Chapter 6

The Hodge theorem

6.1 Riemannian metrics

Let X be C^∞-manifold. A Riemannian metric on X is a C^∞ family of inner products on the tangent spaces. Here is a more precise definition.

Definition 6.1.1. *A Riemannian metric is family of inner products $g : T_p \otimes T_p \to \mathbb{R}$ such that given two C^∞ vector fields $u, v \in Vect(U)$, $g(u, v) \in C^\infty(U)$.*

Proposition 6.1.2. *Every manifold possesses a Riemannian metric.*

This is standard application of partitions of unity. A proof can be found in any book on differential geometry. The object g is called the metric tensor. In local coordinates

$$g = \sum g_{ij} dx_i \otimes dx_j$$

where $p \mapsto g_{ij}$ is a C^∞ family of symmetric positive definite matrices. By linear algebra, g induces inner products on $\wedge^k T_p^*$ for each k. This gives us a pointwise inner product

$$(\, , \,) : \mathcal{E}^k(X) \times \mathcal{E}^k(X) \to C^\infty(X)$$

(NB: This notation is different from the last chapter.)

Let us now assume that X is an oriented n-manifold. Then we define the volume form locally by

$$dvol = \sqrt{\det(g_{ij})} dx_1 \wedge \ldots \wedge dx_n$$

where we order the coordinates so that the expression is positively oriented. This is globally well defined, and it is not exact in spite of the notation. This form defines a measure on X by $\int_X f dvol$. *Let us also now assume that X is compact.* Then this is a finite measure i.e. $\int_X dvol < \infty$. We can define an actual inner product on $\mathcal{E}^k(X)$ by

$$\langle \alpha, \beta \rangle = \int_X (\alpha, \beta) dvol$$

The Hodge star operator $*\mathcal{E}^k(X) \to \mathcal{E}^{n-k}(X)$ is the unique linear transformation satisfying

$$\alpha \wedge *\beta = (\alpha, \beta) dvol$$

Thus

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge *\beta$$

If $e_1, \ldots, e_n$ is a positively ordered orthonormal basis of $\mathcal{E}^1(U)$ (which exists by Gram-Schmid), we can see that

$$dvol = e_1 \wedge \ldots \wedge e_n$$

$$*e_{i_1} \wedge \ldots \wedge e_{i_k} = \pm e_{j_1} \wedge \ldots \wedge e_{j_{n-k}}$$

where $\{j_1, \ldots\} = \{1, \ldots, n\} - \{i_1, \ldots\}$. It follows that $** = \pm 1$. The precise sign is $(-1)^{k(n-k)}$ on k-forms.

Lemma 6.1.3. *For all $\alpha \in \mathcal{E}^k(X), \beta \in \mathcal{E}^{k+1}(X)$,*

$$\langle d\alpha, \beta \rangle = \langle \alpha, (-1)^{k(n-k)} * d * \beta \rangle.$$

In other words, $(-1)^{k(n-k)} * d*$ is the adjoint d^* of d.

Proof. The proof follows by applying Stokes' theorem to the identity

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta \pm \alpha \wedge ** d * \beta$$

(This is nothing other integration by parts.) □

6.2 The Hodge theorem for Riemannian manifolds

Fix an oriented Riemannian manifold X. Let d^* denote the adjoint to d, which equals $\pm * d*$.

Definition 6.2.1. *The Laplacian (or Hodge Laplacian, or Laplace-Beltrami operator) of X is*

$$\Delta = dd^* + d^*d$$

Example 6.2.2. *Let $X = \mathbb{R}^2$ with Euclidian metric $g = dx \otimes dx + dy \otimes dy$. If $f \in C^\infty(X)$, then*

$$\Delta f = - * d * df = -(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2})$$

So it agrees with the usual Laplacian up to sign.

