Chapter 5

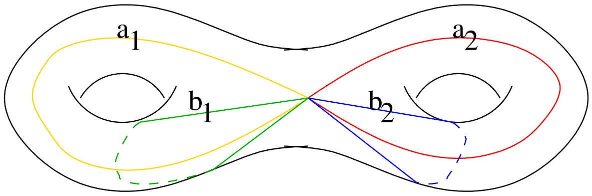
Riemann surfaces

5.1 Topology

Let X be a connected compact Riemann surface. We showed long ago that X is orientable. As a topological space, X is completely understood. The following fact is classical.

Theorem 5.1.1. *A compact orientable* 2*-manifold is homeomorphic to either* S^2 *or a connected sum of a finite number of* 2*-tori.*

See for example [Donaldson, Riemann surfaces] for an explanation of how to prove it. We define the genus of X to be 0 if it is S^2, or g if it is a connected sum of g tori. In more informal terms, X is a g-holed donut.



The first homology group $H_1(X,\mathbb{Z})$ of a space X is defined precisely in any basic book in algebraic topology, such as Hatcher. Very roughly, it is the free abelian group generated by closed paths $\gamma : [0,1] \to X$ modulo the boundaries of embedded "surfaces" in X. See Hatcher for the key computation:

Theorem 5.1.2. *If* X *is a genus* g *Riemann surface, then*

$$H_1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$$

See above picture for a basis of H_1 in the genus 2 case.

De Rham's theorem takes an explicit form. We write $H^i_{dR}(X,\mathbb{R})$ (resp. $H^i_{DR}(X,\mathbb{C})$) for de Rham cohomology using real (resp. complex) valued forms.

Theorem 5.1.3. *If X is a manifold, then*

$$H^1_{dR}(X,\mathbb{R}) \cong Hom(H_1(X,\mathbb{Z}),\mathbb{R})$$

$$H^1_{dR}(X,\mathbb{C}) \cong Hom(H_1(X,\mathbb{Z}),\mathbb{C})$$

where the isomorphisms send a closed form α to

$$\gamma \mapsto \int_\gamma \alpha$$

Corollary 5.1.4. *If X is a genus g Riemann surface,*

$$H^1_{dR}(X,\mathbb{C}) \cong \mathbb{C}^{2g}$$

Finally, we note that $H^1(X,\mathbb{C})$ has a bilinear form given by

$$(\alpha,\beta) = \int_X \alpha \wedge \beta$$

This is skew symmetric. Poincaré duality tells that this is nondegenerate, i.e. any matrix representing it is nonsingular. Under the de Rham isomorphism above, the above pairing is compatible with intersection pairing on homology.

5.2 Sheaf cohomology

Let X be a compact Riemann surface of genus g. Then

$$H^i(X,\mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } i = 0,2 \\ \mathbb{C}^{2g} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

The case $i = 0$ is elementary, $i = 1$ was explained earlier, and the remaining cases follow (for example) from Poincaré duality. In this case, Dolbeault's theorem amounts to the following statements

$$H^1(X,\mathcal{O}_X) = \frac{\mathcal{E}^{(0,1)}(X)}{\bar{\partial}C^\infty(X)}$$

$$H^1(X,\Omega^1_X) = \frac{\mathcal{E}^{(1,1)}(X)}{\bar{\partial}\mathcal{E}^{(1,0)}_X(X)}$$

and

$$H^i(X,\mathcal{O}_X) = H^i(X,\Omega^1_X) = 0$$

if $i > 1$.

Next, we give a holomorphic analogue of the de Rham complex.

Proposition 5.2.1. *There is an exact sequence of sheaves*

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \to 0$$

Proof. The only nontrivial part of the assertion is that $\mathcal{O}_X \to \Omega^1_X$ is an epimorphism. We can check this by replacing X by a disk D. A holomorphic 1-form α on D is automatically closed, therefore $\alpha = df$ by the usual Poincaré lemma. Since df is holomorphic, $\bar\partial f = 0$. Therefore f is holomorphic. □

Corollary 5.2.2. *There is a long exact sequence*

$$0 \to H^0(X, \mathbb{C}) \to H^0(X, \mathcal{O}_X) \to H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C}) \dots$$

Holomorphic 1-forms are closed, and

$$H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C})$$

is the map which sends a holomorphic form to its class in (complex valued) de Rham cohomology.

