Chapter 3

Differential forms

3.1 Introduction

Let us assume that $U \subset \mathbb{R}^n$ is a open with coordinates x_i. Then a differential form of degree 1, or simply a 1-form, on U is an expression

$$\sum f_i dx_i$$

where $f_i \in C^\infty(U)$. The (exterior) derivative of a C^∞ function is

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

So this plays the role of the gradient. A 1-form can be integrated along a path $\gamma : [0, 1] \to U$ by the usual rules of calculus. In order to define surface integrals etc., we need differential forms of higher degree. A 2-form is an expression

$$\sum f_{ij} dx_i \wedge dx_j$$

The symbol $\wedge$ is a distributive and anticommutative product. Anticommutiivity means $dx_i \wedge dx_j = -dx_j \wedge dx_i$. This might seem like a strange rule to impose, but it is extremely useful to do so. Given a 1-form

$$\alpha = \sum f_j dx_j$$

define its exterior derivative

$$d\alpha = \sum_j df_j \wedge dx_j = \sum_i \sum_j \frac{\partial f_j}{\partial x_i} dx_i \wedge dx_j$$

When $n = 3$, a basis of 2-forms consists of $dx_1 \wedge dx_2, dx_1 \wedge dx_3, dx_2 \wedge dx_3$ by anticommutivity. Simplifying $d\alpha$ shows that it is essentially the curl. So a single operation d replaces and generalizes the standard operations of vector calculus.

3.2 1-forms on manifolds

Let X be C^∞ n-manifold. A 1-form at $x \in X$ is simply a cotangent vector in T_x^*. A 1-form on X is a collection of cotangent vectors

$$\alpha \in \prod_{x \in X} T_x^*$$

which varies in C^∞ fashion. Here is the precise condition.

Definition 3.2.1. *A 1-form on X is a collection α as above such that*

$$\forall v \in Vect(U), \langle v, \alpha \rangle \in C^\infty(U)$$

where $\langle , \rangle$ is the obvious pairing between tangent and contangent vectors. Let $\mathcal{E}^1(X)$ denote the set of 1-forms.

Recall that we defined vector fields as simply derivations. So given $v \in Vect(X)$ and $f \in C^\infty(X)$, let $v(f) \in C^\infty(X)$ be the result of applying v as a derivation.

Definition 3.2.2. *Given $f \in C^\infty(X)$, the exterior derivative is the 1-form defined by*

$$\langle v, df \rangle = v(f)$$

Thus we have now given meaning to the symbols dx_i used before. It should be clear that

Lemma 3.2.3. *$d : C^\infty(X) \to \mathcal{E}^1(X)$ is an $\mathbb{R}$-linear derivation.*

Since we basically saying that

$$\mathcal{E}(X) \subset Map_{\prod T_x^*}(X)$$

we can ask whether $U \mapsto \mathcal{E}^1(U)$ is a sheaf of functions, and it is.

Proposition 3.2.4. *The sheaf $\mathcal{E}_X^1$ defined by $U \mapsto \mathcal{E}^1(U)$ is a sheaf of functions.*

Proof. It should be clear that the restriction of a 1-form is 1-form by the way we defined it. So $\mathcal{E}^1$ is a presheaf. Suppose $\{U_i\}$ is an open cover of U, and the restriction of $\alpha \in \prod_{x \in U} T_x^*$ to each U_i is a 1-form. Suppose that $v \in Vect(V)$, then $\langle v|_{U_i \cap V}, \alpha|_{U_i \cap V} \rangle \in C^\infty(U_i \cap V)$. Therefore $\langle v|, \alpha \rangle \in C^\infty(V)$. □

In fact, more is true

Theorem 3.2.5. *$\mathcal{E}_X^1$ is a locally free sheaf of modules over C^∞ of rank n.*

Sketch. The fact that $\mathcal{E}_X^1$ is a sheaf of modules should be clear. If U is a coordinate nbhd, one checks that $\mathcal{E}_U^1$ is has basis given by dx_i. □

1-forms can also be understood as sections of the contangent bundle. The construction is similar to the tangent bundle. Details can be found elsewhere.