Definition 6.2.3. *A C^∞ form is harmonic if $\Delta\alpha = 0$.*

Lemma 6.2.4. *A form α is harmonic iff $d\alpha = d * \alpha = 0$.*

Proof. Suppose $\Delta\alpha = 0$, then

$$0 = \langle \Delta\alpha, \alpha \rangle = ||d\alpha||^2 + ||*d*\alpha||^2$$

Therefore $d\alpha = d*\alpha = 0$. The other direction is clear. $\square$

We now consider a problem in PDEs. Given a form α, can we always solve Poisson's equation

$$\Delta\gamma = \alpha?$$

The answer is almost always. Here is the precise statement.

Theorem 6.2.5 (Hodge theorem)**.** *Suppose that X is a compact oriented Riemannian manifold. Then the space of harmonic forms is finite dimensional. Any form α can be written as $\beta + \Delta\gamma$, where β is harmonic.*

We won't give a proof, but just make a few comments. The first step is to complete the inner product space $\mathcal{E}^k(X)$ to a Hilbert space. Methods of functional analysis can be used to prove the existence of a weak solution to $\alpha = \beta + \Delta\gamma$ with β, γ in this Hilbert space. The second step is to prove that this weak solution is in fact a C^∞ solution. This depends crucially on the fact that Δ is a so called elliptic partial differential operator.

Let's look at an example.

Example 6.2.6. *Let $X = \mathbb{R}^n/\mathbb{Z}^n$ with the Euclidean metric. A differential form α can be expanded in a Fourier series*

$$\alpha = \sum_{\lambda\in\mathbb{Z}^n} \sum_{|I|=p} a_{\lambda,I} e^{2\pi i\lambda\cdot\mathbf{x}} dx_I \tag{6.1}$$

where $dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_p}$. By direct calculation, one finds the Laplacian

$$\Delta = -\sum \frac{\partial^2}{\partial x_i^2} \text{ (on coefficients)}$$

Then $\alpha = \beta + \Delta\gamma$ with

$$\beta = \sum_I a_{0,I} dx_I$$

and

$$\gamma = \sum_{\lambda\in\mathbb{Z}^n-\{0\}} \sum_I \frac{a_{\lambda,I}}{4\pi^2|\lambda|^2} e^{2\pi i\lambda\cdot\mathbf{x}} dx_I$$

Since β has constant coefficients, it's harmonic.

We already used this theorem in the last chapter, and we will give more applications shortly. The following corollary is what most people would call the Hodge theorem.

Corollary 6.2.7. *Any de Rham cohomology class has a unique harmonic representative.*

Proof. We proved this statement for Riemann surfaces earlier, and the general case is the same. Suppose α is closed then write $\alpha = \beta + \Delta\gamma = d(d^*\gamma) + d^*(d\gamma)$ as above.

$$||d^* d\gamma||^2 = \langle d\alpha, d\gamma\rangle - \langle d^2 d^*\gamma, d\gamma\rangle = 0$$

So α and the harmonic form β lie in the same cohomology class. Uniqueness of β is proved similarly. □

Here is another proof of Poincaré duality.

Corollary 6.2.8. *If* $\dim X = n$*, then* $H^k_{dR}(X) \cong H^{n-k}_{dR}(X)$

Proof. $*$ takes harmonic forms to harmonic forms, so it induces the above isomorphism. □

6.3 Kähler manifolds

Let X be a complex manifold.

Definition 6.3.1. *A Hermitian metric on X is a family of Hermitian inner products on the complex tangent spaces which vary in C^∞ fashion. More precisely, H would be given by a section of $\mathcal{E}_X^{(1,0)} \otimes \mathcal{E}_X^{(0,1)}$, such that in some (any) locally coordinate system $z_i = x_i + \sqrt{-1}y_i$ around each point, H is given by*

$$H = \sum h_{ij} dz_i \otimes d\bar{z}_j$$

with h_{ij} positive definite Hermitian.

