Example 4.4.5. *From the exact sequence*

$$0 \to \mathbb{R}_{S^1} \to C^\infty_{S^1} \xrightarrow{d} \mathcal{E}^1_{S^1} \to 0$$

we deduce

$$C^\infty(S^1) \to \mathcal{E}^1(S^1) \to H^1(S^1, \mathbb{R}_{S^1}) \to 0$$

and

$$H^i(S^1, \mathbb{R}_{S^1}) = 0,\ i > 1$$

In other words, sheaf cohomology with these coefficients equals de Rham.

$$H^i(S^1, \mathbb{R}_{S^1}) \cong H^i_{DR}(S^1)$$

4.5 de Rham's theorem

We want to generalize the calculation in the last example. But first we need a method for computing all sheaf cohomology in one go.

Definition 4.5.1. *An acyclic resolution of a sheaf $\mathcal{F}$ is an exact sequence*

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \dots$$

of sheaves such that each $\mathcal{F}^i$ is acyclic (e.g. flasque or soft).

Let us write

$$\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$$

This a functor. Given an exact sequence of sheaves $\mathcal{F}^\bullet$, the induced sequence

$$\Gamma(X, \mathcal{F}^0) \to \Gamma(X, \mathcal{F}^1) \to \dots$$

may fail to be exact (as we've seen). But it is necessarily a complex of Abelian groups in the sense that the composition of any two consecutive maps is 0. Define the cohomology of this (or any complex) by

$$\mathcal{H}^i(\Gamma(X, \mathcal{F}^\bullet)) = \frac{\Gamma(X, \mathcal{F}^i) \to \Gamma(X, \mathcal{F}^{i+1})}{\Gamma(X, \mathcal{F}^{i-1}) \to \Gamma(X, \mathcal{F}^i)}$$

This measure the failure of exactness.

Theorem 4.5.2 (Theorem on acyclic resolutions)**.** *Given an acyclic resolution $\mathcal{F}^\bullet$ of $\mathcal{F}$,*

$$H^i(X, \mathcal{F}) \cong \mathcal{H}^i(\Gamma(X, \mathcal{F}^\bullet))$$

Proof. Let $\mathcal{K}^{-1} = \mathcal{F}$ and $\mathcal{K}^i = \ker(\mathcal{F}^{i+1} \to \mathcal{F}^{i+2})$ for $i \geq 0$. Then there are exact sequences

$$0 \to \mathcal{K}^{i-1} \to \mathcal{F}^i \to \mathcal{K}^i \to 0$$

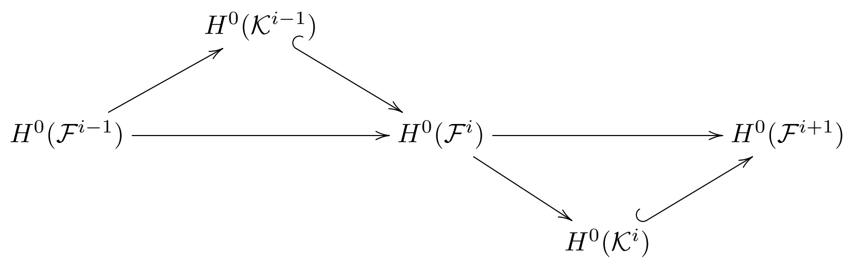
for $i \geq 0$. Since each $\mathcal{F}^i$ is acyclic, Theorem 4.3.1 implies that

$$0 \to H^0(\mathcal{K}^{i-1}) \to H^0(\mathcal{F}^i) \to H^0(\mathcal{K}^i) \to H^1(\mathcal{K}^{i-1}) \to 0 \tag{4.1}$$

is exact, and

$$H^j(\mathcal{K}^i) \cong H^{j+1}(\mathcal{K}^{i-1}) \tag{4.2}$$

for $j > 0$. We have a diagram



which is commutative since the morphism $\mathcal{F}^{i-1} \to \mathcal{F}^i$ factors through $\mathcal{K}^{i-1}$ and so on. The oblique line in the diagram is part of (4.1), so it is exact. In particular, the first hooked arrow is injective. The injectivity of the second hooked arrow follows for similar reasons. Thus

$$\text{im}[H^0(\mathcal{F}^i) \to H^0(\mathcal{K}^i)] = \text{im}[H^0(\mathcal{F}^i) \to H^0(\mathcal{F}^{i+1})] \tag{4.3}$$