3.3 p-forms

Before discussing differential forms of higher degree, we need to pause for some algebra. Let R be a commutative ring, and M an R-module. Define

$$T^*(M) = R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \ldots$$

This an R-module, with a product given by tensor product. This makes $T^*(M)$ into a *noncommutative* associative R-algebra called the *tensor algebra* of M. Note that it is also a graded algebra with the nth graded piece

$$T^n(M) = M \otimes \ldots \otimes M \ (n \text{ times})$$

We define the *exterior algebra* (sometimes called the Grassman algebra)

$$\wedge^* M = T^*(M)/I$$

where I is the 2-sided ideal generated by $\{m \otimes m \mid m \in M\}$. The operation $\otimes$ induces a product denoted by $\wedge$, which makes $\wedge^* M$ into a graded algebra again. Notice that we are forcing

$$m \wedge m = 0$$

which implies

$$(m_1 + m_2) \wedge (m_1 + m_2) = m_1 \wedge m_2 + m_2 \wedge m_1 = 0$$

for $m_1, m_2 \in M$ Therefore $\wedge$ is anticommutative. (The technically correct term is graded commutative.)

Lemma 3.3.1. *When M is a free module with basis $m_1, \ldots, m_n$, then $\wedge^d M$ is free with a basis $m_{i_1} \wedge \ldots m_{i_d}$ with $i_1 < \ldots i_d$. In particular, $\wedge^d M = 0$ when $d > n$.*

This has been pretty abstract, so let's specialize to the case of a finite dimensional vector space M over a field k. Essentially by the definition of tensor product,

$$Hom_k(T^d M, k)$$

is the space of multilinear functions

$$f : M \times \ldots M \to k$$

in d-variables. The space

$$Hom_k(\wedge^d M, k)$$

consists of multilinear functions which are antisymmetric in the sense that switching two arguments results in a sign change. Many basic books start with this point view, but the drawback is the wedge product looks really complicated and unatural when described this way.

Definition 3.3.2. *Given a C^∞ manifold X, a 0-form is simply a C^∞-function. When $p > 0$, a p-form is an element of*

$$\alpha \in \prod_{x\in X} Hom(\wedge^p T_x, \mathbb{R})$$

such that given vector fields $v_1, \ldots, v_p \in Vect(U)$,

$$\alpha(v_1, \ldots, v_p) \in C^\infty(U)$$

Let $\mathcal{E}^p(X)$ be the set of p-forms

The following can be proved as we did before.

Theorem 3.3.3. *The assignment $\mathcal{E}^p_X : U \to \mathcal{E}^p(U)$ forms a locally free sheaf of modules.*

The sheaf property allows us to describe a differential form locally in coordinates, if we need to. One case where this useful is in describing the exterior derivative.

Theorem 3.3.4. *There exists a linear operation $d : \mathcal{E}^p(X) \to \mathcal{E}^{p+1}(X)$ compatible with restriction and satisfying the Leibnitz rule*

$$d(f\alpha) = df \wedge \alpha + f d\alpha, \quad f \in C^\infty(X), \alpha \in \mathcal{E}^p(X)$$

and

$$d^2 f = 0$$

Proof. Let $\{U_j\}$ be a covering by charts. The above rules imply that in any coordinate system on U_j, d must be computed by

$$d(\sum_{i_1<\ldots i_p} f_{i_1\ldots i_p} dx_{i_1} \wedge \ldots dx_{i_p}) = (\sum\sum_j \frac{\partial f_{i_1\ldots i_p}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots dx_{i_p})$$

Given this formula, linearity and Leibnitz is clear. We check that

$$d^2 f = d(\sum_i f_i dx_i) = \sum_{i,j} f_{ij} dx_j \wedge dx_i = \sum_{i<j} (f_{ij} - f_{ji}) dx_i \wedge dx_j = 0$$

where $f_i = \partial f / \partial x_i$ etc. It follows that there is exactly one such operation $d : \mathcal{E}^p(U_j) \to \mathcal{E}^{p+1}(U_j)$. It follows that these local expressions $d(\alpha|_{U_j})$ must patch to yield a well defined $p+1$ form $d\alpha$. □

3.4 de Rham cohomology

Lemma 3.4.1. *Let X be a manifold. The operation $d^2 : \mathcal{E}^p(X) \to \mathcal{E}^{p+2}(X)$ is zero.*

Proof. This is easily checked in coordinates. □

A differential form is called *exact* if it lies in the image of d, and *closed* if its derivative is zero.