Lemma 5.2.3. *This map is an injection.*

Proof. Since global holomorphic functions on X are constant

$$H^0(X, \mathbb{C}) \to H^0(X, \mathcal{O}_X)$$

is surjective. □

It follows that $\dim H^0(X, \Omega^1) \leq 2g$. In fact, we can we will show later that

Theorem 5.2.4. *If X is a compact Riemann surface of genus g, then*

$$\dim H^0(X, \Omega) = H^1(X, \mathcal{O}_X) = g$$

So this will give another interpretation of genus. For now we prove a weaker statement.

Lemma 5.2.5. $\dim H^0(X, \Omega^1) \leq g$

Proof. If $\alpha, \beta \in H^0(X, \Omega^1)$, then $\alpha \wedge \beta = 0$ because it would be a $(2, 0)$ form on a 1 dimensional complex manifold. Therefore $(\alpha, \beta) = 0$. This says that $H^0(X, \Omega^1)$ is an *isotropic* subspace. The bound follows from the following fact from linear algebra:

Theorem 5.2.6. *A finite dimensional vector space V with a nondegenerate skew symmetric form is even dimensional. An isotropic subspace has at most half the dimension of V.*

□

5.3 Harmonic forms

We will prove theorem 5.2.4. We start with a seemingly unrelated problem. Recall that de Rham cohomology

$$H^1_{dR}(X,\mathbb{C}) = \frac{\{\alpha \in \mathcal{E}^1(X) \mid d\alpha = 0\}}{\{df \mid f \in C^\infty(X)\}}$$

where $\mathcal{E}^1 = \mathcal{E}^1_{\mathbb{C}}$. So an element of it is really an equivalence class. *Does such a class have a distinguished representative?* The answer will turn out to be yes. To describe it, let us introduce a $\mathbb{C}$-linear operation called the Hodge star given locally by $*dx = dy$, $*dy = -dx$. This is amounts to multiplication by i in the cotangent plane, so it is globally well defined operation. It follows that

Lemma 5.3.1.

$$\langle \alpha, \beta \rangle = (\alpha, *\bar{\beta}) = \int_X \alpha \wedge *\bar{\beta}$$

defines an inner product on $\mathcal{E}^1(X)$.

Proof. One can see that

$$(fdx + gdy) \wedge *\overline{(fdx + gdy)} = (|f|^2 + |g|^2) dx \wedge dy$$

This implies positive definiteness. The other properties are routine. ☐

Definition 5.3.2. *We define a* 1*-form* α *to be co-closed if* $d(*\alpha) = 0$. *It is harmonic if it is both closed and co-closed, i.e.* $d\alpha = d(*\alpha) = 0$. *A form is called co-exact if it equals* $*df$.

We will explain why these are called harmonic later. The basic properties are given by:

Proposition 5.3.3.

- *(a) A harmonic* 1*-form is a sum of a* $(1,0)$ *harmonic form and* $(0,1)$ *harmonic form.*
- *(b) A* $(1,0)$*-form is holomorphic iff it is closed iff is harmonic.*
- *(c) A* $(0,1)$*-form is harmonic iff it is antiholomorphic i.e. its complex conjugate is holomorphic.*
- *(d) The space of co-closed (closed) forms is orthogonal to the space of exact (co-exact) forms. In particular, the space of harmonic forms is orthogonal to both spaces.*

Proof. If α is a harmonic 1-form, then $\alpha = \alpha' + \alpha''$, where $\alpha' = \frac{1}{2}(\alpha + i * \alpha)$ is a harmonic $(1,0)$-form and $\alpha'' = \frac{1}{2}(\alpha - i * \alpha)$ is a harmonic $(0,1)$-form.

If α is $(1,0)$, then $d\alpha = \bar{\partial}\alpha$. This implies the first half (b). For the second half, use the identity

$$*dz = *(dx + idy) = dy - idx = -idz$$

Finally, note that the harmoncity condition is invariant under conjugation, so the (c) follows from (b).

For (d), suppose α is co-closed. Then so is $\bar{\alpha}$. Integration by parts (essentially Stokes' theorem) implies

$$\langle df, \alpha\rangle = \int_X df \wedge *\bar{\alpha} = \int d(f * \bar{\alpha}) - \int_X f d * \bar{\alpha} = 0$$

Similarly, for α closed,

$$\langle \alpha, *df\rangle = \int_X \alpha \wedge * * d\bar{f} = -\int_X d(\bar{f}\alpha) + \int_X \bar{f} d\alpha = 0$$

□

Here is the key fact. We will say more about this in later on.

