal{O}_X) = 0, i > 1$$

established earlier.

Now assume $m(D) > 0$ and that the theorem is true for $m(D') < m(D)$. Given p, we have an exact sequence

$$0 \to \mathcal{O}_X(-p) \to \mathcal{O}_X \to \mathbb{C}_p \to 0$$

Tensoring by $\mathcal{O}_X(D)$ gives

$$0 \to \mathcal{O}_X(D - p) \to \mathcal{O}_X(D) \to \mathbb{C}_p \to 0$$

because $\mathcal{O}(D) \otimes \mathbb{C}_p \cong \mathbb{C}_p$. Observe that $H^0(X, \mathbb{C}_p) = \mathbb{C}$ by definition and $H^i(X, \mathbb{C}_p) = 0$ for $i > 0$ because $\mathbb{C}_p$ is flasque. Thus

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D - p)) + \chi(\mathbb{C}_p) = \chi(\mathcal{O}_X(D - p)) + 1$$

Therefore

$$\chi(\mathcal{O}(D)) - \deg D = \chi(\mathcal{O}(D-p)) - \deg(D-p)$$

or by changing variable and writing the formula backwards

$$\chi(\mathcal{O}(D)) - \deg D = \chi(\mathcal{O}(D+p)) - \deg(D+p)$$

We can choose p such that $m(D \pm p) < m(D)$. So one of these two formulas shows that $\chi(\mathcal{O}_X(D)) - \deg D$ equals $1-g$.

□

Corollary 5.4.2 (Riemann's inequality)**.**

$$h^0(\mathcal{O}_X(D)) \geq \deg D + 1 - g$$

In particular, this contains a nonzero function when $\deg D \geq g$.

To appreciate what this tells, let us classify surfaces of genus 0. We have at least one example, namely $\mathbb{P}^1_{\mathbb{C}}$ (= the Riemann sphere).

Theorem 5.4.3. *A compact Riemann surface of genus* 0 *is isomorphic to* $\mathbb{P}^1_{\mathbb{C}}$.

Sketch. Let X have genus 0. Choose a point $p \in X$. Riemann's inequality tells us there exists a meromorphic function f with a simple pole at p and no other singularities. This can be viewed as holomorphic map

$$f : X \to \mathbb{P}^1$$

such that $f^{-1}(\infty) = p$. Given $y \in \mathbb{C}$, let $g(x) = f(x) - y$. The preimage $f^{-1}(y) = g^{-1}(0)$ is a finite set $\{q_1, \ldots\}$. Let $n_1, \ldots$ be the order of the zeros of g these points. We claim that $\sum n_k = 1$. To see this, consider the meromorphic differential form $\alpha = dg/g$. This has poles at $\{p, q_1, \ldots\}$. We can choose a coordinate at q_k, so that g is locally $z \mapsto z^{n_k}$. It follows that $\alpha = n_k dz/z$. This means that the residue of α at q_k is n_k. Similarly, the residue at p is -1. To prove the claim, observe that by Stokes, the sum of residues

$$-1 + \sum n_k = \frac{1}{2\pi i} \iint_{X - \cup D_k} d\alpha = 0$$

With the claim in hand, we can see that f is a bijection. Using open mapping theorem from complex analysis shows f^{-1} is also holomorphic. Therefore $X \cong \mathbb{P}^1$. □

5.5 The Jacobian

Let X be Riemann surface of genus g. Let C^∞_X denote sheaf of complex valued C^∞ functions, and $\mathcal{O}_X$ is the subsheaf of holomorphic functions. For our purposes a *holomorphic line bundle* is a rank one locally free sheaf L over $\mathcal{O}_X$. A C^∞ (complex) line bundle is defined analogously. A holomorphic line bundle gives rise to C^∞ line bundle by "extending scalars". In particular, this remark applies to $\mathcal{O}_X(D)$.

Lemma 5.5.1. *$\mathcal{O}_X(D)$ is a holomorphic line bundle.*

Proof. Let $\{U_i\}$ be a covering by coordinate nbhds containing at most one point of D. If U_i contains no points of D, then $\mathcal{O}(D)|_{U_i} = \mathcal{O}_{U_i}$. If U_i contains $p_j \in D$, choose a coordinate so that p_j is $z = 0$. Then

$$\mathcal{O}(D)|_{U_i} = \mathcal{O}_{U_i} z^{-n_j}$$

□

Let L be a C^∞ or holomorphic line bundle. In either case, we have an open cover $\{U_i\}$ and isomorphisms

$$\sigma_i : C^\infty_{U_i} \cong L|_{U_i}, \quad \text{or } \mathcal{O}_{U_i} \cong L|_{U_i}$$

Note that σ_i need not be compatible with σ_j. We can measure the difference by taking

$$\phi_{ij} = \sigma_j^{-1} \circ \sigma_i : \mathcal{O}(U_{ij}) \cong \mathcal{O}(U_{ij})$$

in either case, but we just wrote the second case. An automorphism of a commutative ring is just multiplication by a unit. So we can view

$$\phi_{ij} \in \mathcal{O}^*(U_{ij})$$

We can see from the definition that ϕ_{ij} is a 1-cocycle in $Z^1(\{U_i\}, \mathcal{O}^*_X)$.

Theorem 5.5.2. *The map $L \mapsto \phi_{ij}$ induces a bijection between the set of isomorphism classes of holomorphic (resp. C^∞) line bundles and $H^1(X, \mathcal{O}^*_X)$ (resp. $H^1(X, {C^\infty}^*_X)$).*

Both sides are abelian groups, where tensor product is an operation on the left. The above statement can be improved to an isomorphism of groups. The group of holomorphic line bundles is called the Picard group, and denoted by $Pic(X)$.

Proposition 5.5.3. *Let $e(f) = e^{2\pi i f}$, then the sequences of sheaves*

$$0 \to \mathbb{Z}_X \to C^\infty_X \xrightarrow{e} {C^\infty}^*_X \to 1$$

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \xrightarrow{e} \mathcal{O}^*_X \to 1$$

are exact.

Proof. Exactness is local, so we can reduce to the disk D. Since D is simply connected, a branch of the logarithm can be chosen to get surjectivity $e : \mathcal{O}(D) \to \mathcal{O}^*(D)$. The rest is straigtforward. □

Taking the first sequence gives

$$H^1(X, C^\infty_X) \to H^1(X, {C^\infty}^*_X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^1(X, C^\infty_X)$$

The map labelled c_1 is called the first Chern class. Since C^∞_X is soft, we obtain

Lemma 5.5.4. *The first Chern class induces an isomorphism*

$$H^1(X, {C^\infty}^*_X) \cong H^2(X, \mathbb{Z})$$

This says that c_1 is a complete invariant for C^∞ line bundles. Note that $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$, so c_1 is really just a number, called the first Chern number. We omit the details, but this can be computed to obtain

Theorem 5.5.5. *The first Chern number $c_1(\mathcal{O}(D)) = \deg D$.*

Corollary 5.5.6. *The C^∞ line bundles associated to two divisors are isomorphic iff they have the same degree,.*

The holomorphic side is much more interesting. We now get a sequence

$$\ldots \to H^1(X, \mathcal{O}_X) \to Pic(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to 0$$

because $H^2(X, \mathcal{O}_X) = 0$. The kernel of c_1 is denoted by $Pic^0(X)$.

Theorem 5.5.7. *$Pic^0(X)$ has the structure of a complex torus of dimension g.*

Proof. From the above exact sequence

$$Pic^0(X) \cong \frac{H^1(X, \mathcal{O}_X)}{\operatorname{im} H^1(X, \mathbb{Z})}$$

We saw earlier that the numerator on the right is a g dimensional complex vector space. It is enough to prove the claim that denominator is a lattice (a discrete subgroup of maximal rank).

Since $H^1(X, \mathbb{Z})$ sits a lattice inside $H^1(X, \mathbb{R})$, to prove the claim it is enough to show that the natural map

$$r : H^1(X, \mathbb{R}) \to H^1(X, \mathcal{O}_X)$$

is injective. It α is a harmonic form representing a nonzero element of $H^1(X, \mathbb{R})$. Then we can uniquely decompose $\alpha = \alpha^{0,1} + \alpha^{1,0}$ into a sum of a holomorphic and antiholomorphic forms. Note that $r(\alpha) = \alpha^{0,1}$. Since α is real, $\alpha^{0,1} = \overline{\alpha^{1,0}}$. Therefore $r(\alpha) \neq 0$.

□

The torus $Pic^0(X)$ is called the *Jacobian* of X, and also denoted by $J(X)$. This is a fundamental invariant of X. It has the effect of "linearizing" X. What makes story more interesting is that X also maps to $J(X)$. Fix a base point $p_0 \in X$ and a positive integer n. The Abel-Jacobi map $AJ : X^n \to J(X)$ sends

$$(p_1, \ldots, p_n) \mapsto \mathcal{O}(p_1 + \ldots + p_n - np_0)$$

We summarize the key result without proof.

