Chapter 2

Sheaves of functions

2.1 Sheaves

It is convenient at this point to introduce the language of sheaves, although in the limited way. We say that such a collection of functions is a presheaf if it is closed under restriction. Given sets X and T, let $Map_T(X)$ denote the set of maps from X to T. Here is the precise definition of a presheaf.

Definition 2.1.1. *Suppose that X is a topological space and T a nonempty set. A presheaf of T-valued functions on X is a collection of subsets $\mathcal{P}(U) \subseteq Map_T(U)$, for each nonempty open $U \subseteq X$, such that the restriction $f|_V \in \mathcal{P}(V)$ whenever $f \in \mathcal{P}(U)$ and $V \subset U$.*

The collection of all functions $Map_T(U)$ is of course a presheaf. Less trivially:

Example 2.1.2. *Let T be a topological space, then the set of continuous functions $C_{X,T}(U)$ from $U \subseteq X$ to T is a presheaf.*

Example 2.1.3. *Let X be a topological space and T be a set. The set $T^{pre}(U)$ of constant functions from U to T is a presheaf called the constant presheaf.*

Upon comparing these two examples, we see an essential difference. Continuity is a local condition, which means that it can be checked in a neighbourhood of a point. Constancy is, however, not local. A presheaf is called a sheaf if the defining condition is local as in the first example. More precisely:

Definition 2.1.4. *A presheaf of functions $\mathcal{P}$ is called a sheaf if given any open set U with an open cover $\{U_i\}$, a function f on U lies in $\mathcal{P}(U)$ if $f|_{U_i} \in \mathcal{P}(U_i)$ for all i.*

The first example $C_{X,T}(U)$ is certainly a sheaf, while the second is not in general. Suppose that T has at least two elements t_1, t_2, and that X contains a disconnected open set U. Then we can write $U = U_1 \cup U_2$ as a union of two disjoint open sets. The function τ taking the value of t_i on U_i is not in $T^{pre}(U)$, but $\tau|_{U_i} \in T^{pre}(U_i)$. Therefore T^{pre} is not sheaf.

However, there is a simple remedy.

Example 2.1.5. *A function is locally constant if it is constant in a neighbourhood of a point. For instance, the function τ constructed above is locally constant but not constant. The set of locally constant functions, denoted by $T(U)$ or $T_X(U)$, is a now sheaf, precisely because the condition can be checked locally. A sheaf of this form is called a constant sheaf.*

There are a number of further examples that will come up frequently.

Example 2.1.6. *Let $X = \mathbb{R}^n$ or a C^∞ manifold, the sets $C^\infty(U)$ of C^∞ real valued functions form a sheaf.*

Example 2.1.7. *Let $X = \mathbb{C}^n$ or a complex manifold, the sets $\mathcal{O}(U)$ of holomorphic functions on U form a sheaf.*

Example 2.1.8. *Let L be a linear differential operator on $\mathbb{R}^n$ with C^∞ coefficients (e. g. $\sum \partial^2/\partial x_i^2$). Let $S(U)$ denote the space of C^∞ solutions in U. This is a sheaf.*

Example 2.1.9. *Let $X = \mathbb{R}^n$, the sets $L^1(U)$ of L^1 or summable functions forms a presheaf which is not a sheaf, because summability is a global condition and not a local one.*

We can always create a sheaf from a presheaf by the following construction.

Example 2.1.10. *Given a presheaf $\mathcal{P}$ of functions from X to T. Define the*

$$\mathcal{P}^s(U) = \{f : U \to T \mid \forall x \in U, \exists \text{ a neighbourhood } U_x \text{ of } x, \text{ such that } f|_{U_x} \in \mathcal{P}(U_x)\}$$

This is a sheaf called the sheafification of $\mathcal{P}$.

When $\mathcal{P}$ is a presheaf of constant functions, $\mathcal{P}^s$ is exactly the sheaf of locally constant functions. When this construction is applied to the presheaf L^1, we obtain the sheaf of locally L^1 functions.