Suppose that $\alpha \in H^0(\mathcal{F}^i)$ maps to 0 in $H^0(\mathcal{F}^{i+1})$, then it maps to 0 in $H^0(\mathcal{K}^i)$. Therefore α lies in the image of $H^0(\mathcal{K}^{i-1})$. Thus

$$H^0(\mathcal{K}^{i-1}) = \ker[H^0(\mathcal{F}^i) \to H^0(\mathcal{F}^{i+1})] \tag{4.4}$$

This already implies the theorem when $i = 0$. Replacing i by $i+1$ in (4.4), and combining it with (4.1) and (4.3) shows that

$$H^1(\mathcal{K}^{i-1}) \cong \frac{H^0(\mathcal{K}^i)}{\text{im}[H^0(\mathcal{F}^i) \to H^0(\mathcal{K}^i)]} = \frac{\ker[H^0(\mathcal{F}^{i+1}) \to H^0(\mathcal{F}^{i+2})]}{\text{im}[H^0(\mathcal{F}^i) \to H^0(\mathcal{F}^{i+1})]}$$

Combining this with the isomorphisms

$$H^{i+1}(\mathcal{F}) = H^{i+1}(\mathcal{K}^{-1}) \cong H^i(\mathcal{K}^0) \cong \dots H^1(\mathcal{K}^{i-1})$$

of (4.2) proves the theorem for positive exponents. □

Let $\mathbb{R}_X$ denote the of locally constant real valued functions on a space X. But ee often just write it as $\mathbb{R}$ when the context makes it clear.

Theorem 4.5.3 (de Rham's theorem)**.** *If X is a C^∞-manifold,*

$$H^i(X, \mathbb{R}_X) \cong H^i_{DR}(X)$$

Proof. We have a sequence of sheaves

$$\mathbb{R}_X \to \mathcal{E}^0_X \xrightarrow{d} \mathcal{E}^1_X \xrightarrow{d} \dots$$

where the first map is simply inclusion. We claim that the sequence

$$0 \to \mathbb{R}_X \to \mathcal{E}_X^0 \xrightarrow{d} \mathcal{E}_X^1 \xrightarrow{d} \dots \tag{4.5}$$

is exact. Exactness at the first step is clear, since the map is inclusion. For the rest of the sequence, we check exactness at stalks. Then we can replace X by a ball. Since this is diffeomorphic to $\mathbb{R}^n$, exactness follows from the the Poincaré lemma. Since $\mathcal{E}_X^i$ are soft, tells us that (4.5) is an acyclic resolution. Therefore this theorem follows from the previous theorem. □

One might complain that de Rham's theorem is supposed to say that de Rham cohomology is the same as *singular* cohomology with real coefficients. It is easy to deduce this too by showing singular cohomology equals sheaf cohomology. This can be proved by another acyclic resolution. Finally, we note that we can define complex valued de Rham cohomology $H_{dR}^*(X, \mathbb{C})$ by using complex valued forms. The argument as above shows

$$H^i(X, \mathbb{C}_X) \cong H_{dR}^i(X, \mathbb{C})$$

4.6 Dolbeault's theorem

In this section, X will be a complex manifold and $\mathcal{E}^\bullet(X)$ will stand for the space of complex valued forms. Recall that Dolbeault cohomology is the cohomology of the complex

$$\mathcal{E}^{p0}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p1}(X) \xrightarrow{\bar\partial} \dots$$

The key fact that we need is an analogue of Poincaré's lemma for the Cauchy-Riemann operator.

Theorem 4.6.1. *If Δ is a polydisk, then given $q > 0$ and $\alpha \in \mathcal{E}^{pq}(\bar\Delta)$ satisfying $\bar\partial\alpha = 0$, there exists $\beta \in \mathcal{E}^{p,q-1}(\Delta)$ such that $\alpha = \bar\partial\beta$.*

A proof can be found on p 25 of Griffiths-Harris for example. We just indicate how it goes in one variable. Let $\Delta \subset \mathbb{C}$ be an open disk. Given $f \in C^\infty(\bar\Delta)$, we need to find a function $g \in C^\infty(\Delta)$ such that $\frac{\partial g}{\partial \bar z} = f$. A version of Cauchy's formula shows that

$$g(\zeta) = \frac{1}{2\pi i} \iint_\Delta \frac{f(z)}{z - \zeta}\, dz \wedge d\bar z$$

gives the desired solution.