Theorem 5.5.8 (Abel-Jacobi)**.** *The Abel-Jacobi map is holomorphic, and surjective when $n \geq g$, and injective when $n = 1$ and $g \geq 1$.*

Corollary 5.5.9. *When $n = g = 1$, AJ is a holomorphic isomorphism. It follows that a genus 1 Riemann surface is isomorphic to a quotient of $\mathbb{C}$ by a lattice.*

Chapter 6

The Hodge theorem

6.1 Riemannian metrics

Let X be C^∞-manifold. A Riemannian metric on X is a C^∞ family of inner products on the tangent spaces. Here is a more precise definition.

Definition 6.1.1. *A Riemannian metric is family of inner products $g : T_p \otimes T_p \to \mathbb{R}$ such that given two C^∞ vector fields $u, v \in Vect(U)$, $g(u,v) \in C^\infty(U)$.*

Proposition 6.1.2. *Every manifold possesses a Riemannian metric.*

This is standard application of partitions of unity. A proof can be found in any book on differential geometry. The object g is called the metric tensor. In local coordinates

$$g = \sum g_{ij} dx_i \otimes dx_j$$

where $p \mapsto g_{ij}$ is a C^∞ family of symmetric positive definite matrices. By linear algebra, g induces inner products on $\wedge^k T_p^*$ for each k. This gives us a pointwise inner product

$$(\ ,\ ) : \mathcal{E}^k(X) \times \mathcal{E}^k(X) \to C^\infty(X)$$

(NB: This notation is different from the last chapter.)

Let us now assume that X is an oriented n-manifold. Then we define the volume form locally by

$$dvol = \sqrt{\det(g_{ij})} dx_1 \wedge \ldots \wedge dx_n$$

where we order the coordinates so that the expression is positively oriented. This is globally well defined, and it is not exact in spite of the notation. This form defines a measure on X by $\int_X f dvol$. *Let us also now assume that X is compact.* Then this is a finite measure i.e. $\int_X dvol < \infty$. We can define an actual inner product on $\mathcal{E}^k(X)$ by

$$\langle \alpha, \beta \rangle = \int_X (\alpha, \beta) dvol$$

The Hodge star operator $*\mathcal{E}^k(X) \to \mathcal{E}^{n-k}(X)$ is the unique linear transformation satisfying

$$\alpha \wedge *\beta = (\alpha, \beta) dvol$$

Thus

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge *\beta$$

If $e_1, \ldots, e_n$ is a positively ordered orthonormal basis of $\mathcal{E}^1(U)$ (which exists by Gram-Schmid), we can see that

$$dvol = e_1 \wedge \ldots \wedge e_n$$

$$*e_{i_1} \wedge \ldots \wedge e_{i_k} = \pm e_{j_1} \wedge \ldots \wedge e_{j_{n-k}}$$

where $\{j_1, \ldots\} = \{1, \ldots, n\} - \{i_1, \ldots\}$. It follows that $** = \pm 1$. The precise sign is $(-1)^{k(n-k)}$ on k-forms.

Lemma 6.1.3. *For all $\alpha \in \mathcal{E}^k(X), \beta \in \mathcal{E}^{k+1}(X)$,*

$$\langle d\alpha, \beta \rangle = \langle \alpha, (-1)^{k(n-k)} * d * \beta \rangle.$$

In other words, $(-1)^{k(n-k)} * d*$ is the adjoint d^* of d.

Proof. The proof follows by applying Stokes' theorem to the identity

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta \pm \alpha \wedge ** d * \beta$$

(This is nothing other integration by parts.) □

6.2 The Hodge theorem for Riemannian manifolds

Fix an oriented Riemannian manifold X. Let d^* denote the adjoint to d, which equals $\pm * d*$.

Definition 6.2.1. *The Laplacian (or Hodge Laplacian, or Laplace-Beltrami operator) of X is*

$$\Delta = dd^* + d^*d$$

Example 6.2.2. *Let $X = \mathbb{R}^2$ with Euclidian metric $g = dx \otimes dx + dy \otimes dy$. If $f \in C^\infty(X)$, then*

$$\Delta f = - * d * df = -(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2})$$

So it agrees with the usual Laplacian up to sign.

Definition 6.2.3. *A C^∞ form is harmonic if $\Delta\alpha = 0$.*

Lemma 6.2.4. *A form α is harmonic iff $d\alpha = d * \alpha = 0$.*

Proof. Suppose $\Delta\alpha = 0$, then

$$0 = \langle \Delta\alpha, \alpha\rangle = ||d\alpha||^2 + ||*d*\alpha||^2$$

Therefore $d\alpha = d*\alpha = 0$. The other direction is clear. $\square$

We now consider a problem in PDEs. Given a form α, can we always solve Poisson's equation

$$\Delta\gamma = \alpha?$$

The answer is almost always. Here is the precise statement.

Theorem 6.2.5 (Hodge theorem)**.** *Suppose that X is a compact oriented Riemannian manifold. Then the space of harmonic forms is finite dimensional. Any form α can be written as $\beta + \Delta\gamma$, where β is harmonic.*

We won't give a proof, but just make a few comments. The first step is to complete the inner product space $\mathcal{E}^k(X)$ to a Hilbert space. Methods of functional analysis can be used to prove the existence of a weak solution to $\alpha = \beta + \Delta\gamma$ with β, γ in this Hilbert space. The second step is to prove that this weak solution is in fact a C^∞ solution. This depends crucially on the fact that Δ is a so called elliptic partial differential operator.

Let's look at an example.

Example 6.2.6. *Let $X = \mathbb{R}^n/\mathbb{Z}^n$ with the Euclidean metric. A differential form α can be expanded in a Fourier series*

$$\alpha = \sum_{\lambda\in\mathbb{Z}^n}\sum_{|I|=p} a_{\lambda,I} e^{2\pi i\lambda\cdot\mathbf{x}} dx_I \tag{6.1}$$

where $dx_I = dx_{i_1}\wedge\ldots\wedge dx_{i_p}$. By direct calculation, one finds the Laplacian

$$\Delta = -\sum \frac{\partial^2}{\partial x_i^2} \text{ (on coefficients)}$$

Then $\alpha = \beta + \Delta\gamma$ with

$$\beta = \sum_I a_{0,I} dx_I$$

and

$$\gamma = \sum_{\lambda\in\mathbb{Z}^n-\{0\}}\sum_I \frac{a_{\lambda,I}}{4\pi^2|\lambda|^2} e^{2\pi i\lambda\cdot\mathbf{x}} dx_I$$

Since β has constant coefficients, it's harmonic.

We already used this theorem in the last chapter, and we will give more applications shortly. The following corollary is what most people would call the Hodge theorem.

Corollary 6.2.7. *Any de Rham cohomology class has a unique harmonic representative.*

Proof. We proved this statement for Riemann surfaces earlier, and the general case is the same. Suppose α is closed then write $\alpha = \beta + \Delta\gamma = d(d^*\gamma) + d^*(d\gamma)$ as above.

$$||d^* d\gamma||^2 = \langle d\alpha, d\gamma\rangle - \langle d^2 d^*\gamma, d\gamma\rangle = 0$$

So α and the harmonic form β lie in the same cohomology class. Uniqueness of β is proved similarly. □

Here is another proof of Poincaré duality.

Corollary 6.2.8. *If* $\dim X = n$, *then* $H^k_{dR}(X) \cong H^{n-k}_{dR}(X)$

Proof. $*$ takes harmonic forms to harmonic forms, so it induces the above isomorphism. □

6.3 Kähler manifolds

Let X be a complex manifold.

Definition 6.3.1. *A Hermitian metric on X is a family of Hermitian inner products on the complex tangent spaces which vary in C^∞ fashion. More precisely, H would be given by a section of $\mathcal{E}_X^{(1,0)} \otimes \mathcal{E}_X^{(0,1)}$, such that in some (any) locally coordinate system $z_i = x_i + \sqrt{-1}y_i$ around each point, H is given by*

$$H = \sum h_{ij} dz_i \otimes d\bar{z}_j$$

with h_{ij} positive definite Hermitian.

The real part of the matrix H is positive definite symmetric, and the tensor

$$\sum Re(h_{ij})(dx_i \otimes dx_i + dy_i \otimes dy_i)$$

gives a globally defined Riemannian structure on X. We also have a $(1,1)$-form ω called the *Kähler form* which is the normalized image of H under $\mathcal{E}_X^{(1,0)} \otimes \mathcal{E}_X^{(0,1)} \to \mathcal{E}_X^{(1,0)} \wedge \mathcal{E}_X^{(0,1)}$. In coordinates

$$\omega = \frac{\sqrt{-1}}{2} \sum h_{ij} dz_i \wedge d\bar{z}_j.$$

The normalization makes ω real, i.e. $\bar{\omega} = \omega$. It is clear from this formula, that ω determines the metric. Sometimes we just refer to ω as the metric.

In Riemannian geometry it is always possible to choose coordinates about a point which make it Euclidean up to second derivatives. These so called *normal coordinates* are often useful for computations. In the analytic world, such coordinates are not always possible, and this leads to a definition.

