Chapter 4

Sheaf Cohomology

4.1 Presheaves and Sheaves

It will be convenient to define presheaves of things other than functions. For instance, one might consider sheaves of equivalence classes of functions, distributions and so on. For this more general notion of presheaf, the restrictions maps have to be included as part the datum:

Definition 4.1.1. *A presheaf $\mathcal{P}$ of sets (respectively groups or rings) on a topological space X consists of a set (respectively group or ring) $\mathcal{P}(U)$ for each open set U, and maps (respectively homomorphisms) $\rho_{UV} : \mathcal{P}(U) \to \mathcal{P}(V)$ for each inclusion $V \subseteq U$ such that:*

1. $\rho_{UU} = id_{\mathcal{P}(U)}$
2. $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$ *if* $W \subseteq V \subseteq U$.

We will usually write $f|_V = \rho_{UV}(f)$.

Definition 4.1.2. *A sheaf $\mathcal{P}$ is a presheaf such that for any open cover $\{U_i\}$ of U and $f_i \in \mathcal{P}(U_i)$ satisfying $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, there exists a unique $f \in \mathcal{P}(U)$ with $f|_{U_i} = f_i$.*

In English, this says that a collection of local sections can be patched together provided they agree on the intersections. Here is an example constructed abstractly.

Example 4.1.3. *Given a space X, a point $p \in X$ and a group A, the skyscraper sheaf*

$$A_p(U) = \begin{cases} A & \text{if } p \in U \\ 0 & \text{otherwise} \end{cases}$$

The restriction $\rho_{UV} = id$ if both sets contain p, otherwise it's 0.

Definition 4.1.4. *Given presheaves of sets (respectively groups) $\mathcal{P}, \mathcal{P}'$ on the same topological space X, a morphism $f : \mathcal{P} \to \mathcal{P}'$ is collection of maps (respectively homomorphisms) $f_U : \mathcal{P}(U) \to \mathcal{P}'(U)$ which commute with the restrictions. Given morphisms $f : \mathcal{P} \to \mathcal{P}'$ and $g : \mathcal{P}' \to \mathcal{P}''$, the compositions $g_U \circ f_U$ determine a morphism from $\mathcal{P} \to \mathcal{P}''$. The collection of presheaves of Abelian groups and morphisms with this notion of composition constitutes a category $PAb(X)$.*

Definition 4.1.5. *The category $Ab(X)$ is the full subcategory of $PAb(X)$ generated by sheaves of Abelian groups on X. In other words, objects of $Ab(X)$ are sheaves, and morphisms are defined in the same way as for presheaves.*

Example 4.1.6. *The exterior derivative $d : \mathcal{E}^p_X \to \mathcal{E}^{p+1}_X$ is a morphism of sheaves.*

A special case of a morphism is the notion of a *subsheaf* of a sheaf. This is a morphism of sheaves $f : \mathcal{P} \to \mathcal{P}'$ where each $f_U : \mathcal{P}(U) \subseteq \mathcal{P}'(U)$ is an inclusion.

Example 4.1.7. *The sheaf of C^∞-functions on $\mathbb{R}^n$ is a sub sheaf of the sheaf of continuous functions.*

Example 4.1.8. *Let Y be a closed subset of a complex manifold $(X, \mathcal{O}_X)$, the ideal sheaf associated to Y,*

$$\mathcal{I}_Y(U) = \{f \in \mathcal{O}_X(U) \mid f|_Y = 0\},$$

is a subsheaf of $\mathcal{O}_X$

Example 4.1.9. *Given a sheaf of rings of functions $\mathcal{R}$ over X, and $f \in \mathcal{R}(X)$, the collection of maps $\mathcal{R}(U) \to \mathcal{R}(U)$ given by multiplication by $f|_U$ is a morphism.*

Example 4.1.10. *Let X be a C^∞ manifold, then $d : C^\infty_X \to \mathcal{E}^1_X$ is a morphism of sheaves.*

Fix a space X and a point $x \in X$. We define the *stalk* $\mathcal{P}_x$, of a presheaf $\mathcal{P}$ at x, as the direct limit $\varinjlim \mathcal{P}(U)$ over neighbourhoods of x. The elements can be represented by germs of sections of $\mathcal{P}$ in concrete cases considered earlier. Given a morphism $\phi : \mathcal{P} \to \mathcal{P}'$, we get an induced map $\mathcal{P}_x \to \mathcal{P}'_x$ taking the germ of f to the germ of $\phi(f)$. This gives a functor from $PAb(X) \to Ab$.