The real part of the matrix H is positive definite symmetric, and the tensor

$$\sum Re(h_{ij})(dx_i \otimes dx_i + dy_i \otimes dy_i)$$

gives a globally defined Riemannian structure on X. We also have a $(1,1)$-form ω called the *Kähler form* which is the normalized image of H under $\mathcal{E}_X^{(1,0)} \otimes \mathcal{E}_X^{(0,1)} \to \mathcal{E}_X^{(1,0)} \wedge \mathcal{E}_X^{(0,1)}$. In coordinates

$$\omega = \frac{\sqrt{-1}}{2} \sum h_{ij} dz_i \wedge d\bar{z}_j.$$

The normalization makes ω real, i.e. $\bar{\omega} = \omega$. It is clear from this formula, that ω determines the metric. Sometimes we just refer to ω as the metric.

In Riemannian geometry it is always possible to choose coordinates about a point which make it Euclidean up to second derivatives. These so called *normal coordinates* are often useful for computations. In the analytic world, such coordinates are not always possible, and this leads to a definition.

Definition 6.3.2. *A Hermitian metric on X is called a Kähler metric for any $p \in X$ if there exist analytic coordinates $z_1, \dots z_n$ with $z_i = 0$ at p, for which the metric becomes Euclidean up to second order:*

$$h_{ij} \equiv \delta_{ij} \mod (z_1, \dots z_n)^2$$

A Kähler manifold is a complex manifold which admits a Kähler metric. (Sometimes the term is used for a manifold with a fixed Kähler metric.)

In such a coordinate system, a Taylor expansion gives

$$\omega = \frac{\sqrt{-1}}{2} \sum dz_i \wedge d\bar{z}_i + \text{ terms of 2nd order and higher}$$

Therefore $d\omega = 0$ at $z_i = 0$. Since such coordinates can be chosen around point, $d\omega$ is identically zero. This gives a nontrivial obstruction for a Hermitean metric to be Kähler. In fact, this condition characterizes Kähler metrics and often taken as the definition:

Proposition 6.3.3. *Given a Hermitean metric H, the following are equivalent*

(1) H is Kähler.

(2) The Kähler form is closed: $d\omega = 0$.

(3) The Kähler form is locally expressible as $\omega = \partial\bar{\partial} f$.

Proof. That (1) implies (2) was explained above. See p 107 of Griffiths-Harris for the converse. (3) clearly implies (2), The converse will be proved in the exercises. ☐

We will refer the cohomology class of ω as the *Kähler class*. The function f such that $\omega = \sqrt{-1}\partial\bar{\partial} f$ is called a Kähler potential. A function f is *plurisubharmonic* if it is a Kähler potential, or equivalently in coordinates this means

$$\sum \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \xi_i \bar{\xi}_j > 0$$

for any nonzero vector ξ.

Basic examples of compact Kähler manifolds are:

Example 6.3.4. *Any Hermitian metric on a Riemann surface is Kähler since $d\omega$ vanishes for trivial reasons.*

Example 6.3.5. *Complex tori are Kähler. Any flat (Euclidean) metric will do.*

Before describing the next example, we note that the usual "round" metrics on the sphere are completely characterized by the fact that they are rotationally invariant. This can be made unique by fixing the area. We can extend this to other spaces. First observe that the unitary group $U(n+1)$ acts transitively on $\mathbb{P}^n$ via the standard action on $\mathbb{C}^{n+1}$.

Lemma 6.3.6. *A $U(n+1)$-invariant Hermitian metric on $\mathbb{P}^n$ is unique up to a positive scalar multiple.*

Proof. The isotropy group of $p = [1, 0, \ldots, 0]$ is $U(1) \times U(n)$, and the action of the last factor $U(n)$ on T_p can be identified with the standard one on $\mathbb{C}^n$ using the basis $\frac{\partial}{\partial(z_i/z_0)}$. A $U(n+1)$-invariant Hermitian metric on $\mathbb{P}^n$ is determined by a $U(n)$-invariant inner product on $T_p = \mathbb{C}^n$. Such an inner product must be a positive multiple of the standard one. $\square$

Technically, we haven't proved that such a metric exists. We do that next. With an appropriate normalization, this metric is called the *Fubini-Study* metric, and it is of fundamental importance.