Theorem 5.3.4 (Hodge theorem)**.** *Any form in $\mathcal{E}^1(X)$ can be decomposed into a sum $\beta + df + *dg$, where β is harmonic and f, g are C^∞ functions.*

Corollary 5.3.5. *Every de Rham cohomology class has a unique harmonic representative.*

Proof. Suppose α is closed. Write $\alpha = \beta + df + *dg$ as above. Part d of the last proposition implies

$$|| * dg||^2 = \langle \alpha, *dg\rangle = 0$$

So α and the harmonic form β lie in the same cohomology class. Suppose $\alpha = \beta' + dg$ with β' harmonic. Then $\beta - \beta'$ is harmonic and exact. Applying part d again shows that it's zero.

□

Proposition 5.3.6. *$H^1(X, \mathcal{O}_X)$ is isomorphic to the space of antiholomorphic forms.*

Proof. Let $H \subset \mathcal{E}^{01}(X)$ denote the space of antiholomorphic forms. We will show that

$$\pi : H \to \mathcal{E}^{01}(X)/\operatorname{im}\bar{\partial}$$

is an isomorphism. Suppose that $\alpha \in \mathcal{E}^{01}(X)$. By theorem 5.3.4, we may choose a harmonic form β such that $\beta = \alpha + df + *dg$ for some $f, g \in C^\infty(X)$. Then the $(0,1)$ part of β gives an element $\beta' \in H$ such that $\beta' = \alpha + \bar{\partial}(f + ig)$. This shows that π is surjective.

Suppose that $\alpha \in \ker \pi$. Then $\alpha = \bar{\partial} f$ for some f. Therefore $\alpha + \bar{\alpha} = df$. Consequently $\alpha + \bar{\alpha}$ is exact and harmonic, so $\alpha + \bar{\alpha} = 0$. This implies $\alpha = 0$. □

Corollary 5.3.7. *$\alpha \mapsto \bar{\alpha}$ gives a conjugate linear isomorphism*

$$H^0(X, \Omega^1_X) \cong H^1(X, \mathcal{O}_X)$$

The last result is a special case of Serre duality.

Proof of theorem 5.2.4. Combining previous results shows that

$$\dim H^0(X, \Omega^1_X) = \dim H^1(X, \mathcal{O}_X) \leq g$$

We have an exact sequence

$$H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X)$$

which forces

$$2g \leq 2 \dim H^0(X, \Omega^1_X)$$

□

5.4 Riemann-Roch

Let X be Riemann surface of genus g.. A classical problem, sometimes called the Riemann-Roch problem, is construct a meromorphic function with prescribed zeros and poles.This data is a choice of finitely many points $p_1, \ldots, p_k$ with multiplicities $n_1, \ldots, n_k$, which we write as a formal sum $D = \sum n_i p_i$. D is called a *divisor*. The degree $\deg D = \sum n_i$. Write $M(U)$ for the field of meromorphic functions on $U \subseteq X$, and define the sheaf

$$\mathcal{O}_X(D)(U) = \{f \in M(U) \mid ord_{p_i} f \geq -n_i, \forall p_i \in U\}$$

A more precise form of the Riemann-Roch problem is to try calculate the dimension of the global sections of the above sheaf. The key result is as follows:

Theorem 5.4.1 (Riemann-Roch)**.** *The dimensions* $h^i(\mathcal{O}_X(D)) = \dim H^i(X, \mathcal{O}_X(D))$ *are finite for* $i = 0, 1$ *and zero for* $i \geq 2$. *We have*

$$\chi(\mathcal{O}_X(D)) := h^0(\mathcal{O}_X(D)) - h^1(\mathcal{O}_X(D)) = \deg D + 1 - g$$

Proof. Let $m(D) = \sum |n_i|$ denote the "mass" of D. The proof proceeds by induction on $m(D)$. The base case $m(D) = 0$ follows from the computations

$$h^0(\mathcal{O}_X) = 1, \quad h^1(\mathcal{O}_X) = g, \quad h^i(\mathcal{O}_X) = 0, i > 1$$

established earlier.