2.2 Manifolds again

We can now reinterpret the notion of a C^∞ or complex manifold. We start with a metrizable space X and a sheaf of C^∞ or holomorphic functions. We require that each point has a nbhd homeomorphic to a ball in $\mathbb{R}^n$ or $\mathbb{C}^n$ so that the homeomorphism preserves this special class of functions. To be more precise, let us fix a field k such as $k = \mathbb{R}$ or $\mathbb{C}$. Then $Map_k(X)$ is a commutative k-algebra with pointwise addition and multplication.

Definition 2.2.1. *Let $\mathcal{R}$ be a sheaf of k-valued functions on X. We say that $\mathcal{R}$ is a sheaf of algebras if each $R(U) \subseteq Map_k(U)$ is a subalgebra. Call the pair $(X, \mathcal{R})$ a concrete ringed space over k simply a k-space. We will sometimes refer to elements of $\mathcal{R}(U)$ as distinguished functions.*

The sheaf $\mathcal{R}$ is called the structure sheaf of X. Basic examples of $\mathbb{R}$-spaces are $(\mathbb{R}^n, C_{\mathbb{R}^n,\mathbb{R}})$, $(\mathbb{R}^n, C^\infty)$, and $(\mathbb{C}^n, \mathcal{O})$ is an example of a $\mathbb{C}$-space.

Definition 2.2.2. *A morphism of k-spaces $(X, \mathcal{R}) \to (Y, \mathcal{S})$ is a continuous map $F : X \to Y$ such that the pullback of distinguished functions are distinguished. More precisely, the condition is that if $f \in \mathcal{S}(U)$ then $F^*f \in \mathcal{R}(F^{-1}U)$, where $F^*f = f \circ F|_{F^{-1}U}$.*

For example, a C^∞ map $F : \mathbb{R}^n \to \mathbb{R}^m$ induces a morphism $(\mathbb{R}^n, C^\infty) \to (\mathbb{R}^m, C^\infty)$ of $\mathbb{R}$-spaces, and a holomorphic map $F : \mathbb{C}^n \to \mathbb{C}^m$ induces a morphism of $\mathbb{C}$-spaces. The converse is also true, and will be left for the exercises.

We note that the collection of k-spaces and morphisms forms a category. In any category, we have a notion of isomorphism. We will spell this out for k-spaces.

Definition 2.2.3. *An isomorphism of k-spaces $(X, \mathcal{R}) \cong (Y, \mathcal{S})$ is a homeomorphism $F : X \to Y$ such that $f \in \mathcal{S}(U)$ if and only if $F^*f \in \mathcal{R}(F^{-1}U)$.*

Given a sheaf S on X and open set $U \subset X$, let $S|_U$ denote the sheaf on U defined by $V \mapsto S(V)$ for each $V \subseteq U$. The following gives an alternative approach to manifolds.

Proposition 2.2.4. *An n-dimensional C^∞ manifold (resp. complex manifold) is the same thing as an $\mathbb{R}$-space (X, C^∞_X) (resp. $\mathbb{C}$-space $(X, \mathcal{O}_X)$) such that*

1. *X is metrizable*
2. *X admits an open cover $\{U_i\}$ such that each $(U_i, C^\infty_X|_{U_i})$ is isomorphic to $(B_i, C^\infty_{B_i})$ for open balls $B_i \subset \mathbb{R}^n$ (resp. ... isomorphic to $(B_i, \mathcal{O}_{B_i})$ for balls in $B_i \subset \mathbb{C}^n$).*

A C^∞ (resp. holomorphic) map between manifolds is the same thing as a morphism of the corresponding $\mathbb{R}$-spaces. (resp. $\mathbb{C}$-spaces).

We omit the proof which is not hard. We give one further example of a complex manifold with the help of this characterization.

Example 2.2.5. *The complex Grassmanian $G = Gr(2, n)$ is the set of 2 dimensional subspaces of $\mathbb{C}^n$. Let $M \subset \mathbb{C}^{2n}$ be the open set of $2 \times n$ matrices of rank 2. Let $\pi : M \to G$ be the surjective map which sends a matrix to the span of its rows. Give G the quotient topology induced from M, and define $f \in \mathcal{O}_G(U)$ if and only if $\pi \circ f \in \mathcal{O}_M(\pi^{-1}U)$. For $i \neq j$, let $U_{ij} \subset M$ be the set of matrices with $(1, 0)^t$ and $(0, 1)^t$ for the ith and jth columns. One can check that*

$$\mathbb{C}^{2n-4} \cong U_{ij} \cong \pi(U_{ij})$$

as ringed spaces. Since the images $\pi(U_{ij})$ cover G, we conclude that G is a $2n - 4$ dimensional complex manifold.