Definition 6.3.2. *A Hermitian metric on X is called a Kähler metric for any $p \in X$ if there exist analytic coordinates $z_1, \dots z_n$ with $z_i = 0$ at p, for which the metric becomes Euclidean up to second order:*

$$h_{ij} \equiv \delta_{ij} \mod (z_1, \dots z_n)^2$$

A Kähler manifold is a complex manifold which admits a Kähler metric. (Sometimes the term is used for a manifold with a fixed Kähler metric.)

In such a coordinate system, a Taylor expansion gives

$$\omega = \frac{\sqrt{-1}}{2} \sum dz_i \wedge d\bar{z}_i + \text{ terms of 2nd order and higher}$$

Therefore $d\omega = 0$ at $z_i = 0$. Since such coordinates can be chosen around point, $d\omega$ is identically zero. This gives a nontrivial obstruction for a Hermitean metric to be Kähler. In fact, this condition characterizes Kähler metrics and often taken as the definition:

Proposition 6.3.3. *Given a Hermitean metric H, the following are equivalent*

(1) H is Kähler.

(2) The Kähler form is closed: $d\omega = 0$.

(3) The Kähler form is locally expressible as $\omega = \partial\bar{\partial} f$.

Proof. That (1) implies (2) was explained above. See p 107 of Griffiths-Harris for the converse. (3) clearly implies (2), The converse will be proved in the exercises. □

We will refer the cohomology class of ω as the *Kähler class*. The function f such that $\omega = \sqrt{-1}\partial\bar{\partial} f$ is called a Kähler potential. A function f is *plurisubharmonic* if it is a Kähler potential, or equivalently in coordinates this means

$$\sum \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \xi_i \bar{\xi}_j > 0$$

for any nonzero vector ξ.

Basic examples of compact Kähler manifolds are:

Example 6.3.4. *Any Hermitian metric on a Riemann surface is Kähler since $d\omega$ vanishes for trivial reasons.*

Example 6.3.5. *Complex tori are Kähler. Any flat (Euclidean) metric will do.*

Before describing the next example, we note that the usual "round" metrics on the sphere are completely characterized by the fact that they are rotationally invariant. This can be made unique by fixing the area. We can extend this to other spaces. First observe that the unitary group $U(n+1)$ acts transitively on $\mathbb{P}^n$ via the standard action on $\mathbb{C}^{n+1}$.

Lemma 6.3.6. *A $U(n+1)$-invariant Hermitian metric on $\mathbb{P}^n$ is unique up to a positive scalar multiple.*

Proof. The isotropy group of $p = [1, 0, \ldots, 0]$ is $U(1) \times U(n)$, and the action of the last factor $U(n)$ on T_p can be identified with the standard one on $\mathbb{C}^n$ using the basis $\frac{\partial}{\partial(z_i/z_0)}$. A $U(n+1)$-invariant Hermitian metric on $\mathbb{P}^n$ is determined by a $U(n)$-invariant inner product on $T_p = \mathbb{C}^n$. Such an inner product must be a positive multiple of the standard one. □

Technically, we haven't proved that such a metric exists. We do that next. With an appropriate normalization, this metric is called the *Fubini-Study* metric, and it is of fundamental importance.

Proposition 6.3.7. *There exists is a unique choice of invariant Hermitian metric, called the Fubini-Study metric, such that $\int_{\mathbb{P}^1} \omega = 1$ for any line $\mathbb{P}^1 \subset \mathbb{P}^n$, where ω is the Kähler form. This metric is Kähler.*

Proof. Let $z_0, \ldots, z_n$ denote homogeneous coordinates. On a chart $U_i = \{z_i \neq 0\}$, the ratios $z_0/z_i, \ldots$ form true coordinates. For simplicity, let $i = 0$, and write $\zeta = (z_1/z_0, \ldots, z_n/z_0)$. We set

$$\omega_0 = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(1 + |\zeta|^2)$$

Computing the matrix of coefficients of ω_0 gives

$$\frac{(1 + |\zeta|^2)I - \zeta^\dagger \zeta}{\pi(1 + |\zeta|^2)^2}$$

where $\zeta^\dagger$ denotes the conjugate transpose. It is positive definite at the origin, because it reduces to $\frac{1}{\pi} I$ there. We have similarly defined forms ω_i on each U_i. Let $\mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ be the projection. Then we can see that

$$\pi^* \omega_i = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(|z_0|^2 + \ldots + |z_n|^2)$$

because the difference between these expressions is proportional to $\partial\bar{\partial} \log |z_i|^2 = 0$. Since the right side is independent of i, we see that the ω_i's patch to a yield a 2-form ω on $\mathbb{P}^n$. It is clear from the last formula that this form is invariant under $U(n+1)$. Since the matrix of coefficients is positive definite at one point, it is positive definite everywhere. Therefore ω defines a Kähler metric.

One can check by direct calculation that $\int_{\mathbb{P}^1} \omega = 1$. □

We note that $H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$, and it is generated by the first Chern class of line bundle $\mathcal{O}(1)$ called the tautological bundle. This group has two generators $\mathcal{O}(\pm 1)$. The preferred one $\mathcal{O}(1)$ is distinguished by the fact that $H^0(\mathbb{P}^n, \mathcal{O}(1)) \neq 0$. In fact, it is isomorphic to the space of linear polynomials in the homogeneous coordinates $z_0, \ldots, z_n$. Since $H^2(\mathbb{P}^n, \mathbb{R}) = H^2(\mathbb{P}^n, \mathbb{Z}) \otimes \mathbb{R}$ is one dimensional, the Kähler class $[\omega]$ would have to be a nonzero multiple of $c_1(\mathcal{O}(1))$. In fact, the constants in the formulas are *chosen* so that these coincide.

Lemma 6.3.8. *A complex submanifold of a Kähler manifold inherits a Kähler metric such that the Kähler class is the restriction of the Kähler class of the ambient manifold.*

Proof. The Kähler form locally has a plurisubharmonic potential f. It follows immediately from the definition, that f restricts to a plurisubharmonic function on complex submanifold. Thus the Kähler form will restrict to a Kähler form. □

There are several reasons why the Kähler condition is a natural and useful. For algebraic geometry, the main reason is as follows.

Theorem 6.3.9. *A smooth projective variety has a Kähler metric.*

Proof. Since $\mathbb{P}^n$ is Kähler, the theorem follows from the previous lemma. □

When X is projective, the cohomology class of the Kähler form ω lies in the image of $H^2(X, \mathbb{Z})$, because it is the restriction of the first Chern class $c_1(\mathcal{O}(1))$. A deep theorem of Kodaira shows that this condition characterizes those Kähler manifolds that come from projective varieties.

Every complex manifold carries a Hermitian metric by a partition of unity argument. However, it is not true that every manifold carries a Kähler metric. There are a number of topological constraints, such as those below.

Theorem 6.3.10. *If X is compact Kähler manifold of dimension n, ω^k defines a nonzero class in $H^{2k}(X, \mathbb{C})$ for $k = 1, \ldots, n$. In particular, the Betti numbers $b_{2k}(X) = \dim H^{2k}(X, \mathbb{C})$ are nonzero.*

Proof. Using analytic normal coordinates, we can see that $\omega^n = Cdvol$, with $C > 0$. This implies that $\int_X \omega^n \neq 0$, so it cannot be exact. Since the class $[\omega^n]$ is the cup product of $[\omega]$ with itself n times, $[\omega^k] \neq 0$ for $k = 1, \ldots, n$. □

When $X \subset \mathbb{P}^N$ is smooth and projective with Fubini-Study metric, there is another way to see this. Bertini's theorem in algebraic geometry guarantees that X contains a smooth subvariety Y of dimension k for any $k \leq n$. Then

$$\int_Y \omega^k = \int_Y c_1(\mathcal{O}(1))^k = \deg Y$$

One definition of $\deg Y$ is that it is the number of points of Y intersected with k general hyperplanes. Further details can be found in Hartshorne or pretty much any book on basic algebraic geometry. The key point is that one always has $\deg Y > 0$. It follows that $[\omega^k] \neq 0$.