Now assume $m(D) > 0$ and that the theorem is true for $m(D') < m(D)$. Given p, we have an exact sequence

$$0 \to \mathcal{O}_X(-p) \to \mathcal{O}_X \to \mathbb{C}_p \to 0$$

Tensoring by $\mathcal{O}_X(D)$ gives

$$0 \to \mathcal{O}_X(D - p) \to \mathcal{O}_X(D) \to \mathbb{C}_p \to 0$$

because $\mathcal{O}(D) \otimes \mathbb{C}_p \cong \mathbb{C}_p$. Observe that $H^0(X, \mathbb{C}_p) = \mathbb{C}$ by definition and $H^i(X, \mathbb{C}_p) = 0$ for $i > 0$ because $\mathbb{C}_p$ is flasque. Thus

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D - p)) + \chi(\mathbb{C}_p) = \chi(\mathcal{O}_X(D - p)) + 1$$

Therefore

$$\chi(\mathcal{O}(D)) - \deg D = \chi(\mathcal{O}(D-p)) - \deg(D-p)$$

or by changing variable and writing the formula backwards

$$\chi(\mathcal{O}(D)) - \deg D = \chi(\mathcal{O}(D+p)) - \deg(D+p)$$

We can choose p such that $m(D \pm p) < m(D)$. So one of these two formulas shows that $\chi(\mathcal{O}_X(D)) - \deg D$ equals $1-g$.

$\square$

Corollary 5.4.2 (Riemann's inequality)**.**

$$h^0(\mathcal{O}_X(D)) \geq \deg D + 1 - g$$

In particular, this contains a nonzero function when $\deg D \geq g$.

To appreciate what this tells, let us classify surfaces of genus 0. We have at least one example, namely $\mathbb{P}^1_{\mathbb{C}}$ (= the Riemann sphere).

Theorem 5.4.3. *A compact Riemann surface of genus* 0 *is isomorphic to* $\mathbb{P}^1_{\mathbb{C}}$.

Sketch. Let X have genus 0. Choose a point $p \in X$. Riemann's inequality tells us there exists a meromorphic function f with a simple pole at p and no other singularities. This can be viewed as holomorphic map

$$f : X \to \mathbb{P}^1$$

such that $f^{-1}(\infty) = p$. Given $y \in \mathbb{C}$, let $g(x) = f(x) - y$. The preimage $f^{-1}(y) = g^{-1}(0)$ is a finite set $\{q_1, \ldots\}$. Let $n_1, \ldots$ be the order of the zeros of g these points. We claim that $\sum n_k = 1$. To see this, consider the meromorphic differential form $\alpha = dg/g$. This has poles at $\{p, q_1, \ldots\}$. We can choose a coordinate at q_k, so that g is locally $z \mapsto z^{n_k}$. It follows that $\alpha = n_k dz/z$. This means that the residue of α at q_k is n_k. Similarly, the residue at p is -1. To prove the claim, observe that by Stokes, the sum of residues

$$-1 + \sum n_k = \frac{1}{2\pi i} \iint_{X - \cup D_k} d\alpha = 0$$

With the claim in hand, we can see that f is a bijection. Using open mapping theorem from complex analysis shows f^{-1} is also holomorphic. Therefore $X \cong \mathbb{P}^1$. $\square$

5.5 The Jacobian

Let X be Riemann surface of genus g. Let C^∞_X denote sheaf of complex valued C^∞ functions, and $\mathcal{O}_X$ is the subsheaf of holomorphic functions. For our purposes a *holomorphic line bundle* is a rank one locally free sheaf L over $\mathcal{O}_X$. A C^∞ (complex) line bundle is defined analogously. A holomorphic line bundle gives rise to C^∞ line bundle by "extending scalars". In particular, this remark applies to $\mathcal{O}_X(D)$.

Lemma 5.5.1. *$\mathcal{O}_X(D)$ is a holomorphic line bundle.*

Proof. Let $\{U_i\}$ be a covering by coordinate nbhds containing at most one point of D. If U_i contains no points of D, then $\mathcal{O}(D)|_{U_i} = \mathcal{O}_{U_i}$. If U_i contains $p_j \in D$, choose a coordinate so that p_j is $z = 0$. Then

$$\mathcal{O}(D)|_{U_i} = \mathcal{O}_{U_i} z^{-n_j}$$

□

Let L be a C^∞ or holomorphic line bundle. In either case, we have an open cover $\{U_i\}$ and isomorphisms

$$\sigma_i : C^\infty_{U_i} \cong L|_{U_i}, \quad \text{or } \mathcal{O}_{U_i} \cong L|_{U_i}$$

Note that σ_i need not be compatible with σ_j. We can measure the difference by taking

$$\phi_{ij} = \sigma_j^{-1} \circ \sigma_i : \mathcal{O}(U_{ij}) \cong \mathcal{O}(U_{ij})$$

in either case, but we just wrote the second case. An automorphism of a commutative ring is just multiplication by a unit. So we can view

$$\phi_{ij} \in \mathcal{O}^*(U_{ij})$$

We can see from the definition that ϕ_{ij} is a 1-cocycle in $Z^1(\{U_i\}, \mathcal{O}^*_X)$.