Corollary 3.4.2. *Exact forms are closed.*

Conversely, we can ask whether closed forms are exact? This is important both in pure math and in various applications. The answer, however, is in general no.

Example 3.4.3. *Let $S^1 = \mathbb{R}/\mathbb{Z}$. We can identify both $C^\infty(S^1)$ and $\mathcal{E}^1(S^1)$ with periodic functions on $\mathbb{R}$, and d with the derivative. Then $1 \in C^\infty(S^1)$, but it is not the derivative of a periodic function.*

We can try to measure the failure of the last question as follows.

Definition 3.4.4. *The nth (real) de Rham cohomology group of a manifold X is*

$$H^n_{dR}(X) = \frac{\{\textit{closed } n\textit{-forms}\}}{\{\textit{exact } n\textit{-forms}\}}$$

We will see that this is very computable. A key fact which helps with computations is the following.

Theorem 3.4.5 (Poincaré's lemma)**.** *For all $i > 0$, $H^i_{dR}(\mathbb{R}^n) = 0$*

Proof. Assume, by induction, that the theorem holds for $n-1$. Consider the maps $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $\iota : \mathbb{R}^{n-1} \to \mathbb{R}^n$ defined by $p(x_1, x_2, \ldots x_n) = (x_2, \ldots x_n)$ and $\iota(x_2, \ldots x_n) = (0, x_2, \ldots x_n)$. Let I be the identity transformation and let $R = (\iota \circ p)^*$. More explicitly, $R : \mathcal{E}^k(\mathbb{R}^n) \to \mathcal{E}^k(\mathbb{R}^n)$ is the $\mathbb{R}$-linear operator defined by

$$R(f(x_1, \ldots x_n)dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = \begin{cases} f(0, x_2, \ldots x_n)dx_{i_1} \wedge \ldots \wedge dx_{i_k} & \text{if } 1 \notin \{i_1, i_2 \ldots\} \\ 0 & \text{otherwise} \end{cases}$$

where we always choose $i_1 < i_2 < \ldots$. The image of R can be identified with $p^*\mathcal{E}^k(\mathbb{R}^{n-1})$. Note that R commutes with d. Therefore if $\alpha \in \mathcal{E}^k(\mathbb{R}^n)$ is closed, $dR\alpha = Rd\alpha = 0$. By the induction assumption, $R\alpha$ is exact.

For each k, define a linear map $h : \mathcal{E}^k(\mathbb{R}^n) \to \mathcal{E}^{k-1}(\mathbb{R}^n)$ by

$$h(f(x_1, \ldots x_n)dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = \begin{cases} (\int_0^{x_1} f dx_1)dx_{i_2} \wedge \ldots \wedge dx_{i_k} & \text{if } i_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then the fundamental theorem of calculus shows that $dh + hd = I - R$. (In other words, h is homotopy from I to R.) Given $\alpha \in \mathcal{E}^k(\mathbb{R}^n)$ satisfying $d\alpha = 0$. We have $\alpha = dh\alpha + R\alpha$, which is exact. □

3.5 Complex manifolds

Let X be a complex manifold of dimension n. We can treat this as a C^∞-manifold of dimension $2n$. It will be convenient to work with complex valued C^∞ forms. We define

$$\mathcal{E}^k_{\mathbb{C}}(X) = \mathcal{E}^k(X) \otimes_{\mathbb{R}} \mathbb{C}$$

When the context makes it clear, we may drop the subscript $\mathbb{C}$. If $z_1, \ldots, z_n$ are local analytic coordinates in U, we write

$$dz_j = dx_j + idy_j$$

$$d\bar{z}_j = dx_j - idy_j$$

where $x_j = Re\, z_j, y_j = Im\, z_j$. Note that this gives a basis of $\mathcal{E}^1_{\mathbb{C}}(U)$ which is more convenient to work with. We can extend this to a basis of $\mathcal{E}^k_{\mathbb{C}}$ by taking wedge products.