6.4 The Hodge theorem for Kähler manifolds

Let X be a compact Hermitian complex manifold. Since it is an oriented Riemannian manifold, we can apply the results for a previous section to see that complex valued de Rham cohomology classes can be represented by harmonic forms. We saw that for Riemann surfaces, there is a close connection between harmonic and holomorphic forms. This is no longer true for a general Hermitian manifold, but it is for Kähler manifolds. The reason behind this is a set of identities called the Kähler identities. In order to explain the key identity, let

$$\bar{\partial}^*\alpha = \pm\overline{*\bar{\partial}*\overline{\alpha}}$$

Then with appropriate sign, this is the adjoint of $\bar{\partial}$ with respect to the inner product $\langle,\rangle$. We define a new Laplacian by

$$\Delta_{\bar{\partial}} = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$$

Lemma 6.4.1. *$\Delta_{\bar{\partial}}$ preserves type, i.e. if $\alpha \in \mathcal{E}^{p,q}(X)$, then $\Delta_{\bar{\partial}}\alpha \in \mathcal{E}^{p,q}(X)$.*

Proof. $\bar{\partial}$ has bidegree $(0,1)$, and $\bar{\partial}^*$ has bidegree $(0,-1)$. □

Theorem 6.4.2. *If X is Kähler, then $\Delta = 2\Delta_{\bar{\partial}}$.*

Since X is Kähler, we can use analytic normal coordinates to reduce theorem to Euclidean space. However, the reduction is more complicated than it sounds because it is Euclidean only up to second order. Details can be found in Griffiths-Harris... Let us check this for the Euclidean metric.

$$\begin{aligned}
\Delta_{\bar{\partial}}(\alpha) &= -2\sum_{I,J,i} \frac{\partial^2\alpha_{IJ}}{\partial z_i \partial \bar{z}_i} dz_I \wedge d\bar{z}_J \\
&= -\frac{1}{2}\sum_{I,J,i}\left(\frac{\partial^2\alpha_{IJ}}{\partial x_i^2} + \frac{\partial^2\alpha_{IJ}}{\partial y_i^2}\right) dz_I \wedge d\bar{z}_J \\
&= \frac{1}{2}\Delta(\alpha)
\end{aligned}$$

Theorem 6.4.3 (The Hodge decomposition)**.** *Suppose that X is a compact Kähler manifold. Then a differential form is harmonic if and only if its (p,q) components are. Consequently we have noncanonical isomorphisms*

$$H^i(X,\mathbb{C}) \cong \bigoplus_{p+q=i} H^q(X,\Omega_X^p).$$

Furthermore, complex conjugation induces $\mathbb{R}$-linear isomorphisms between the space of harmonic (p,q) and (q,p) forms. Therefore

$$H^q(X,\Omega_X^p) \cong H^p(X,\Omega_X^q).$$

Proof. Since $\Delta = 2\Delta_{\bar{\partial}}$, a form is harmonic if and only if its (p,q) components are. Since complex conjugation commutes with Δ, conjugation preserves harmonicity. This shows that

$$H^i(X,\mathbb{C}) \cong \bigoplus_{p+q=i} H^{pq}, \quad \overline{H^{pq}} = H^{qp}$$

where H^{pq} is the space of harmonic (p,q)-forms. To finish, we need to establish an isomorphism

$$H^{pq} \cong H^q(X, \Omega_X^p)$$

Suppose $\alpha \in H^{pq}$. Then

$$0 = \langle \Delta_{\bar\partial}\alpha, \alpha\rangle = ||\bar\partial\alpha||^2 + ||\bar\partial^*\alpha||^2$$

Therefore α is $\bar\partial$-closed. When combined with Dolbeault's theorem, this produces a map

$$\pi : H^{pq} \to H^q(X, \Omega_X^p)$$

We claim that π is surjective, let α be a $\bar\partial$-closed (p,q)-form. Decompose

$$\alpha = \beta + \Delta\gamma = \beta + 2\Delta_{\bar\partial}\gamma = \beta + \bar\partial\gamma_1 + \bar\partial^*\gamma_2$$

with β harmonic. We have

$$||\bar\partial^*\gamma_2||^2 = \langle \gamma_2, \bar\partial\bar\partial^*\gamma_2\rangle = \langle \gamma_2, \bar\partial\alpha\rangle = 0.$$

This proves the claim.

We also claim that π is injective. To see this, suppose that $\alpha \in H^{pq}$ equals $\bar\partial\beta$. Then

$$||\alpha||^2 = \langle \beta, \bar\partial^*\alpha\rangle = 0$$

□

Corollary 6.4.4. *If X is compact Kähler and i is odd, then the Betti number $b_i = \dim H^i(X, \mathbb{C})$ is even.*

Proof. Let $h^{pq} = \dim H^q(X, \Omega_X^p)$. Then

$$b_i = \sum_{p+q=i} h^{pq} = 2 \sum_{p+q=i, p<q} h^{pq}$$

□

6.5 Functorial Hodge structure

The goal of this section is to show hat the Hodge decomposition for compact manifolds can be made independent of the metric. To make a precise statement, we need to formalize things.

Definition 6.5.1. *A Hodge structure of weight i consists of a finitely generated abelian group $H_{\mathbb{Z}}$ and a decomposition*

$$H_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=i} H^{pq}$$

such that $\overline{H^{pq}} = H^{qp}$. A morphism of Hodge structures $f : H_{\mathbb{Z}} \to G_{\mathbb{Z}}$ is a homomorphism of groups, such that the induced map $f \otimes \mathbb{C}$ preserves the bigrading.

Example 6.5.2. *Let X be compact Kähler. Then the isomorphism*

$$H^i(X,\mathbb{Z})\otimes\mathbb{C}\cong \text{ space of harmonic } i \text{ forms}$$

together with the decomposition constructed in the last section gives a natural example of a Hodge structure of weight i.

This seems to depend on the metric, but the surprise is that it doesn't.

Theorem 6.5.3 (Deligne)**.** *There exists functor from the category of compact Kähler manifolds and holomorphic maps to the category of Hodge structures of weight i, such that for any choice of metric, it is isomorphic to the example constructed above.*

First, we reformulate Hodge structures as follows.

Lemma 6.5.4. *The category of Hodge structures of weight i is equivalent to the category of finitely generated abelian groups $H_{\mathbb{Z}}$ with a decreasing filtration $F^\bullet$ on $H = H_{\mathbb{Z}}\otimes\mathbb{C}$ such that for all p, $F^p\oplus\overline{F}^{i-p+1} = H$.*

Sketch. In one direction, given a Hodge structure, set $F^p = H^{pq}\oplus H^{p+1,q-1}\dots$. In the other direction, $H^{pq} = F^p\cap\overline{F}^{i-p}$. □

Given a complex manifold, we define a filtration on differential forms by

$$F^p\mathcal{E}^i(X) = \mathcal{E}^{pq}(X)\oplus\mathcal{E}^{p+1,q-1}(X)\dots$$

In others $\alpha\in F^p$ if there are at least p dz's when written in coordinates.

Lemma 6.5.5. *We have $dF^p\mathcal{E}^i(X)\subseteq F^p\mathcal{E}^{i+1}(X)$, or in other words $F^p\mathcal{E}^\bullet(X)$ is a subcomplex of the de Rham complex.*

Proof. If $\alpha\in\mathcal{E}^{ab}(X)$, then $d\alpha\in\mathcal{E}^{a,b+1}(X)\oplus\mathcal{E}^{a+1,b}(X)$. □

We define

$$F^pH^i(X) = \mathrm{im}[H^i(F^p\mathcal{E}^\bullet(X))\to H^i(X)]$$

We want to show that this filtration defines a Hodge structure when X is compact Kähler. As a first step observe that the complex conjugate

$$\overline{F}^pH^i(X) = \mathrm{im}[H^i(\overline{F}^p\mathcal{E}^\bullet(X))\to H^i(X)]$$

where

$$\overline{F}^p\mathcal{E}^i(X) = \mathcal{E}^{pq}(X)\oplus\mathcal{E}^{p-1,q+1}(X)\dots$$

is also a subcomplex. It is clear that

$$\mathcal{E}^i(X) = F^p\mathcal{E}^i(X)\oplus\overline{F}^{i-p+1}\mathcal{E}^i(X)$$

This easily implies

Lemma 6.5.6. *If X is compact complex manifold,*

$$H^i(X) = F^p H^i(X) + \overline{F}^{i-p+1} H^i(X)$$

Note the sum on the right is not necessarily a direct sum. It is possible the two subspaces have a nontrivial intersection. To handle this, we need some homological algebra.

Lemma 6.5.7. *Let $C^\bullet$ be a bounded complex of vector spaces over a field, with a finite filtration $F^\bullet C^\bullet$ by subcomplexes. Suppose that for all i, p*

$$\sum_p \dim H^i(Gr^p C^\bullet) = \dim H^i(C^\bullet)$$

where the dimensions are assumed finite, and $Gr^p C^\bullet = F^p C^\bullet / F^{p+1} C^\bullet$. Then

$$\dim F^p H^i(C^\bullet) = \dim H^i(Gr^p C^\bullet) + \dim H^i(Gr^{p+1} C^\bullet) + \dots$$

The filtration is called *strict* if the above conditions hold. This is an extremely strong condition. It is equivalent to degeneration of the spectral sequence associated to F at the first page.

Lemma 6.5.8. *If X is compact Kähler, $F^\bullet \mathcal{E}^\bullet(X)$ and $\overline{F}^\bullet \mathcal{E}^\bullet(X)$ are strict. We have*

$$H^i(X) = F^p H^i(X) \oplus \overline{F}^{i-p+1} H^i(X)$$

Therefore the $F^\bullet$ defines a Hodge structure.