We note that the sheaf theory approach to manifolds is not commonly discussed in most references, but it has some advantages in algebraic or complex geometry where we consider more general kinds of spaces.

Example 2.2.6. *Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be a nonconstant polynomial, and let $X = f^{-1}(0)$. Since we didn't impose a condition on the gradient, X need not be a topological manifold. However, we can still introduce a sheaf of holomorphic functions, where $\mathcal{O}_X(U)$ consists of restrictions of holomorphic functions from an open subset of $\mathbb{C}^n$ containing U. (This is more accurately the sheaf of holomorphic functions on the reduced space corresponding to X.)*

2.3 Stalks

Given two functions defined in possibly different neighbourhoods of a point $x \in X$, we say they have the same *germ* at x if their restrictions to some common neighbourhood agree. This is is an equivalence relation. The germ at x of a function f defined near X is the equivalence class containing f. We denote this by f_x.

Definition 2.3.1. *Given a presheaf of functions $\mathcal{P}$, its stalk $\mathcal{P}_x$ at x is the set of germs of functions contained in some $\mathcal{P}(U)$ with $x \in U$.*

It will be useful to give a more abstract characterization of the stalk using *direct limits* (which are also called inductive limits, or filtered colimits). We explain direct limits in the present context. Suppose that a set L is equipped with a family of maps $\mathcal{P}(U) \to L$, where U ranges over open neighbourhoods of x. We will say that the family is a compatible family if $\mathcal{P}(U) \to L$ factors through $\mathcal{P}(V)$, whenever $V \subset U$. The maps $\mathcal{P}(U) \to \mathcal{P}_x$ given by $f \mapsto f_x$ forms a compatible family. A set L equipped with a compatible family of maps is called a direct limit of $\mathcal{P}(U)$ if and only if for any M with a compatible family $\mathcal{P}(U) \to M$, there is a unique map $L \to M$ making the obvious diagrams commute. This property characterizes L up to isomorphism, so we may speak of *the* direct limit

$$\varinjlim_{x \in U} \mathcal{P}(U).$$

Lemma 2.3.2. $\mathcal{P}_x = \varinjlim_{x \in U} \mathcal{P}(U)$.

Proof. Suppose that $\phi : \mathcal{P}(U) \to M$ is a compatible family. Then $\phi(f) = \phi(f|_V)$ whenever $f \in \mathcal{P}(U)$ and $x \in V \subset U$. Therefore $\phi(f)$ depends only on the germ f_x. Thus ϕ induces a map $\mathcal{P}_x \to M$ as required. □

All the examples of k-spaces encountered so far satisfy the following additional property.

Definition 2.3.3. *We will say that a concrete k-space $(X, \mathcal{R})$ is* locally ringed *if $1/f \in \mathcal{R}(U)$ when $f \in \mathcal{R}(U)$ is nowhere zero.*

Recall that a ring R is *local* if it has a unique maximal ideal, say m. The quotient R/m is called the residue field.

Lemma 2.3.4. *Suppose that $k = \mathbb{R}$ or $\mathbb{C}$, and $(X, \mathcal{R})$ is a ringed space, such that $\mathcal{R}(U)$ consists of continuous functions and $1/f \in \mathcal{R}(U)$ when $f \in \mathcal{R}(U)$ is nowhere zero. Then for any $x \in X$ $\mathcal{R}_x$ is a local ring with residue field isomorphic to k. In particular, this applies to C^∞ and complex manifolds.*

Proof. Let m_x be the set of germs of functions vanishing at x. For $\mathcal{R}_x$ to be local with maximal ideal m_x, it is necessary and sufficient that each $f \in \mathcal{R}_x \setminus m_x$ is invertible. This is clear since $1/f|_U \in R(U)$ for some $x \in U$.