Theorem 4.6.2 (Dolbeault's theorem)**.**

$$H^q(X, \Omega_X^p) \cong H_{Dol}^{pq}(X)$$

Proof. Let $\mathcal{E}_X^{pq}$ denote the sheaf of C^∞ (p, q) forms. This is soft. We have sequence of sheaves

$$0 \to \Omega_X^p \to \mathcal{E}_X^{p0} \xrightarrow{\bar\partial} \mathcal{E}_X^{p1}(X) \xrightarrow{\bar\partial} \dots$$

where the first morphism is inclusion. We claim this sequence is exact. We check this at stalks. For this, we can replace X by a polydisk and apply the previous theorem. Therefore we have an acyclic resolution, and the theorem follows. $\square$

4.7 Poincaré duality

Let X be a C^∞ manifold. Let $\mathcal{E}_c^k(X)$ denote the set of C^∞ k-forms with compact support. Since $d\mathcal{E}_c^k(X) \subset \mathcal{E}_c^{k+1}(X)$, these form a complex.

Definition 4.7.1. *Compactly supported de Rham cohomology is defined by* $H^k_{cdR}(X) = \mathcal{H}^k(\mathcal{E}_c^\bullet(X))$.

Lemma 4.7.2. *For all n,*

$$H^k_{cdR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} \text{ if } k = n \\ 0 \text{ otherwise} \end{cases}$$

Proof. See Spivak. $\square$

This computation suggests that these groups are roughly opposite to the usual de Rham groups. There is another piece of evidence, which is that H_{cdR} behaves covariantly in certain cases. For example, given an open set $U \subset X$, a form in $\mathcal{E}_c^k(U)$ can be extended by zero to $\mathcal{E}_c^k(X)$. This induces a map $H^k_{cdR}(U) \to H^k_{cdR}(X)$.

The precise statement of duality requires the notion of orientation. An orientation on an n dimensional real vector space V is a connected component of $\wedge^n V - \{0\}$ (there are two). An ordered basis $v_1, \ldots v_n$ is positively oriented if $v_1 \wedge \ldots \wedge v_n$ lies in the given component. If V were to vary, there is no guarantee that we could choose a orientations consistently. So we make a definition:

Definition 4.7.3. *An n dimensional manifold X is called orientable if $\wedge^n T_X$ minus its zero section has two components. If this is the case, an orientation is a choice of one of these components.*

This is equivalent to the definition we gave earlier, but more convenient. The following test is immediate.

Lemma 4.7.4. *An n-manifold is orientable if it has a nonwhere zero C^∞ n-form.*

Theorem 4.7.5 (Poincaré duality, version I)**.** *Let X be a connected oriented n-dimensional manifold. Then*

$$H^k_{cdR}(X) \cong H^{n-k}(X, \mathbb{R})^*$$

There is a standard proof of this using currents, which are to forms what distributions are to functions. However, we can get by with something much weaker. We define the space of *pseudocurrents* of degree k on an open set $U \subset X$ to be

$$\mathcal{C}^k(U) = \mathcal{E}_c^{n-k}(U)^* := Hom(\mathcal{E}_c^{n-k}(U), \mathbb{R}).$$

This is "pseudo" because we are using the ordinary (as opposed to topological) dual. We make this into a presheaf as follows. Given $V \subseteq U$, $\alpha \in \mathcal{C}_X^k(U)$, $\beta \in \mathcal{E}_c^{n-k}(V)$, define $\alpha|_V(\beta) = \alpha(\tilde{\beta})$ where $\tilde{\beta}$ is the extension of β by 0.