Proof. Dolbeault's theorem implies that

$$H^i(Gr_F^p \mathcal{E}^\bullet(X)) = H^i(\mathcal{E}^{p,\bullet}(X)[-p]) \cong H^{i-p}(X, \Omega_X^p)$$

The symbol $[-p]$ means shift the complex p places to the right. The Hodge decomposition theorem now implies that $F^\bullet \mathcal{E}^\bullet(X)$ is strict. Strictness of the conjugate filtration is similar. We know that

$$H^i(X) = F^p H^i(X) + \overline{F}^{i-p+1} H^i(X)$$

To prove that this is a direct sum, it suffces to show that the sum of dimensions on the right is the dimension on the left. By the previous lemma, it follows that

$$\dim F^p H^i(X) = h^{pq} + h^{p+1,q-1} + \dots$$

$$\dim \overline{F}^{q+1} H^i(X) = h^{p-1,q+1} + h^{p-2,q+2} + \dots$$

where $q = i - p$ and $h^{pq} = \dim H^q(X, \Omega^p)$. This implies the desired equality. □

This last lemma proves the theorem.

6.6 Hodge cycles

Let X be a complex nonsingular projective variety of dimension n, and let $Y \subset X$ be a closed nonsingular subvariety of dimension m. The difference $p = n - m$ is the codimension. We define a functional

$$\int_Y \in H^{2m}(X, \mathbb{R})^*$$

which sends a real m-form $\alpha \in \mathcal{E}^m(X)$ representing the class to

$$\int_Y \alpha$$

Since

$$\int_Y (\alpha + d\beta) = \int_Y \alpha$$

by Stokes' theorem, this is well defined. By Poincaré duality

$$H^{2m}(X, \mathbb{R})^* \cong H^{2p}(X, \mathbb{R})$$

this gives an element $[Y] \in H^{2p}(X, \mathbb{R})$ called the fundamental class. The duality isomorphism can be made more explicit, and this leads to a more explicit description of the class.

Lemma 6.6.1. *$[Y]$ can be represented by a closed $2p$-form $\eta_Y \in \mathcal{E}^{2p}(X)$ such that*

$$\int_Y \alpha = \int_X \eta_Y \wedge \alpha$$

for all closed $2m$-forms α.

Lemma 6.6.2. *$[Y]$ can be represented by a closed (p,p)-form.*

Proof. We can choose a form η_Y as above, which is harmonic. We want to show this is of type (p,p). Suppose not. Then the (a,b)-part $\eta^{a,b} \neq 0$, for some $a < p, b = p - a$. Let $\alpha = \bar{*}\eta^{a,b}$. This is of type $(n-b, n-a)$ and satisfies

$$\int_X \eta_Y \wedge \alpha = ||\alpha||^2 \neq 0$$

This implies

$$\int_Y \alpha \neq 0$$

However, this is impossible because $\alpha|_Y = 0$. □

In fact, the construction can be extended to define the fundamental class of possibly *singular* subvariety Y. One way to do this is appeal to a deep theorem of Hironaka that says there is a nonsingular variety $\tilde{Y}$ and a morphism $\pi : \tilde{Y} \to Y$ which is an isomorphism over the nonsingular part of Y. The map $\tilde{Y} \to Y$

is called a *resolution of singularities.* Then $[Y]$ can be defined as the class in $H^{2m}(X,\mathbb{R})$ dual to $\int_{\tilde{Y}}$. The resolution is not unique, but the class $[Y]$ can be seen to be well defined.

Although we won't prove it, there are alternative constructions which show that

Proposition 6.6.3. *The fundamental class of a subvariety lies in the image of* $H^{2p}(X,\mathbb{Z})$.

Putting these facts together, we find that fundamental class lies in

$$H^{2p}(X,\mathbb{Z}) \cap H^{p,p}(X)$$

where we take $H^{p,p}(X) = F^p \cap \bar{F}^p$ associated to the functorial Hodge structure. (NB: $H^{2p}(X,\mathbb{Z})$ might have torsion, which should be understood as lying in the intersection.) An element of this intersection will be called an *integral Hodge cycle*. A *rational Hodge cycle* or simply a *Hodge cycle* is an element of

$$H^{2p}(X,\mathbb{Q}) \cap H^{p,p}(X)$$

We now come to the famous:

Conjecture 6.6.4 (Hodge conjecture)**.** *A Hodge cycle is a linear combination of fundamental classes of subvarieties. Such a linear combination is called an algebraic cycle.*

Historical Remarks:

1. Hodge didn't actually use the word "conjecture", but he formulated it as a problem in his 1950 ICM talk. In fact, he expected it should hold with integer coefficients. But by 1960 counter-examples were found to the integral version by Atiyah-Hirzebruch, and it has since been formulated as above.

2. The conjecture gained importance in the 1960's partly because of its relation to Grothendieck's theory of motives. In this connection, he wrote a paper in English, famously entitled "Hodge's general conjecture is false for trivial reasons". Although, he quickly points out that he didn't mean the conjecture every thinks of, but instead something related.

3. Although the statement of the Hodge conjecture makes sense for compact Kähler manifolds, it is known to be false (Zucker, Voisin).

At the time Hodge formulated the conjecture, he had one important piece of evidence

Theorem 6.6.5 (Lefschetz $(1,1)$ theorem)**.** *An integral Hodge cycle* $H^2(X)$ *is an algebraic cycle.*

Before proving it, we give an equivalent formulation due to Kodaira and Spencer, which makes the proof easy. Recall that we have the first Chern class map

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

coming from the exponential sequence. The group on the left is the set of isomorphism classes of line bundles. The image is called the *Neron-Severi* group $NS(X)$.

Theorem 6.6.6 (Lefschetz $(1,1)$ theorem, version 2)**.** *An integral Hodge cycle in $H^2(X)$ is the first Chern class of a line bundle.*

Proof. Consider the exact sequence

$$H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O})$$

The last map can be interpreted as the composition

$$H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C}) \xrightarrow{\pi} H^2(X, \mathbb{C})/F^1 \cong H^2(X, \mathcal{O})$$

An integral Hodge cycle α is an element of $H^2(X, \mathbb{Z}) \cap H^{11}(X)$, so it would map to zero under π . Therefore α lies in the image of $H^1(X, \mathcal{O}^*)$. $\square$

We still have to prove version 1 of the theorem. A divisor is a finite linear combination $\sum n_i D_i$, where $D_i \subset X$ are irreducible varieties of codimension one. The line bundle $\mathcal{O}_X(D)$ constructed earlier for Riemann surfaces can be generalized to divisors in this sense. The first version follows from the next lemma.

Lemma 6.6.7.

1. *Any line bundle is isomorphic to $\mathcal{O}_X(D)$ for some divisor D.*
2. *If $D = \sum n_i D_i$, then $c_1(\mathcal{O}_X(D)) = \sum n_i [D_i]$.*

As a corollary, we give a useful criterion to the check the conjecture in some examples. First, observe that cohomology $H^*(X, \mathbb{C})$ forms a ring under cup product, and the space of Hodge cycles forms a subring.

Corollary 6.6.8. *If the ring Hodge cycles on X is generated by divisors, then the Hodge conjecture holds for X.*

6.7 Hodge cycles on self products of elliptic curves

Recall that an elliptic curve is a quotient of $\mathbb{C}$ by a lattice L. One can always normal the lattice to the form $L = \mathbb{Z} + \mathbb{Z}\tau$, where $Im\,\tau > 0$. A useful fact is

Theorem 6.7.1. *Any elliptic curve can be embedded into $\mathbb{P}^2$ as cubic curve.*

A proof can be found in Silverman's book on elliptic curves. A consequence of this and some basic algebraic geometry is that if E is an elliptic curve, then $E^n = E \times E \ldots$ is nonsingular and projective.

Theorem 6.7.2 (Tate)**.** *If E is an elliptic curve, the Hodge conjecture holds for E^n.*

We will prove this under an extra assumption. We say $\mathbb{C}/L$ has *complex multiplication* or is CM if there $\alpha \in \mathbb{C} - \mathbb{Z}$ such that $\alpha L \subseteq L$.

Example 6.7.3. *The curve corresponding to $L = \mathbb{Z}+\mathbb{Z}i$ is CM because $iL \subseteq L$.*

Lemma 6.7.4. *If $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ is CM then τ is algebraic, and in fact it is contained in an imaginary quadratic extension of $\mathbb{Q}$ (an extension of the form $\mathbb{Q}(\sqrt{-d}), d > 0$).*

Proof. The CM condition implies that $\alpha = a + b\tau$ and $\alpha\tau = c + d\tau$ for integers a, b, c, d. Therefore τ satisfies a quadratic equation with integer coefficients. Since $\tau \notin \mathbb{R}$, it lies in an imaginary quadratic extension of $\mathbb{Q}$. □

This shows that the CM curves are atypical, but they are nevertheless very important. We will outline to proof of Tate's theorem for CM curves. We start by analyzing E^2. We need on fact from topology which we state without proof.

Theorem 6.7.5 (Künneth formula)**.** *If X and Y are manifolds, de Rham cohomology satisfies*

$$H^i(X \times Y) \cong \bigoplus_{j+k=i} H^j(X) \otimes H^k(X)$$

where the isomorphism is induced by

$$\mathcal{E}^j(X) \otimes \mathcal{E}^k(Y) \to \mathcal{E}^k(X \times Y), \quad \alpha \otimes \beta \mapsto p_1^*\alpha \wedge p_2^*\beta$$

with p_i denoting projections.