Theorem 4.1.11. *There is a functor $\mathcal{P} \mapsto \mathcal{P}^+$ from $PAb(X) \to Ab(X)$ called sheafification, with the following properties:*

(a) There is a canonical morphism $\mathcal{P} \to \mathcal{P}^+$.

(b) If $\mathcal{P}$ is a sheaf then the morphism $\mathcal{P} \to \mathcal{P}^+$ is an isomorphism.

(c) Any morphism from $\mathcal{P}$ to a sheaf factors uniquely through $\mathcal{P} \to \mathcal{P}^+$

(d) The map $\mathcal{P} \to \mathcal{P}^+$ induces an isomorphism on stalks.

We just explain a construction when $\mathcal{P}$ is a presheaf of functions. We can define

$$\mathcal{P}^+(U) = \{f : U \to T \mid \forall x \in U, \exists \text{ a nbhd } U_x \text{ of } x, \text{ such that } f|_{U_x} \in \mathcal{P}(U_x)\}$$

In other words, we add enough to make it into a sheaf. Properties (a) and (b) should be clear, and (d) can also be checked with a bit of thought.

4.2 Exact Sequences

The categories $PAb(X)$ and $Ab(X)$ are *additive* which means among other things that $Hom(A, B)$ has an Abelian group structure such that composition is bilinear. Actually, more is true. These categories are *Abelian* (see Weibel or any other modern book on homological algebra for the definition). This implies that they possess many of the basic constructions and properties of the category of Abelian groups. In particular, there are intrinsic notions of exactness in these categories. We give a non intrinsic, but equivalent, formulation of this notion for $Ab(X)$.

Definition 4.2.1. *A sequence of sheaves on X*

$$\ldots \mathcal{A} \to \mathcal{B} \to \mathcal{C} \ldots$$

is called exact in the if and only if the sequence of stalks

$$\ldots \mathcal{A}_x \to \mathcal{B}_x \to \mathcal{C}_x \ldots$$

is exact for every $x \in X$.

The key point is that no matter how complicated X and the sheaves are globally, exactness is a local issue, and this what gives the notion its power. We will let the symbols $\mathcal{A}, \mathcal{B}, \mathcal{C}$ stand for sheaves for the remainder of this section unless stated otherwise. We will also say that morphism $\mathcal{A} \to \mathcal{B}$ is a *monomorphism* (respectively *epimorphism*) if $\mathcal{A}_x \to \mathcal{B}_x$ is injective (respectively surjective) for all $x \in X$

Lemma 4.2.2. *Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$, then $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is exact if and only if for any open $U \subseteq X$*

(1) $g_U \circ f_U = 0$.

(2) Given $b \in \mathcal{B}(U)$ with $g(b) = 0$, there exists an open cover $\{U_i\}$ of U and $a_i \in \mathcal{A}(U_i)$ such that $f_{U_i}(a_i) = b|_{U_i}$.

Proof. We will prove one direction, leaving the other as an exercise. Suppose that $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is exact. To simply notation, we write suppress the subscript U. Given $a \in \mathcal{A}(U)$, $g(f(a)) = 0$, since $g(f(a))_x = g(f(a_x)) = 0$ for all $x \in U$. This shows (1).

Given $b \in \mathcal{B}(U)$ with $g(b) = 0$, then for each $x \in U$, b_x is the image of a germ in $\mathcal{A}_x$. Choose a representative a_i for this germ in some $\mathcal{A}(U_i)$ where U_i is a neighbourhood of x. After shrinking U_i if necessary, we have $f(a_i) = b|_{U_i}$. As x varies, we get an open cover $\{U_i\}$, and sections $a_i \in \mathcal{A}(U_i)$ as required. □

Corollary 4.2.3. *If $\mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U)$ is exact for every open set U, then $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is exact.*

The converse is false, but we do have:

Lemma 4.2.4. *If*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is an exact sequence of sheaves, then

$$0 \to \mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U)$$

is exact for every open set U.