Lemma 4.7.6. *$\mathcal{C}_X^k$ is a sheaf.*

Proof. Let $\{U_i\}$ be an open cover of U, which we may assume is locally finite. Suppose that $\alpha_i \in \mathcal{C}_X^k(U_i)$ is a collection of sections such that $\alpha_i|_{U_i \cap U_j} = \alpha_j|_{U_i \cap U_j}$. This means that $\alpha_i(\beta) = \alpha_j(\beta)$ if β has support in $U_i \cap U_j$. Let $\{\rho_i\}$ be a C^∞ partition of unity subordinate to $\{U_i\}$. Then define $\alpha \in \mathcal{C}_X^k(U)$ by

$$\alpha(\beta) = \sum_i \alpha_i(\rho_i \beta|_{U_i})$$

We have to show that $\alpha(\tilde{\beta}) = \alpha_j(\beta)$ for any $\beta \in \mathcal{E}_c^{n-k}(U_j)$ with $\tilde{\beta}$ its extension to U by 0. The support of $\rho_i\tilde{\beta}$ lies in $U_i \cap supp(\beta) \subset U_i \cap U_j$, so only finitely many of these are nonzero. Therefore

$$\alpha(\tilde{\beta}) = \sum_i \alpha_i(\rho_i\tilde{\beta}) = \sum_i \alpha_j(\rho_i\tilde{\beta}) = \alpha_j(\beta)$$

as required. We leave it to the reader to check that α is the unique current with this property. ☐

Define a map $\delta : \mathcal{C}_X^k(U) \to \mathcal{C}_X^{k+1}(U)$ by $\delta(\alpha)(\beta) = (-1)^{k+1}\alpha(d\beta)$. One automatically has $\delta^2 = 0$. Thus one has a complex of sheaves.

Let X be an oriented n-dimensional manifold. Then we will recall [Spivak,...] that one can define an integral $\int_X \alpha$ for any n-form $\alpha \in \mathcal{E}_c^n(X)$. Using a partition of unity, the definition can reduced to the case where α is supported in a coordinate neighbourhood U. Then we can write $\alpha = f(x_1, \ldots x_n)dx_1 \wedge \ldots \wedge dx_n$, where the order of the coordinates are chosen so that $\partial/\partial x_1, \ldots, \partial/\partial x_n$ gives a positive orientation of T_X. Then

$$\int_X \alpha = \int_{\mathbb{R}^n} f(x_1, \ldots x_n)dx_1 \ldots dx_n$$

The functional $\int_X$ defines a canonical global section of $\mathcal{C}_X^0$.

Theorem 4.7.7 (Stokes' theorem)**.** *Let X be an oriented n-dimensional manifold, then $\int_X d\beta = 0$.*

Proof. See Spivak or almost any book on manifolds. ☐

Corollary 4.7.8. $\int_X \in ker[\delta]$.

We define a map $\mathbb{R}_X \to \mathcal{C}^0_X$ by sending $r \to r\int_X$. The key lemma to establish Theorem 4.7.5 is:

Lemma 4.7.9.
$$0 \to \mathbb{R}_X \to \mathcal{C}^0_X \to \mathcal{C}^1_X \to \dots$$
is an acyclic resolution.

Proof. Lemma 4.7.2 implies that this complex is exact. Given $f \in C^\infty(U)$ and $\alpha \in \mathcal{C}^k(U)$ define
$$f\alpha(\beta) = \alpha(f\beta)$$
This makes $\mathcal{C}^k$ into a C^∞-module, and it follows that it is soft and therefore acyclic. □

Proof of theorem 4.7.5. We can now use the complex $\mathcal{C}^\bullet_X$ to compute the cohomology of $\mathbb{R}_X$ to obtain
$$H^i(X,\mathbb{R}) \cong \mathcal{H}^i(\mathcal{C}^\bullet_X(X)) = \mathcal{H}^i(\mathcal{E}_c^{n-\bullet}(X)^*).$$
The right hand space is isomorphic to $H^i_{cdR}(X,\mathbb{R})^*$. This completes the proof of the theorem. □

Corollary 4.7.10. *If X is a compact oriented n-dimensional manifold. Then*
$$H^k(X,\mathbb{R}) \cong H^{n-k}(X,\mathbb{R})^*$$

The following is really a corollary of the proof.