Therefore

$$H^2(E^2) = [H^2(E) \otimes H^0(E)] \oplus [H^1(E) \otimes H^1(E)] \oplus [H^0(E) \otimes H^2(E)] \quad (6.2)$$

This is compatible with the Hodge decomposition. In terms of the Hodge numbers, this means

$$h^{11}(E^2) = h^{11}(E)h^{00}(E) + [h^{10}(E)h^{01}(E) + h^{01}(E)h^{10}(E)] + h^{00}(E)h^{11}(E)$$
$$= 1 + [1 + 1] + 1 = 4$$

In fact, we don't need anything fancy to see this. E^2 is a quotient ot $\mathbb{C}^2$. Letting z_1, z_2 denote the coordinates on $\mathbb{C}^2$. Then $H^{11}(E^2)$ has a basis given the 4 elements $dz_i \wedge d\bar{z}_j$. It's convenient to define a new basis

$$\beta_1 = cdz_1 \wedge d\bar{z}_1, \beta_2 = cdz_2 \wedge d\bar{z}_2, \beta_3 = cdz_1 \wedge d\bar{z}_2, \beta_4 = cdz_2 \wedge d\bar{z}_1$$

where c is chosen so that β_1 integrates to 1 over $E \times o$ with $o \in E$.

Lemma 6.7.6. *The rank of $NS(E^2)$ is at least 3, and it is 4 if E is CM.*

Proof. There are 3 divisors D_1, D_2, D_3 given by $E \times o$, $o \times E$ and the diagonal. In terms of the above basis, $[D_1] = \beta_1$. To see this, note that D_1 is the pullback of the divisor o under the first projection $E \times E \to E$, and $[o]$ is given by this formula. For similar reasons, $[D_2] = \beta_2$. Write

$$[D_3] = \sum a_i \beta_i$$

Define an involution by $\beta_1' = \beta_2, \beta_3' = \beta_4$. Then

$$a_i = \int_{E^2} [D_3] \wedge \beta_i' = \int_{D_3} \beta_i' = \int_E c dz \wedge d\bar{z} = 1$$

So that $[D_3] = \beta_1 + \beta_2 + \beta_3 + \beta_4$. It is then clear that the 3 divisors are linearly independent. Therefore $rank NS \geq 3$.

Suppose that E is CM. Then multiplication by α gives a morphism $\alpha : E \to E$. The graph D_4 of α gives a fourth divisor D_4. We omit the details, but we can expand D_4 in the basis as above, and check independence of all 4 divisors. □

In the CM case, we find that $H^{11}(E^2)$ is generated by divisors. The key point is that this generalizes.

Lemma 6.7.7. *If E is CM then $H^{pp}(E^n)$ is spanned by algebraic cycles.*

Sketch. Using the Künneth formula, one checks that $h^{11}(E^n) = n^2$, and $H^{pp}(X) = \wedge^p H^{11}(X)$. If $p_{ij} : E^n \to E^2$ denote the various projections, one checks that $\{p_{ij}^* D_k\}$ contains n^2 linearly independent elements of H^{11}. This proves the lemma for $p = 1$. The general case follows from the equation $H^{pp}(X) = \wedge^p H^{11}(X)$ by taking products. □

2nd proof. We give an alternative argument for $NS = H^{11}$, which is cleaner but less elementary in that it uses some facts about abelian varieties. An abelian variety is a complex torus $A = V/L$ with an isogeny (an almost isomorphism) between A and the dual V^*/L^*. This isomorphism induces an involution $*$ on the algebra $End_{\mathbb{Q}}(A) = End(A) \otimes \mathbb{Q}$. On page 190 of Mumford's Abelian Varieties, one finds an isomorphism

$$NS(A) \otimes \mathbb{Q} \cong \{M \in End_{\mathbb{Q}}(A) \mid M = M^*\}$$

where the map from left to right can be understood as form of c_1. When $A = E^n$ with E CM, $End_{\mathbb{Q}}(E^n)$ is the algebra of $n \times n$ matrices over the imaginary quadratic field K containing τ. The involution is the conjugate transpose. A matrix satisfying $M^* = M$ is determined by $n(n-1)/2$ entries above the diagonal in K, and n entries along the diagonal in $\mathbb{Q}$, giving

$$\dim NS(E^n) = [K : \mathbb{Q}] \frac{n(n-1)}{2} + n = n^2$$

as required. □

Corollary 6.7.8. *Tate's theorem is true for CM elliptic curves.*

In the non CM case $H^{pp}(E^n)$ is not generated by algebraic cycles. So in this case, the proof requires a completely different strategy, which we discuss later.

Chapter 7

The Lefschetz theorems

7.1 Weak Lefschetz

Given a smooth projective variety $X \subset \mathbb{P}^N$, Bertini's theorem (c.f. Hartshorne) shows that there exists a hyperplane $H \subset \mathbb{P}^N$ such that $Y = X \cap H$ is smooth. What is the relationship between the topology of X and Y? Obviously, they are not homeomorphic, because they have different dimensions, nevertheless there is surprisingly close connection.

Theorem 7.1.1 (Weak Lefschetz)**.** *Let $n = \dim X$.*

(a) The restriction map $H^i(X, \mathbb{R}) \to H^i(Y, \mathbb{R})$ is an isomomorphism for $i < n-1$ and an injection for $i = n-1$.

(b) If $U = X - Y$, then $H^i(U, \mathbb{R}) = 0$ for $i > n$.

Corollary 7.1.2. *Let $Y \subset \mathbb{P}^n$ be a smooth hypersurface. The Betti numbers*

$$b_i(Y) = \begin{cases} 0 & \text{if } i \text{ odd} \\ 1 & \text{if } i \text{ even} \end{cases}$$

for $i < n-1$ or $n-1 < i \leq 2n-2$

Proof. Let $X = \mathbb{P}^n$ and $d = \deg Y$. We can find an embedding $X \subset \mathbb{P}^N$ for $N = \binom{n+d}{d} - 1$, called the Veronese embedding, such that $Y = X \cap H$ for a hyperplane $H \subset \mathbb{P}^N$. (Details can be found in any basic AG book; Hartshorne calls this the "d-uple" embedding for some reason.) The theorem and Poincaré duality tells us that $b_{2n-2-i}(Y) = b_i(Y) = b_i(\mathbb{P}^n)$ for $i < n-1$. The Betti numbers of projective space are known and given as above. □

Although, we stated this with real coefficients, the theorem holds with integer coefficients. Statement (b) implies (a) by an exact sequence

$$\ldots H^i_c(U) \to H^i(X) \to H^i(Y) \to H^{i+1}_c(U) \ldots$$

plus Poincaré duality

$$H^i_c(U) \to H^{2n-i}(U)^*$$

So it remains to prove (b).

We outline a proof due to Andreotti-Frankel, "The Lefschetz theorem on hyperplane sections", Ann Math 1959. This uses Morse theory, which is explained in Milnor's book of the same title. The idea is given a C^∞ manifold M and a suitable C^∞ function $f : M \to \mathbb{R}$, we can see how the topology changes for $M_t = f^{-1}(-\infty, t]$ as t increases. A function f is called *Morse* if it is proper (preimages of compact sets are compact), the critical points where the derivative vanishes are isolated, and at each critical point the Hessian $H_f = (\frac{\partial^2 f}{\partial x_i \partial x_j})$ is nonsingular. The number of negative eigenvalues of H_f at a critical point is called its *index*.

Theorem 7.1.3. *Suppose that $f : M \to \mathbb{R}$ is Morse. If there are no critical points between t_1 and t_2, then M_{t_1} and M_{t_2} are homotopy equivalent. If there is exactly one critical point between t_1 and t_2, and it has index k, then M_{t_2} is homotopy equivalent to M_{t_1} with a k cell attached.*

A proof can be found in Milnor. Here the terms "homotopy equivalent" can be found in any book in algebraic topology such as Hatcher; it should be understood as "the same as" for our purposes. In particular, the cohomology groups with $\mathbb{R}$ or even $\mathbb{Z}$ coefficients would be isomorphic. Attaching a k cell to M_{t_1} means to glue the boundary of the unit ball in $\mathbb{R}^k$ to the preceding space. This won't affect cohomology in degree larger than k. Therefore

Corollary 7.1.4. *If the indices of f are less than or equal to K, then M is homotopy equivalent to a CW complex of dimension $\leq K$. In particular, $H^i(M, \mathbb{Z}) = 0$ for $i > K$.*

Suppose $M \subset \mathbb{R}^N$ is a proper submanifold. Then we have the following facts (see Milnor):

1. For almost all (in the sense of measure theory) $p_0 \in \mathbb{R}^N - M$, $f(p) = |p - p_0|^2$ is a Morse function.

2. A point $p \in M$ is a critical point of f iff there is a focal point lying on the line segment $\overline{pp_0}$ from p to p_0, where $q \in \mathbb{R}^N$ is a focal point if (roughly) normal lines to M intersect at q. More formally, q is a focal point if there exist $p \in M$, such that $\overline{pq}$ is normal to M and the derivative of $(p, q) \mapsto q$ has a nontrivial kernel, the dimension of which gives the multiplicity.