Proposition 6.3.7. *There exists is a unique choice of invariant Hermitian metric, called the Fubini-Study metric, such that $\int_{\mathbb{P}^1} \omega = 1$ for any line $\mathbb{P}^1 \subset \mathbb{P}^n$, where ω is the Kähler form. This metric is Kähler.*

Proof. Let $z_0, \ldots, z_n$ denote homogeneous coordinates. On a chart $U_i = \{z_i \neq 0\}$, the ratios $z_0/z_i, \ldots$ form true coordinates. For simplicity, let $i = 0$, and write $\zeta = (z_1/z_0, \ldots, z_n/z_0)$. We set

$$\omega_0 = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(1 + |\zeta|^2)$$

Computing the matrix of coefficients of ω_0 gives

$$\frac{(1 + |\zeta|^2)I - \zeta^\dagger \zeta}{\pi(1 + |\zeta|^2)^2}$$

where $\zeta^\dagger$ denotes the conjugate transpose. It is positive definite at the origin, because it reduces to $\frac{1}{\pi}I$ there. We have similarly defined forms ω_i on each U_i. Let $\mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ be the projection. Then we can see that

$$\pi^*\omega_i = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(|z_0|^2 + \ldots + |z_n|^2)$$

because the difference between these expressions is proportional to $\partial\bar{\partial} \log |z_i|^2 = 0$. Since the right side is independent of i, we see that the ω_i's patch to a yield a 2-form ω on $\mathbb{P}^n$. It is clear from the last formula that this form is invariant under $U(n+1)$. Since the matrix of coefficients is positive definite at one point, it is positive definite everywhere. Therefore ω defines a Kähler metric.

One can check by direct calculation that $\int_{\mathbb{P}^1} \omega = 1$. $\square$

We note that $H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$, and it is generated by the first Chern class of line bundle $\mathcal{O}(1)$ called the tautological bundle. This group has two generators $\mathcal{O}(\pm 1)$. The preferred one $\mathcal{O}(1)$ is distinguished by the fact that $H^0(\mathbb{P}^n, \mathcal{O}(1)) \neq 0$. In fact, it is isomorphic to the space of linear polynomials in the homogeneous coordinates $z_0, \ldots, z_n$. Since $H^2(\mathbb{P}^n, \mathbb{R}) = H^2(\mathbb{P}^n, \mathbb{Z}) \otimes \mathbb{R}$ is one dimensional, the Kähler class $[\omega]$ would have to be a nonzero multiple of $c_1(\mathcal{O}(1))$. In fact, the constants in the formulas are *chosen* so that these coincide.

Lemma 6.3.8. *A complex submanifold of a Kähler manifold inherits a Kähler metric such that the Kähler class is the restriction of the Kähler class of the ambient manifold.*

Proof. The Kähler form locally has a plurisubharmonic potential f. It follows immediately from the definition, that f restricts to a plurisubharmonic function on complex submanifold. Thus the Kähler form will restrict to a Kähler form. $\square$

There are several reasons why the Kähler condition is a natural and useful. For algebraic geometry, the main reason is as follows.

Theorem 6.3.9. *A smooth projective variety has a Kähler metric.*

Proof. Since $\mathbb{P}^n$ is Kähler, the theorem follows from the previous lemma. $\square$

When X is projective, the cohomology class of the Kähler form ω lies in the image of $H^2(X, \mathbb{Z})$, because it is the restriction of the first Chern class $c_1(\mathcal{O}(1))$. A deep theorem of Kodaira shows that this condition characterizes those Kähler manifolds that come from projective varieties.

Every complex manifold carries a Hermitian metric by a partition of unity argument. However, it is not true that every manifold carries a Kähler metric. There are a number of topological constraints, such as those below.