To see that $\mathcal{R}_x/m_x = k$, it is enough to observe that the ideal m_x is the kernel of the evaluation map $ev : \mathcal{R}_x \to k$ given by $ev(f) = f(x)$, and the map is surjective, because $ev(a) = a$ when $a \in k$. □

Proposition 2.3.5. *When $(X, \mathcal{O}_X)$ is an n-dimensional complex manifold, the local ring $\mathcal{O}_{X,x}$ can be identified with ring of convergent power series in n variables.*

Proof. We can replace (X, x) by $(\mathbb{C}^n, 0)$. Then the germ of a holomorphic function at 0 is completely determined by its Taylor series, which converges in a nbhd of 0. □

We write

$$\mathbb{C}\{z_1, \ldots, z_n\} \subset \mathbb{C}[z_1, \ldots, z_n]$$

for the rings of convergent and formal power series in the above variables. Both rings are local with maximal ideal $m = (z_1, \ldots, z_n)$. Also both rings are known to be noetherian [see for example Zariski-Samuel Vol II], so standard results from commutative algebra can be applied. By contrast, when X is a C^∞-manifold, the stalks are non-noetherian local rings. This is because $\cap_n m^n$ contains nonzero functions such as

$$\begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

when $X = \mathbb{R}$, so it violates Krull's theorem [Atiyah-Macdonald, pp 110-111]. Nevertheless, the maximal ideals are finitely generated.

Proposition 2.3.6. *If R is the ring of germs at 0 of C^∞ functions on $\mathbb{R}^n$, then its maximal ideal m is generated by the coordinate functions $x_1, \ldots x_n$.*

Proof. One checks that if $f \in m$, then

$$f = \sum_i x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots tx_n) dt$$

□

If R is a local ring with maximal ideal m then $k = R/m$ is a field called the *residue field.* We will often convey all this by referring to the triple (R, m, k) as a local ring. For stalks of C^∞ and complex manifolds, the residue fields are respectively, $\mathbb{R}$ and $\mathbb{C}$. We note the following properties hold in these cases.

1. There is an inclusion $k \subset R$ which gives a splitting of the natural map $R \to k$.

2. The ideal m is finitely generated.

2.4 Tangent spaces

The tangent space to a manifold at a point is the best linear approximation to it. For a hypersurface $X = f^{-1}(0) \subset \mathbb{R}^n$ with $\nabla f|_X \neq 0$. We could use the definition from calculus: the tangent space at $p \in X$ is

$$T_p = \{v \in \mathbb{R}^n \mid v \cdot \nabla(f)(p) = 0\}$$

But this depends on the embedding. It is better to use a more intrinsic approach. The idea is that to each tangent vector $v \in T_p$ we can associate a directional derivative δ_v which operates on germs of functions at p. Since $v \mapsto \delta_v$ involves no loss of information, we may as well identify them. Thus we arrive the following abstract definition.

Definition 2.4.1. *Let (R, m, k) be a local ring of a C^∞ or complex manifold X at a point p, or more generally a ring satisfying the conditions at the end of the last section. Define the tangent space $T_p = T_R$ to be the set of k-linear derivations $Der_k(R, k)$ i.e. linear maps $\delta : R \to k$ satisfying $\delta(fg) = f(p)\delta g + g(p)\delta f$.*

When (R, m, k) satisfies the above conditions, R/m^2 splits canonically as $k \oplus m/m^2$. The second factor m/m^2 is finite dimensional. Let us focus on $R = C_p^\infty$ for now. The decomposition is given by $f \mapsto (f(p), f - f(p))$. Set $df = f - f(p)$. To get a better sense of what this means, expand f using Taylor's formula

$$f(x_1, \ldots, x_n) = f(p) + \sum \left.\frac{\partial f}{\partial x_i}\right|_p x_i + r(x_1, \ldots, x_n)$$

where the remainder r lies in m^2. We thus

$$df = \sum \left.\frac{\partial f}{\partial x_i}\right|_p \tag{2.1}$$

as the notation suggests.

Lemma 2.4.2. *$d : R \to m/m^2$ is a $\mathbb{R}$-linear derivation. There is an isomorphism*

$$T_p = Hom(m/m^2, \mathbb{R})$$

given by $\delta \mapsto \delta|_{m/m^2}$.