Proof. Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ denote the maps. By lemma 4.2.2, $g_u \circ f_U = 0$. Suppose $a \in \mathcal{A}(U)$ maps to 0 under f, then $f(a_x) = f(a)_x = 0$ for each $x \in U$ (we are suppressing the subscript U once again). Therefore $a_x = 0$ for each $x \in U$, and this implies that $a = 0$.

Suppose $b \in \mathcal{B}(U)$ satisfies $g(b) = 0$. Then by lemma 4.2.2, there exists an open cover $\{U_i\}$ of U and $a_i \in \mathcal{A}(U_i)$ such that $f(a_i) = b|_{U_i}$. Then $f(a_i|_{U_i \cap U_j} - a_j|_{U_i \cap U_j}) = 0$, which implies $a_i|_{U_i \cap U_j} - a_j|_{U_i \cap U_j} = 0$ by the first paragraph. Therefore $\{a_i\}$ patch together to yield an element of $a \in \mathcal{A}(U)$ such that $f(a) = b$.

□

We give some *natural* examples to show that $\mathcal{B}(X) \to \mathcal{C}(X)$ is not usually surjective.

Example 4.2.5. *Consider the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Let $\mathbb{R}_{S^1}$ denote the sheaf of locally constant $\mathbb{R}$-valued function. Then*

$$0 \to \mathbb{R}_{S^1} \to C^\infty_{S^1} \xrightarrow{d} \mathcal{E}^1_{S^1} \to 0$$

is exact. However $C^\infty(S^1) \to \mathcal{E}^1(S^1)$ is not surjective.

To see the first statement, let $U \subset S^1$ be an open set diffeomorphic to an open interval. Then the sequence

$$0 \to \mathbb{R} \to C^\infty(U) \xrightarrow{f \to f'} C^\infty(U)dx \to 0$$

is exact by calculus. Thus we get exactness on stalks. The 1-form dx gives a global section of $\mathcal{E}^1(S^1)$ since it is translation invariant. However, it is not the differential a periodic function. Therefore $C^\infty(S^1) \to \mathcal{E}^1(S^1)$ is not surjective.

Example 4.2.6. *Let $(X, \mathcal{O}_X)$ be a complex manifold and $Y \subset X$ a submanifold. Let*

$$\mathcal{I}_Y(U) = \{f \in \mathcal{O}_X(U) \mid f|_Y = 0\}$$

then this is a sheaf called the ideal sheaf of Y, and

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

is exact. The map $\mathcal{O}_X(X) \to \mathcal{O}_Y(X)$ need not be surjective. For example, let $X = \mathbb{P}^1_{\mathbb{C}}$ with $\mathcal{O}_X$ the sheaf of holomorphic functions. Let $Y = \{p_1, p_2\} \subset \mathbb{P}^1$ be a set of distinct points. Then the function $f \in \mathcal{O}_Y(X)$ which takes the value 1 on p_1 and 0 on p_2 cannot be extended to a global holomorphic function on $\mathbb{P}^1$ since all such functions are constant by Liouville's theorem.

4.3 Sheaf cohomology

Let Ab denote the category of abelian groups. Here is the basic result

Theorem 4.3.1. *Given a space X, there exists a sequence of functors*

$$H^i(X, -) : Ab(X) \to Ab, i \in \mathbb{N}$$

with the following properties:

1. *$H^0(X, \mathcal{A}) \cong \mathcal{A}(X)$*
2. *Given a short exact sequence*

   $$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

   of sheaves, there is a long exact sequence

   $$0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to H^0(X, \mathcal{C}) \to H^1(X, \mathcal{A}) \to H^1(X, \mathcal{B}) \to \dots$$

   extending what we found in lemma 4.2.4.
3. *There is a class of sheaves called injective sheaves, such that*
   (a) *any sheaf can be embedded (via a monomorphism) into an injective sheaf,*
   (b) *an injective sheaf $\mathcal{B}$ is acyclic, which means $H^i(X, \mathcal{B}) = 0$ for all $i > 0$.*

We omit the proof, or for that matter the definition of an injective object, since it is better left for a class in homological algebra. As an abstract statement it certainly does have content. However, to be truly useful, we need to better understand what these higher cohomology groups are, and how to compute them in good cases. In particular, we would like to know when $H^1(X, \mathcal{A}) = 0$, as this would tell us that $\mathcal{B}(X) \to \mathcal{C}(X)$ is surjective. Of course, injective sheaves have this property, but it is hard to find examples "in nature". Fortunately, there is more accessible class of sheaves with this property.