Corollary 4.7.11. *If X is a connected oriented n-dimensional manifold. Then the map $\alpha \mapsto \int_X \alpha$ induces an isomorphism*
$$\int_X : H^n_{cdR}(X,\mathbb{R}) \cong \mathbb{R}$$

We can make the Poincaré duality isomorphism more explicit:

Theorem 4.7.12 (Poincaré duality, version II)**.** *If $f \in H^{n-k}_{cdR}(X)^*$, then there exists a closed form $\alpha \in \mathcal{E}^k(X)$ such that $f([\beta]) = \int_X \alpha\wedge\beta$. Moreover the class $[\alpha] \in H^k_{dR}(X)$ is unique.*

If $\alpha \in \mathcal{E}^i(X)$ and $\beta \in \mathcal{E}^j_c(X)$ are closed forms, then $\alpha\wedge\beta$ is also closed by the Leibnitz rule. The *cup product* of the associated cohomology classes is defined by $[\alpha]\cup[\beta] = [\alpha\wedge\beta]$. This is a well defined operation which makes de Rham cohomology into a graded ring when X is compact. The theorem tells us that:

Corollary 4.7.13. *The cup product followed by integration gives a nondegenerate pairing*
$$H^k_{dR}(X) \times H^{n-k}_{cdR}(X) \to H^n_{cdR}(X) \cong \mathbb{R}$$

Here is a simple example to illustrate of this.

Example 4.7.14. *Consider the torus $T = \mathbb{R}^n/\mathbb{Z}^n$. We will show later that: Every de Rham cohomology class on T contains a unique form with constant coefficients. This will imply that there is an algebra isomorphism $H^*(T, \mathbb{R}) \cong \wedge^*\mathbb{R}^n$. Poincaré duality becomes the standard isomorphism*

$$\wedge^k\mathbb{R}^n \cong \wedge^{n-k}\mathbb{R}^n.$$

4.8 Cech interpretation for H^1

There is another approach to sheaf cohomology which is quite explicit, and this makes it useful for many computations. Let us start with a sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

on X. Given a section $\gamma \in \mathcal{C}(X)$, let us try to directly understand when it lifts to a section $\beta \in \mathcal{B}(X)$. We can find an open cover $\mathcal{U} = \{U_i\}$ and sections $\beta_i \in \mathcal{B}(U_i)$ which map to $\gamma|_{U_i}$. Let $U_{ij} = U_i \cap U_j$, then

$$\alpha_{ij} = \beta_i|_{U_{ij}} - \beta_j|_{U_{ij}}$$

can viewed as a collection of sections of $\mathcal{A}(U_{ij})$. These satisfy the 1-cocycle identies

$$\alpha_{ii} = 0$$

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 0$$

We say that it is 1-coboundary if we can find a collection $\alpha_i \in \mathcal{A}(U_i)$ such that

$$\alpha_{ij} = \alpha_i|_{U_{ij}} - \alpha_j|_{U_{ij}}$$

We now come to be key observation:

Lemma 4.8.1. *If α_{ij} is a coboundary, then the sections $\beta_i - \alpha_i$ will patch to form a global section of $\mathcal{B}$ lifting γ.*

We can put of all this together Let $Z^1(\mathcal{U}, \mathcal{A})$ denote the set of 1-cocyles. It is naturally an abelian group. Let $B^1(\mathcal{U}, \mathcal{A})$ denote subgroup of 1-coboundaries.

Definition 4.8.2. *The first Cech cohomology groups*

$$\check{H}^1(\mathcal{U}, \mathcal{A}) = Z^1(\mathcal{U}, \mathcal{A})/B^1(\mathcal{U}, \mathcal{A})$$

$$\check{H}^1(X, \mathcal{A}) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{A})$$

where covers $\mathcal{U}$ are ordered by refinement in the direct limit.

Lemma 4.8.3. *Sending γ above to the class of $\{\alpha_{ij}\}$ yields a map*

$$H^0(X, \mathcal{C}) \xrightarrow{\partial} \check{H}^1(X, \mathcal{A})$$

such that $\partial(\gamma) = 0$ iff γ lifts to $H^0(X, \mathcal{B})$.

Corollary 4.8.4. *There is an exact sequence*

$$0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to H^0(X, \mathcal{C}) \to \check{H}^1(X, \mathcal{A})$$

This says the first Cech group is doing the same job as the first sheaf cohomology. In fact:

Theorem 4.8.5. *There is an isomorphism $H^1(X, \mathcal{A}) \cong \check{H}^1(X, \mathcal{A})$ compatible with the connecting maps ∂.*

There are also higher Cech cohomology groups, where cocycles for a fixed cover are collections $\alpha_{i_0,\ldots,i_n} \in \mathcal{A}(U_{i_0} \cap \ldots U_{i_n})$ satisfying appropriate conditions. But the story gets more complicated. Cech groups agree with sheaf cohomology groups under mild assumptions, but need not agree in all cases.