Theorem 6.3.10. *If X is compact Kähler manifold of dimension n, ω^k defines a nonzero class in $H^{2k}(X, \mathbb{C})$ for $k = 1, \ldots, n$. In particular, the Betti numbers $b_{2k}(X) = \dim H^{2k}(X, \mathbb{C})$ are nonzero.*

Proof. Using analytic normal coordinates, we can see that $\omega^n = Cdvol$, with $C > 0$. This implies that $\int_X \omega^n \neq 0$, so it cannot be exact. Since the class $[\omega^n]$ is the cup product of $[\omega]$ with itself n times, $[\omega^k] \neq 0$ for $k = 1, \ldots, n$. $\square$

When $X \subset \mathbb{P}^N$ is smooth and projective with Fubini-Study metric, there is another way to see this. Bertini's theorem in algebraic geometry guarantees that X contains a smooth subvariety Y of dimension k for any $k \leq n$. Then

$$\int_Y \omega^k = \int_Y c_1(\mathcal{O}(1))^k = \deg Y$$

One definition of $\deg Y$ is that it is the number of points of Y intersected with k general hyperplanes. Further details can be found in Hartshorne or pretty much any book on basic algebraic geometry. The key point is that one always has $\deg Y > 0$. It follows that $[\omega^k] \neq 0$.

6.4 The Hodge theorem for Kähler manifolds

Let X be a compact Hermitian complex manifold. Since it is an oriented Riemannian manifold, we can apply the results for a previous section to see that complex valued de Rham cohomology classes can be represented by harmonic forms. We saw that for Riemann surfaces, there is a close connection between harmonic and holomorphic forms. This is no longer true for a general Hermitian manifold, but it is for Kähler manifolds. The reason behind this is a set of identities called the Kähler identities. In order to explain the key identity, let

$$\bar{\partial}^* \alpha = \pm \overline{* \bar{\partial} * \overline{\alpha}}$$

Then with appropriate sign, this is the adjoint of $\bar{\partial}$ with respect to the inner product $\langle , \rangle$. We define a new Laplacian by

$$\Delta_{\bar{\partial}} = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$$

Lemma 6.4.1. *$\Delta_{\bar{\partial}}$ preserves type, i.e. if $\alpha \in \mathcal{E}^{p,q}(X)$, then $\Delta_{\bar{\partial}} \alpha \in \mathcal{E}^{p,q}(X)$.*

Proof. $\bar{\partial}$ has bidegree $(0,1)$, and $\bar{\partial}^*$ has bidegree $(0,-1)$. $\square$

Theorem 6.4.2. *If X is Kähler, then $\Delta = 2\Delta_{\bar{\partial}}$.*

Since X is Kähler, we can use analytic normal coordinates to reduce theorem to Euclidean space. However, the reduction is more complicated than it sounds because it is Euclidean only up to second order. Details can be found in Griffiths-Harris... Let us check this for the Euclidean metric.

$$\begin{aligned} \Delta_{\bar{\partial}}(\alpha) &= -2 \sum_{I,J,i} \frac{\partial^2 \alpha_{IJ}}{\partial z_i \partial \bar{z}_i} dz_I \wedge d\bar{z}_J \\ &= -\frac{1}{2} \sum_{I,J,i} \left(\frac{\partial^2 \alpha_{IJ}}{\partial x_i^2} + \frac{\partial^2 \alpha_{IJ}}{\partial y_i^2} \right) dz_I \wedge d\bar{z}_J \\ &= \frac{1}{2} \Delta(\alpha) \end{aligned}$$