Definition 4.3.2. *A sheaf $\mathcal{A}$ is flasque (or flabby) if $\mathcal{A}(X) \to \mathcal{A}(U)$ is surjective for all open U.*

We note that the class of flasque is wider:

Proposition 4.3.3. *Injective sheaves are flasque.*

Proof. Hartshorne, p 207. □

Here are a couple of natural examples.

Example 4.3.4. *Skyscraper sheaves are flasque.*

Example 4.3.5. *Let X be an irreducible algebraic variety some field with its* Zariski topology *(closed sets are finite unions of subvarieties). A preseaf of locally constant functions (with any target) is a flasque sheaf.*

Lemma 4.3.6. *If $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is an exact sequence of sheaves with $\mathcal{A}$ flasque, then $\mathcal{B}(X) \to \mathcal{C}(X)$ is surjective.*

Proof. Let's assume for simplicity that X has a countable basis. (Although this isn't necessary, it holds in all the cases that we care about.) Let $\gamma \in \mathcal{C}(X)$. By assumption, there is an open cover $\{U_i\}_{i\in\mathbb{N}}$, such that $\gamma|_{U_i}$ lifts to a section $\beta_i \in \mathcal{B}(U_i)$. We will define

$$\sigma_i \in \mathcal{B}(\bigcup_{j<i} U_j)$$

inductively, so that it maps to the restriction of γ. Set $\sigma_1 = \beta_0$. Now suppose that σ_i exists. Let $U = U_i \cap (\cup_{j<i} U_j)$. Then $\beta_i|_U - \sigma_i|_U$ is the image of a section $\alpha_i' \in \mathcal{A}(U)$. By hypothesis α_i' extends to a global section $\alpha_i \in \mathcal{A}(X)$. Then set

$$\sigma_{i+1} = \begin{cases} \sigma_i \text{ on } \bigcup_{j<i} U_j \\ \beta_i - \alpha_i|_{U_i} \text{ on } U_i \end{cases}$$

□

Corollary 4.3.7. *If $\mathcal{A}$ is flasque, then $H^1(X, \mathcal{A}) = 0$.*

Proof. Embed $\mathcal{A}$ into an injective sheaf $\mathcal{B}$, we can complete this to an exact sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

by taking $\mathcal{C}$ to be the sheafification of $U \mapsto \mathcal{B}(U)/\mathcal{A}(U)$. Then we have

$$\mathcal{B}(X) \to \mathcal{C}(X) \to H^1(X, \mathcal{A}) \to 0$$

Since the first map is surjective, the corollary follows. □

More is true.

Theorem 4.3.8. *Flasque sheaves are acyclic.*

Proof. We show that $H^i(\mathcal{A}) = 0$ for $i > 0$ and $\mathcal{A}$ flasque by induction. The case of $i = 1$ was just proved. Now suppose that it holds for i. Let $\mathcal{A}$ be flasque. Embed it into an injective sheaf $\mathcal{B}$, and complete this to a short exact sequence

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

as above. By the previous lemma $\mathcal{B}(U) \to \mathcal{C}(U)$ is surjective. Since $\mathcal{B}$ is flasque, $\mathcal{B}(X) \to \mathcal{C}(X)$ is surjective. This forces $\mathcal{C}(X) \to \mathcal{C}(U)$ to be surjective. Therefore $\mathcal{C}$ is flasque. So by induction

$$0 = H^i(X, \mathcal{C}) \to H^{i+1}(X, \mathcal{A}) \to H^{i+1}(X, \mathcal{B}) = 0$$

□

For applications to number theory, Weil proposed in 1949 that there should exist a cohomology theory for algebraic varieties over finite fields, which behaves like singular cohomology, i.e. the kind of cohomology one learns in a topology class. Such theories (étale and crystalline cohomology) were constructed by Grothendieck and his collaborators in the early 60's. However, sheaf cohomology with the Zariski topology won't work, because you'll just zero by the previous theorem.