3. The index of f at p is the number of focal points on line segment $\overline{pp_0}$ counted with multiplicity. The number of focal points (with multiplicity) on the whole line joining p and p_0 is at most $\dim M$.

Let us turn to the proof of part (b) of Weak Lefschetz. U is a closed submanifold of $\mathbb{P}^N - H = \mathbb{C}^N$ Therefore it suffices to prove the more general statement:

Theorem 7.1.5. *If M is a n dimensional closed complex submanifold of $\mathbb{C}^N$, then $H^i(M, \mathbb{Z}) = 0$ for $i > n$.*

Idea. An elementary fact from linear algebra is that given an $n \times n$ complex symmetric matrix, its real part, viewed as a $2n \times 2n$ real matrix, has eigenvalues in $\pm$ pairs. Using this, one checks that if ℓ is a real normal line through $p \in M$, then focal points (and their multiplicities) occur symmetrically about p. Identify $\mathbb{C}^N = \mathbb{R}^{2N}$, and choose a Morse function f given as the square of the distance from p_0 as above. Suppose that p is a critical point with index $k \leq 2n$. This implies that half of the possible $2n$ focal points on the real line joining p and p_0 lie on $\overline{pp_0}$. Therefore $k \leq n$.

□

7.2 The hard Lefschetz theorem

Fix a smooth projective variety $X \subset \mathbb{P}^N$ of dimension n. If H is a general hyperplane, then the weak Lefschetz tells us that $H^{n-1}(X, \mathbb{Z}) \to H^{n-1}(Y, \mathbb{Z})$ is injective, but it gives no further information. However, Lefschetz analyzed this case further by choosing a suitable family of hyperplanes. Rather going in this direction, we reformulate things in a different, and now standard, way which works for an arbitrary compact Kähler manifold.

Theorem 7.2.1 (Hard Lefschetz theorem)**.** *Let X be a compact Kähler manifold of dimension n. Then for all $i < n$, cup product with $[\omega^i]$ gives an isomorphism*

$$H^{n-i}(X, \mathbb{R}) \xrightarrow{\sim} H^{n+i}(X, \mathbb{R})$$

This is deduced from a more complicated, but equally important, statement called the Lefschetz decomposition. Let L denote cup product with $[\omega]$. The space

$$P^i(X) = \ker[L^{n-i+1} : H^i(X, \mathbb{C}) \to H^{2n-i+2}(X, \mathbb{C})]$$

is called the *primitive* cohomology.

Theorem 7.2.2 (Lefschetz decomposition)**.** *For every i,*

$$H^i(X, \mathbb{C}) = \bigoplus_{j=0}^{[i/2]} L^j P^{i-2j}(X)$$

To see that this implies the hard Lefschetz, observe that it implies that

$$L^i : H^{n-i}(X) \to H^{n+i}(X)$$

is surjective. Therefore it is an isomorphism, since by Poincaré duality, they have the same dimension.

We outline the proof of the Lefschetz decomposition. It depends on an additional set of Kähler identities. Define the following maps from $\mathcal{E}^\bullet(X) \to \mathcal{E}^\bullet(X)$: $L = \omega \wedge -$, $\Lambda = -\bar{*} L \bar{*}$, and H by multiplication by $n - k$ on $\mathcal{E}^k(X)$.

Proposition 7.2.3. *The following hold:*

(1) $[\Lambda, L] = H$

(2) $[H, L] = -2L$

(3) $[H, \Lambda] = 2\Lambda$

Furthermore these operators commute with Δ*.*

Proof. See p 115, 121 of Griffiths-Harris. □

The first three relations imply that L, Λ, H together define a representation of the Lie algebra $sl_2(\mathbb{C})$. This proposition plus the following fact from representation theory will prove the Lefschetz decomposition.

Proposition 7.2.4. *Let* V *be a possibly infinite dimensional complex vector space with endomorphisms* L, Λ, H *satisfying the above identities. If* $P = \ker(\Lambda)$*, then*

$$V = P \oplus LP \oplus L^2 P \oplus \ldots$$

Proof. Let us explain how to reduce the proof to two standard facts in representation theory, which can be found in Humphries Lie Algebras for example.

1. Representation of $sl_2(\mathbb{C})$ are directs sums of irreducible representations.
2. An irreducible representation is generated by a single vector – the highest weight vector – by repeatedly applying L. This vector is annihilated by Λ, so lies in P.

□

7.3 Consequences of hard Lefschetz

We discuss a few consequences of the previous theorem.

Corollary 7.3.1. *The sequence of even (odd) degree Betti numbers* $b_0 \le b_2 \le \ldots$ *(*$b_1 \le b_3 \le \ldots$*) weakly increases until the degree reaches* n*, after which it weakly decreases.*

Proof. By the theorem, cup product with $[\omega]$ gives an injection (surjection) $H^i(X, \mathbb{R}) \to H^{i+2}(X, \mathbb{R})$ when $i \le n-1$ ($i \ge n-1$). □

Corollary 7.3.2. *If* X *is a smooth projective variety of dimension* n*, then a Hodge cycle in* $H^{2n-2}(X, \mathbb{Q})$ *is an algebraic cycle. In particular, the Hodge conjecture holds for varieties of dimension at most* 3*.*

Proof. We can assume that X is given a Fubini-Study metric associated to an embedding $X \subset \mathbb{P}^N$. Then $[\omega] = c_1(\mathcal{O}(1))$ is an algebraic cycle. In fact, $\omega = [H]$ for any hyperplane. The hard Lefshetz gives an isomorphism

$$L^{n-1} : H^2(X, \mathbb{Q}) \to H^{2n-2}(X, \mathbb{Q})$$

which induces an isomorphism

$$H^{11}(X) \cong H^{n-1,n-1}(X)$$

Therefore a Hodge cycles $\alpha \in H^{2n-1}(X)$ is of the form $[\omega^{n-1}] \cup \beta$, where $\beta \in H^2(X, \mathbb{Q})$ is a Hodge cycle. The Lefschetz $(1,1)$ implies that $\beta = [D]$ for some divisor. One can check that

$$[\omega^{n-1}] \cup \beta = [H_1 \cap \dots H_{n-1} \cap D]$$

for distinct hyperplanes H_i. Therefore α is algebraic.

The last statement follows from the Lefschetz $(1,1)$ theorem and what we just proved.

□

Let X be a smooth projective variety of dimension n with a Fubini-Study metric. Then $[\omega] \in H^2(X, \mathbb{Q})$, so the Lefschetz operator $L = \omega \wedge -$ is rational. Consequently, we define rational primitive cohomology

$$P^i_{\mathbb{Q}}(X) = \ker[L^{n-i+1} : H^i(X, \mathbb{Q}) \to H^{2n-i+2}(X, \mathbb{Q})]$$

We also have $[\omega] \in H^{11}(X)$. It follows that $P^i(X) \subset H^i(X)$ is a rational Hodge substructure. This means that we have a rational vector space with a decomposition

$$P^i(X) := P^i_{\mathbb{Q}}(X) \otimes \mathbb{C} = \bigoplus_{p+q=i} P^{pq}(X), \quad \overline{P}^{qp}(X) = P^{pq}(X) \tag{7.1}$$

where $P^{pq}(X) = H^{pq}(X) \cap P^i(X)$. We define a bilinear form

$$Q(\alpha, \beta) = (-1)^{i(i-1)/2} \int_X L^{n-i} \alpha \wedge \beta$$

on $H^i(X, \mathbb{Q})$. This is symmetric if i is even, and skew-symmetric otherwise.

Theorem 7.3.3 (Hodge index theorem). *The restriction of Q to $P^i(X)$ satisfies*

$$\sqrt{-1}^{p-q} Q(\alpha, \bar{\alpha}) > 0, \quad \alpha \in P^{pq}(X) - \{0\}$$

and the decomposition (7.1) is orthogonal.

A proof can be found in Griffiths-Harris. A *polarization* on a Hodge structure is bilinear form Q as above. The significance of this notion will be clear later. For now we observe

Corollary 7.3.4. *$H^i(X)$ carries a polarization.*

Proof. By the Lefschetz decomposition,

$$H^i(X) \cong P^i(X) \oplus P^{i-2}(X)(-1) \oplus \dots$$

where the operation (-1) called Tate twist means $P(-1)^{p,q} = P^{p-1,q-1}$. The direct sum of the polarizations on the primitive parts will yield a polarization on $H^i(X)$ □

Chapter 8

Hodge structures

8.1 Mumford-Tate groups

A linear algebraic group G over a field k, is subgroup of some $GL_n(k)$ defined by polynomial equations. These include all the standard examples such as $SL_n(k), O_n(k), \ldots$. An algebraic group is an (not necessarily connected) affine algebraic variety such that the group operations are morphisms. The converse is true, although this isn't obvious. Given a bigger field $L \supseteq k$, let $G(L)$ be the set of solutions in $GL_n(L)$. This is also a group. A representation of G is a morphism of algebraic groups $G \to GL_N(k)$. When $k = \mathbb{R}$, we think of a $G(\mathbb{R})$ as a complex algebraic group $G(\mathbb{R})$ together with a conjugation $\sigma : G(\mathbb{C}) \to G(\mathbb{C})$ for which $G(\mathbb{R})$ is the fixed point set. A representation of $G(\mathbb{R}) \to GL_N(V)$ defined over $\mathbb{R}$ can be identified with a representation $G(\mathbb{C}) \to GL_N(V \otimes \mathbb{C})$ which commutes with conjugation.