Proof. The first statement is clear from the formula (2.1). Given $\delta' \in Hom(m/m^2, k)$, let $\delta = \delta' \circ d$. This lies in T_p, and the map $\delta' \mapsto \delta$ gives the inverse to the map above. □

Corollary 2.4.3. *$T_p^* = T_R^* \cong m/m^2$ This is called the cotangent space for obvious reasons.*

Lemma 2.4.4. *When (R, m, k) is the ring of germs at 0 of C^∞ functions on $\mathbb{R}^n$. Then a basis for $Der_k(R, k)$ is given*

$$D_i = \left.\frac{\partial}{\partial x_i}\right|_0 \quad i = 1, \dots n$$

The lemma is straight forward using previous facts. A homomorphism $F : S \to R$ of local rings is called local if it takes the maximal ideal of S to the maximal ideal of R. Under these conditions, we get map of cotangent spaces $T_S^* \to T_R^*$ called the codifferential of F. When residue fields coincide, we can dualize this to get a map $dF : T_R \to T_S$ called the derivative or differential. To see the name is justified, suppose $f : U \to V$ is a C^∞ map, where $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ are open. Given $p \in U$, and $q = f(p)$, we get a homomorphism $f^* : S \to R$ between rings of germs of functions on V and U by $f^* g = g \circ f$. The following is straight forward.

Lemma 2.4.5. *Using standard bases, df^* is represented by the Jacobian matrix*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots \frac{\partial f_1}{\partial x_n} \\ & \dots \\ \frac{\partial f_m}{\partial x_1} & \dots \frac{\partial f_m}{\partial x_n} \end{pmatrix} (p)$$

where f_i are the components of f.

Proof. Writing $y_i = f_i(x_1, \dots, x_n)$. The chain rule gives

$$\frac{\partial}{\partial x_j} = \sum_i \frac{\partial f_i}{\partial x_j} \frac{\partial}{\partial y_i}$$

□

Let X be a complex manifold with $p \in X$, let $\mathcal{O}_{X,p} = \mathcal{O}_p$ be the ring of germs of holomorphic functions, and let $C^\infty_{X,p}$ be the ring of germs of C^∞ functions. Let us define the holomorphic tangent space as $T_p^h = T_{\mathcal{O}_p}$, and the real tangent space $T_p = T_{C_p^\infty}$. We have local homomorphism $\mathcal{O}_{X,p} \to C^\infty_{X,p}$, which induces a map $T_p^h \to T_p$ of real vector spaces. If $z_1, \dots, z_n$ are local analytic coordinates, the previous map is given by

$$\frac{\partial}{\partial z_j} \mapsto \frac{\partial}{\partial x_j}$$

Usually, we identify

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$$

So the map above is essentially the real part. From this description, we see that

$$T_p^h \cong T_p$$

as $\mathbb{R}$-vector spaces. This isomorphism imparts on T_p the structure of a complex vector space.

2.5 Vector fields and the tangent bundle

Let X be a C^∞ manifold. A C^∞ vector field on X is $\mathbb{R}$-linear derivation $Der_{\mathbb{R}}(C^\infty(X), C^\infty(X))$. Let $Vect(X)$ denote the set of these. It is clearly an abelian group. Given $f \in C^\infty(X)$ and $D \in Vect(X)$, $fD = g \mapsto f(x)D(g(x))$ is another vector field. This structure makes $Vect(X)$ into a module over $C^\infty(X)$. The following is not hard.

Proposition 2.5.1. *If $U \subset X$ is a coordinate nbhd of X with coordinates $x_1, \ldots, x_n$, then $\frac{\partial}{\partial x_i}$ gives a basis for $Vect(U)$ as $C^\infty(U)$-module. In particular, $Vect(U)$ is a free module of rank n.*

There is an alternative way to understand what a vector field is; it is simply a "C^∞ family" of vectors $v_p \in T_p$ for each p. The C^∞ requirement can be made precise by choosing coordinates as above. We describe a coordinate free approach in the case of a hypersurface $X = f^{-1}(0) \subset \mathbb{R}^n$. Define the manifold

$$T_X = \{(p, v) \in \mathbb{R}^n \times \mathbb{R}^n \mid \nabla f(p) \cdot v = 0\}$$

This has a projection $\pi : T_X \to X$ given by $\pi(p, v) = p$. The space T_X together with π is called the *tangent bundle* of X. The fibres are $\pi^{-1}(p) = T_p$ (using the calculus definition). Then a vector field is simply a C^∞ map $\sigma : X \to T_X$ such that $\pi \circ \sigma = 1$, because we want $\sigma(p) \in T_p$.