Definition 3.5.1. *A differential form is of type (p, q) if it expressible in coordinates as a linear combination of*

$$dz_{i_1} \wedge \ldots dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots d\bar{z}_{j_q}$$

We let $\mathcal{E}^{pq}(X)$ denote the space of these.

Since the notation above gets quite cumbersome, let's abbreviate it as

$$dz_I \wedge d\bar{z}_J$$

for ordered sets $I = i_1 < i_2 \ldots$, $J = j_1 < j_2 \ldots$.

Given a function f, define

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j})$$

$$\partial f = \sum_{j=0}^{n} \frac{\partial f}{\partial z_j} dz_j$$

$$\frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j})$$

$$\bar{\partial} f = \sum_{j=0}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j$$

Recall that the last operator is the Cauchy-Riemann operator. We extend these to higher degree forms as follows

$$\partial(\sum f_{IJ} dz_I \wedge d\bar{z}_J) = \sum (\partial f_I) \wedge dz_I \wedge d\bar{z}_J$$

$$\bar{\partial}(\sum f_{IJ} dz_I \wedge d\bar{z}_J) = \sum (\bar{\partial} f_I) \wedge dz_I \wedge d\bar{z}_J$$

Although we define these using coordinates:

Theorem 3.5.2. *The operators ∂ and $\bar\partial$ are well defined.*

Proof. We just do this for the original operators on functions. Rather than proving this by calculation, we give an intrinsic description. Recall that we have an isomorphism of real vector space $T_x^h \cong T_x$, where the first space is the holomorphic tangent space. Let $J : T_x \to T_x$ denote the linear map induced by multiplication by i on T_x^h. This induces maps $T_x^* \to T_x^*$ and on the complexification that we also denote by J. In coordinates $J(dx_k) = dy_k$ and $J(dy_k) = -dx_k$. It follows that $Jdz_k = -idz_k$ and $Jd\bar z_k = id\bar z_k$. So these are eigenvectors for J. To define ∂f or $\bar\partial f$, we can project df onto the $-i$ or $+i$ eigenspaces □

Lemma 3.5.3. *The following identities hold*

$$\partial^2 = \bar\partial^2 = \bar\partial\partial + \partial\bar\partial = 0$$

$$d = \partial + \bar\partial$$

Proof. The lemma follows from the identities

$$\frac{\partial^2 f}{\partial z_j \partial z_k} = \frac{\partial^2 f}{\partial z_k \partial z_j}$$

$$\frac{\partial^2 f}{\partial \bar z_j \partial \bar z_k} = \frac{\partial^2 f}{\partial \bar z_k \partial \bar z_j}$$

$$\frac{\partial^2 f}{\partial z_j \partial \bar z_k} = \frac{\partial^2 f}{\partial \bar z_k \partial z_j}$$

which are easily checked. □

An analogue of de Rham cohomology is provided by:

Definition 3.5.4. *The (p,q)th Dolbeault cohomology is*

$$H^{pq}_{Dol}(X) = \frac{\ker \mathcal{E}^{pq}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,q+1}(X)}{\operatorname{im} \mathcal{E}^{p,q-1}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,q}(X)}$$

For people comfortable with homological algebra, the previous lemma says that $\mathcal{E}^{\bullet\bullet}(X)$ forms a double complex. This gives rise to a spectral sequence, which we may study later. Dolbeault cohomology gives first page of the spectral sequence.

For now, we want to understand the simplest case.

Definition 3.5.5. *A p-form $\alpha \in \mathcal{E}^p_{\mathbb{C}}(X)$ is called holomorphic if it can be expressed in local coordinates as*

$$\alpha = \sum f_I dz_I$$

with f_I holomorphic. Let $\Omega^p_X(X) = \Omega^p(X)$ be the space of these.

Lemma 3.5.6. $H^{p,0}(X) = \Omega^p(X)$

Proof. $H^{p,0}(X) = \{\alpha \in \mathcal{E}^p(X) \mid \bar{\partial}\alpha = 0\} = \Omega^p(X)$ □

Similar to the C^∞ case, we have

Theorem 3.5.7. *The assigment $\Omega^p_X : U \mapsto \Omega^p(U)$ is a sheaf, and in fact a locally free $\mathcal{O}_X$-module.*

We will see later that the other Dolbeault groups can be understood as higher cohomology of these sheaves.