Theorem 5.5.2. *The map $L \mapsto \phi_{ij}$ induces a bijection between the set of isomorphism classes of holomorphic (resp. C^∞) line bundles and $H^1(X, \mathcal{O}^*_X)$ (resp. $H^1(X, {C^\infty}^*_X)$).*

Both sides are abelian groups, where tensor product is an operation on the left. The above statement can be improved to an isomorphism of groups. The group of holomorphic line bundles is called the Picard group, and denoted by $Pic(X)$.

Proposition 5.5.3. *Let $e(f) = e^{2\pi i f}$, then the sequences of sheaves*

$$0 \to \mathbb{Z}_X \to C^\infty_X \xrightarrow{e} {C^\infty}^*_X \to 1$$

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \xrightarrow{e} \mathcal{O}^*_X \to 1$$

are exact.

Proof. Exactness is local, so we can reduce to the disk D. Since D is simply connected, a branch of the logarithm can be chosen to get surjectivity $e : \mathcal{O}(D) \to \mathcal{O}^*(D)$. The rest is straigtforward. □

Taking the first sequence gives

$$H^1(X, C^\infty_X) \to H^1(X, {C^\infty}^*_X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^1(X, C^\infty_X)$$

The map labelled c_1 is called the first Chern class. Since C^∞_X is soft, we obtain

Lemma 5.5.4. *The first Chern class induces an isomorphism*

$$H^1(X, {C^\infty}^*_X) \cong H^2(X, \mathbb{Z})$$

This says that c_1 is a complete invariant for C^∞ line bundles. Note that $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$, so c_1 is really just a number, called the first Chern number. We omit the details, but this can be computed to obtain

Theorem 5.5.5. *The first Chern number* $c_1(\mathcal{O}(D)) = \deg D$.

Corollary 5.5.6. *The C^∞ line bundles associated to two divisors are isomorphic iff they have the same degree,.*

The holomorphic side is much more interesting. We now get a sequence

$$\ldots \to H^1(X, \mathcal{O}_X) \to Pic(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to 0$$

because $H^2(X, \mathcal{O}_X) = 0$. The kernel of c_1 is denoted by $Pic^0(X)$.

Theorem 5.5.7. *$Pic^0(X)$ has the structure of a complex torus of dimension g.*

Proof. From the above exact sequence

$$Pic^0(X) \cong \frac{H^1(X, \mathcal{O}_X)}{\operatorname{im} H^1(X, \mathbb{Z})}$$

We saw earlier that the numerator on the right is a g dimensional complex vector space. It is enough to prove the claim that denominator is a lattice (a discrete subgroup of maximal rank).

Since $H^1(X, \mathbb{Z})$ sits a lattice inside $H^1(X, \mathbb{R})$, to prove the claim it is enough to show that the natural map

$$r : H^1(X, \mathbb{R}) \to H^1(X, \mathcal{O}_X)$$

is injective. It α is a harmonic form representing a nonzero element of $H^1(X, \mathbb{R})$. Then we can uniquely decompose $\alpha = \alpha^{0,1} + \alpha^{1,0}$ into a sum of a holomorphic and antiholomorphic forms. Note that $r(\alpha) = \alpha^{0,1}$. Since α is real, $\alpha^{0,1} = \overline{\alpha^{1,0}}$. Therefore $r(\alpha) \neq 0$.

$\square$

The torus $Pic^0(X)$ is called the *Jacobian* of X, and also denoted by $J(X)$. This is a fundamental invariant of X. It has the effect of "linearizing" X. What makes story more interesting is that X also maps to $J(X)$. Fix a base point $p_0 \in X$ and a positive integer n. The Abel-Jacobi map $AJ : X^n \to J(X)$ sends

$$(p_1, \ldots, p_n) \mapsto \mathcal{O}(p_1 + \ldots + p_n - np_0)$$

We summarize the key result without proof.

Theorem 5.5.8 (Abel-Jacobi)**.** *The Abel-Jacobi map is holomorphic, and surjective when $n \geq g$, and injective when $n = 1$ and $g \geq 1$.*

Corollary 5.5.9. *When $n = g = 1$, AJ is a holomorphic isomorphism. It follows that a genus 1 Riemann surface is isomorphic to a quotient of $\mathbb{C}$ by a lattice.*