In general:

Theorem 2.5.2. *Given a C^∞ n-manifold X, there exists a $2n$-manifold T_X with a C^∞ map $\pi : T_X \to X$ such that*

1. *Each fibre $\pi^{-1}(p) \cong T_p$*
2. *There exists an open cover $\{U_i\}$ and isomorphisms*

$$\begin{array}{ccc} \pi^{-1}U_i & \xrightarrow{\sim} & U_i \times \mathbb{R}^n \\ \downarrow \pi & \swarrow & \\ U_i & & \end{array}$$

   which are linear on the fibres.

3. *$Vect(X)$ is isomorphic to the set of C^∞ maps $\sigma : X \to T_X$ (called sections) such that $\pi \circ \sigma = 1$.*

The data in item 2 is called a "local trivialization". One can a choose collection of charts for the cover $\{U_i\}$. A detailed construction can be found in any basic book on manifolds. Here we describe it when we have two charts U_1 and U_2 with coordinates x_i and x'_i respectively. We extend these to coordinates $x_1, \ldots, x_{2n}$ on $U_1 \times \mathbb{R}^n$, and $x'_1, \ldots, x'_{2n}$ on $U_2 \times \mathbb{R}^n$. We want to think of

$x_{n+i} = \partial/\partial x_i$ etc. On the intersection $U_1 \cap U_2$ we are given functions expressing $x'_i = \phi_i(x_1, \ldots, x_n)$ and visa versa. Differentiating gives the transformation rule

$$x'_{n+i} = \frac{\partial}{\partial x'_i} = \sum_j \frac{\partial x_j}{\partial x'_i} \frac{\partial}{\partial x_j} = \sum_j \frac{\partial x_j}{\partial x'_i} x_{n+j}$$

for the remaining coordinates. This allows us to glue $U_1 \times \mathbb{R}^n$ to $U_2 \times \mathbb{R}^n$.

The tangent bundle is called trivial, or X is called parallelizable, if we can take $\{U_i\} = \{X\}$, i.e. there exists an isomorphism $T_X \cong X \times \mathbb{R}^n$ as above. In algebraic terms, this equivalent to $Vect(X)$ to being a free module of rank n. Most manifolds are not parallelizable. The simplest counter example is the sphere S^2.

Finally, let us return to sheaf viewpoint. Given C^∞ n-manifold X, if we view $v \in Vect(U)$ as a section $U \to T_X|_U = T_U$, we can restrict it to any subset $V \subset U$. It should be clear that the assignment $Vect_X : U \mapsto Vect(U)$ forms a sheaf of abelian groups. In fact, restrictions are compatible with the module structure in the following sense. Given $f \in C^\infty(U)$ and $v \in Vect(U)$, $fv|_V = f|_V v|_V$. We say that $U \mapsto Vect(U)$ is a sheaf of modules over the sheaf C^∞_X. Finally, if U is a coordinate nbhd, proposition 2.5.1 implies that

$$Vect_X|_U \cong (C^\infty_U)^n$$

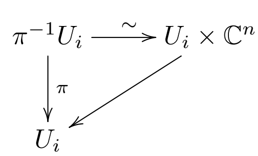
A sheaf of modules with this property is locally free of rank n. In summary:

Proposition 2.5.3. *$Vect_X$ is a locally free sheaf of modules over C^∞_X of rank n.*

There is a parallel story in the holomorphic case.

Theorem 2.5.4. *Given a complex n-manifold X, there exists a complex $2n$-manifold T^h_X with a holomorphic map $\pi : T^h_X \to X$ such that*

1. *Each fibre $\pi^{-1}(p) \cong T^h_p$*
2. *There exists an open cover $\{U_i\}$ and isomorphisms*



   which are linear on the fibres.

3. *As C^∞ manifolds $T^h_X = T_X$.*

Thanks to 3, we will usually drop the h in the future. We can define a holomorphic vector field as holomphic section $\sigma : X \to T_X$. We can form a sheaf of holomorphic vector fields as above.