4.4 Soft sheaves

While flasque sheaves are better than injective sheaves in terms of finding examples, most sheaves won't be flasque. However, if we replace open by closed sets in the definition, this situation improves dramatically.

Definition 4.4.1. *A sheaf $\mathcal{A}$ is soft if for any closed set S, and section $\alpha \in \mathcal{A}(U)$, where $S \subset U$, there exists $\tilde{\alpha} \in \mathcal{A}(X)$ such that $\tilde{\alpha}|_{U'} = \alpha|_{U'}$ for some $S \subset U' \subseteq U$.*

The definition can be restated as saying the germ of α along S can be extended to an element of $\mathcal{A}(X)$. Here the set of germs $\mathcal{A}(S)$ can be defined like we did for stalks by taking a direct limit of $\mathcal{A}(U)$ as $U \supset S$ shrinks down to S.

Lemma 4.4.2. *Suppose that X is metric space (or more generally a paracompact Hausdorff space). If $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is an exact sequence of sheaves with $\mathcal{A}$ soft, then $\mathcal{B}(X) \to \mathcal{C}(X)$ is surjective.*

We should probably recall that a topological space is paracompact if any open cover has a locally finite subcover. Metric spaces are paracompact [Stone, Paracompactness and product spaces, Bull AMS, 1948]. Finally, we note that the conditions on X are essential, and this limits the utility of softness for the sorts of spaces that come in up in abstract algebraic geometry (schemes). This is why books like Hartshorne don't even discuss this concept.

Sketch. The argument is similar to the flasque case, so we just outline it. Let $\gamma \in \mathcal{C}(X)$. By the assumptions, there exists a locally finite open cover $\{U_i\}_{i\in I}$, such that $\gamma|_{U_i}$ lifts to $\beta_i \in \mathcal{B}(U_i)$. Choose another open cover $\{V_i\}$ such that $S_i = \overline{V}_i \subset U_i$. The differences $(\beta_i - \beta_j)|_{S_i\cap S_j}$ define local sections of $\mathcal{A}$. Softness of $\mathcal{A}$ allows us to choose lifts of these sections to global sections. This allows to correct β_i so they patch. □

Theorem 4.4.3. *On a paracompact Hausdorff space, soft sheaves are acyclic.*

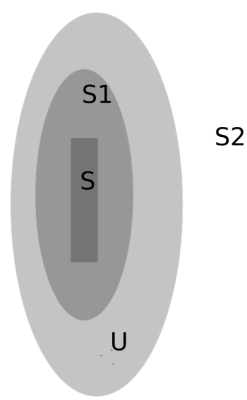
We omit the proof because it is very similar to the corresponding theorem for flasque sheaves.

Now let us consider examples.

Theorem 4.4.4. *The sheaf C_X of continuous real valued functions on a metric space X is soft. The sheaf of C^∞_X of C^∞ functions on a manifold is soft. More generally, a sheaf of modules $\mathcal{M}$ over C_X or C^∞_X is soft.*

Proof. The basic strategy for the proof of all these statements is the construction of a continuous or C^∞ "cutoff" function ρ which is 0 outside a given neighbourhood U of a closed set S, and 1 close to S. Suppose we have ρ. Then given a section $f \in \mathcal{M}(U)$ for a module over one of these rings, ρf can be extended by 0 to all of X. Since f and ρf have the same germ along S, this would prove the surjectivity of $\mathcal{M}(X) \to \mathcal{M}(S)$ as required

We spell out the construction of ρ in the continuous case. Let $S_1 \subset U$ be a closed set containing S in its interior. This can be constructed by expressing U as a union of open balls, and taking the union of closed balls of half the radii. Let $S_2 = X - U$. Then Urysohn's lemma from point set topology guarantees the existence of a continuous ρ, taking a value of 1 on S_1 and 0 on S_2.



□

We are ready to do a serious computation, which gives a special case of (a version of) de Rham's theorem.