A real Hodge structure of weight k is a finite dimensional real vector space $H_{\mathbb{R}}$ together with decomposition

$$H_{\mathbb{R}} \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{pq}$$

such that $\overline{H^{pq}} = H^{qp}$. We can express real Hodge structures in representation theoretic terms by first defining an algebraic group over $\mathbb{R}$, called Deligne's torus, by

$$S = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid d = a, c = -b\} \subset GL_2(\mathbb{R})$$

So that

$$S(\mathbb{R}) = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\} \cong \mathbb{C}^*$$

where the isomorphism sends the matrix to $a + bi$. It is a bit convenient to consider the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a + bi, a - bi)$$

Then this same map gives an isomorphism

$$S(\mathbb{C}) \cong \mathbb{C}^* \times \mathbb{C}^*$$

under which $S(\mathbb{R})$ is fixed points of the conjugation

$$\sigma(z_1, z_2) = (\bar{z}_2, \bar{z}_1)$$

Lemma 8.1.1. *There is an equivalence of categories between the category of real Hodge structures of weight k, and the category of representations of S, defined over $\mathbb{R}$, such that $\rho(t)$ acts by t^k for $t \in \mathbb{R}$.*

Sketch. We start with a representation of S as above. We are given a real vector space $V_{\mathbb{R}}$ and a homomorphism of algebraic groups $\rho : S \to GL(V_{\mathbb{R}})$. We can complexify this to a homomorphism of complex algebraic groups $\rho : S(\mathbb{C}) = (\mathbb{C}^*)^2 \to GL(V)$, where $V = V_{\mathbb{R}} \otimes \mathbb{C}$. Basic representation theory, tells us that $V = \oplus V^{pq}$, where V^{pq} be the space for which $\rho(z_1, z_2)v = z_1^p z_2^q v$. By assumption, $\rho(t, t) = t^k$. Therefore $p + q = k$. Finally, since ρ is defined over $\mathbb{R}$, it commutes with conjugation, i.e.

$$\overline{z_1^p z_2^q v} = \overline{z_2}^p \overline{z_1}^q v$$

This implies that $\overline{V^{pq}} = V^{qp}$.

Conversely, given a real Hodge structure $V_{\mathbb{R}}, \ldots$. We define a representation of $(\mathbb{C}^*)^2 \to GL(V)$ by $\rho(z_1, z_2)v = z_1^p z_2^q v$ for $v \in V^{pq}$. One can see that this defines a representation of $S(\mathbb{R})$. And that this gives an inverse to the previous construction.

□

Given real Hodge structures V_i viewed as representations of $S(\mathbb{R})$, it acts on $V_1 \otimes V_2$ by

$$g(v_1 \otimes v_2) = gv_1 \otimes gv_2$$

Therefore $V_1 \otimes V_2$ becomes a Hodge structure. It is easy enough to describe the (p, q) decomposition directly

$$(V_1 \otimes V_2)^{pq} = \bigoplus V_1^{rs} \otimes V_2^{p-r,q-s}$$

8.2 Mumford-Tate groups

A rational Hodge structure is the same thing as a finite dimensional $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$ and a Hodge structure on $H_{\mathbb{Q}} \otimes \mathbb{R}$.

Definition 8.2.1. *The Mumford-Tate group $MT(H) \subseteq GL(H)$ is the small algebraic group defined over $\mathbb{Q}$ such that $MT(H)(\mathbb{R})$ contains the image of $S(\mathbb{R})$. The special Mumford-Tate group or Hodge group (Mumford's original name) is*

$$SMT(H) = MT(H) \cap SL(H)$$

These groups can be thought of as some sort of symmetry or Galois groups in Hodge theory. A better characterization of SMT is given by

Lemma 8.2.2. *$SMT(H)$ is the smallest $\mathbb{Q}$-algebraic group such that $SMT(H)(\mathbb{R})$ contains the image of the circle*

$$S^1 = \{(t, t^{-1}) \mid |t| = 1\}$$

Given real Hodge structures V_i of weight k_i viewed as representations of $S(\mathbb{R})$, it acts on $V_1 \otimes V_2$ by

$$g(v_1 \otimes v_2) = gv_1 \otimes gv_2$$

Therefore $V_1 \otimes V_2$ becomes a Hodge structure of weight $k_1 + k_2$ It is easy enough to describe the (p, q) decomposition directly

$$(V_1 \otimes V_2)^{pq} = \bigoplus V_1^{rs} \otimes V_2^{p-r,q-s}$$

Similarly $S(\mathbb{R})$ acts on the dual V_1^*, so this becomes a Hodge structure of weight $-k_1$. One has

$$(V_1^*)^{pq} = (V_1^{-p,-q})^*$$

If $V_1 = V_2 = V$ is rational Hodge structure, then we see that $MT(H)$ acts on

$$V^{\otimes n} = V \otimes V \otimes \dots V$$

as above. Here is the key point for us:

Proposition 8.2.3. *If H is a rational Hodge structure, then the Hodge cycles in $H^{\otimes n}$ are exactly the $SMT(H)$-invariant tensors.*

Proof. If $V = H^{\otimes n}$ has odd weight then the result is vacuous. Suppose that it has weight $2k$. Let $\mathbb{Q}(-k)$ be the unique one dimensional Hodge structure of weight $2k$. Let $G = SMT(H)$ act on it trivially. Then the space of Hodge cycles in V is given by

$$Hom_{HS}(\mathbb{Q}(-k), V)$$

One checks easily that

$$Hom_{HS}(\mathbb{Q}(-k), V) = Hom_G(\mathbb{Q}(-k), V) = V^G$$

□

In general, Mumford-Tate groups are difficult to compute. The following gives useful information. Given an algebraic group G defined over $k \subset \mathbb{C}$, we say that G is *reductive* if if $G(\mathbb{C})$ contains a compact Zariski dense subgroup. For example, $GL_n(\mathbb{C})$ is reductive because the unitary subgroup U_n can be seen to be Zariski dense.

Proposition 8.2.4. *$SMT(H)$ is connected. If H has polarization, then $SMT(H)$ is a reductive.*

Sketch. The first statement follows from the fact that circle S^1 is connected in the usual topology. Using the polarization, one sees that the elements of S^1 preserve a positive definite form. Therefore it lies in a unitary group. This subgroup is dense in $SMT(H)(\mathbb{C})$. □

8.3 Tate's theorem: conclusion

Recall that

Theorem 8.3.1 (Tate)**.** *If E is an elliptic curve, the Hodge conjecture holds for E^n.*

We gave the proof when E has CM. For the general case, we need to compute the Mumford-Tate group $G = SMT(H)$ where $H = H^1(E)$). We know that $G(\mathbb{C}) \subseteq SL_2(\mathbb{C})$ is a connected reductive group, so by standard results concerning such groups, there are 3 possibilities

$$G(\mathbb{C}) = 1, \mathbb{C}^*, SL_2(\mathbb{C})$$

In fact only the last two occur

Proposition 8.3.2. *$G(\mathbb{C}) = \mathbb{C}^*$ if and only if E has CM. Otherwise, $G(\mathbb{C}) = SL_2(\mathbb{C})$*

Proof. Earlier, we computed the dimension of the Hodge cycles on

$$H^2(E^2) = \mathbb{Q}(-1)^2 \oplus H \otimes H$$

and we found that it is 4 exactly when E has CM, otherwise it's 3. This means that $(H \otimes H)^{G(\mathbb{C})}$ is 2 or 1 depending on these cases. On the other hand, we can compute

$$\dim(H \otimes H)^{G(\mathbb{C})} = \begin{cases} 3 & \text{if } G = 1 \\ 2 & \text{if } G = \mathbb{C}^* \\ 1 & \text{if } G = SL_2(\mathbb{C}) \end{cases}$$

□

To finish the proof of Tate's theorem, we can assume that $G = SL_2$. We have to compute the invariants of $H^{\otimes n}$, when n is even. Fortunately, this is classical

Theorem 8.3.3 (Weyl)**.** *The G-invariants of $H^{\otimes n}$ is spanned by*

$$\{\sigma(\Delta \otimes \Delta \dots) \mid \sigma \in S_n\}$$

where Δ be the generator of $(H \otimes H)^G$.

Proposition 8.3.4. *Tate's theorem holds in the non CM case.*

Proof. By Künneth's formula $H^{2p}(E^n)$ is a sum tensor products of even powers of H and some other factors, which can be seen to be generated by algebraic cycles. One reduces to checking that Hodge cycles in $H^{\otimes 2k} \subset H^{2k}(E^k)$ are algebraic. When $k = 1$, this true by the Lefschetz $(1,1)$ theorem. In general, by Weyl's theorem, the space of Hodge cycles is generated by products $p_{ij}^*\Delta$, for projections $p_{ij}E^{2k} \to E^2$. These classes are also algebraic. □