Theorem 6.4.3 (The Hodge decomposition)**.** *Suppose that X is a compact Kähler manifold. Then a differential form is harmonic if and only if its (p,q) components are. Consequently we have noncanonical isomorphisms*

$$H^i(X, \mathbb{C}) \cong \bigoplus_{p+q=i} H^q(X, \Omega_X^p).$$

Furthermore, complex conjugation induces $\mathbb{R}$-linear isomorphisms between the space of harmonic (p,q) and (q,p) forms. Therefore

$$H^q(X, \Omega_X^p) \cong H^p(X, \Omega_X^q).$$

Proof. Since $\Delta = 2\Delta_{\bar{\partial}}$, a form is harmonic if and only if its (p,q) components are. Since complex conjugation commutes with Δ, conjugation preserves harmonicity. This shows that

$$H^i(X, \mathbb{C}) \cong \bigoplus_{p+q=i} H^{pq}, \quad \overline{H^{pq}} = H^{qp}$$

where H^{pq} is the space of harmonic (p,q)-forms. To finish, we need to establish an isomorphism

$$H^{pq} \cong H^q(X, \Omega^p_X)$$

Suppose $\alpha \in H^{pq}$. Then

$$0 = \langle \Delta_{\bar\partial}\alpha, \alpha\rangle = ||\bar\partial\alpha||^2 + ||\bar\partial^*\alpha||^2$$

Therefore α is $\bar\partial$-closed. When combined with Dolbeault's theorem, this produces a map

$$\pi : H^{pq} \to H^q(X, \Omega^p_X)$$

We claim that π is surjective, let α be a $\bar\partial$-closed (p,q)-form. Decompose

$$\alpha = \beta + \Delta\gamma = \beta + 2\Delta_{\bar\partial}\gamma = \beta + \bar\partial\gamma_1 + \bar\partial^*\gamma_2$$

with β harmonic. We have

$$||\bar\partial^*\gamma_2||^2 = \langle \gamma_2, \bar\partial\bar\partial^*\gamma_2\rangle = \langle \gamma_2, \bar\partial\alpha\rangle = 0.$$

This proves the claim.

We also claim that π is injective. To see this, suppose that $\alpha \in H^{pq}$ equals $\bar\partial\beta$. Then

$$||\alpha||^2 = \langle \beta, \bar\partial^*\alpha\rangle = 0$$

□

Corollary 6.4.4. *If X is compact Kähler and i is odd, then the Betti number $b_i = \dim H^i(X,\mathbb{C})$ is even.*

Proof. Let $h^{pq} = \dim H^q(X, \Omega^p_X)$. Then

$$b_i = \sum_{p+q=i} h^{pq} = 2 \sum_{p+q=i, p<q} h^{pq}$$

□

6.5 Functorial Hodge structure

The goal of this section is to show hat the Hodge decomposition for compact manifolds can be made independent of the metric. To make a precise statement, we need to formalize things.

Definition 6.5.1. *A Hodge structure of weight i consists of a finitely generated abelian group $H_{\mathbb{Z}}$ and a decomposition*

$$H_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=i} H^{pq}$$

such that $\overline{H^{pq}} = H^{qp}$. A morphism of Hodge structures $f : H_{\mathbb{Z}} \to G_{\mathbb{Z}}$ is a homomorphism of groups, such that the induced map $f \otimes \mathbb{C}$ preserves the bigrading.

Example 6.5.2. *Let X be compact Kähler. Then the isomorphism*

$$H^i(X,\mathbb{Z}) \otimes \mathbb{C} \cong \text{ space of harmonic } i \text{ forms}$$

together with the decomposition constructed in the last section gives a natural example of a Hodge structure of weight i.

This seems to depend on the metric, but the surprise is that it doesn't.

Theorem 6.5.3 (Deligne)**.** *There exists functor from the category of compact Kähler manifolds and holomorphic maps to the category of Hodge structures of weight i, such that for any choice of metric, it is isomorphic to the example constructed above.*

First, we reformulate Hodge structures as follows.

Lemma 6.5.4. *The category of Hodge structures of weight i is equivalent to the category of finitely generated abelian groups $H_{\mathbb{Z}}$ with a decreasing filtration $F^\bullet$ on $H = H_{\mathbb{Z}} \otimes \mathbb{C}$ such that for all p, $F^p \oplus \overline{F}^{i-p+1} = H$.*

Sketch. In one direction, given a Hodge structure, set $F^p = H^{pq} \oplus H^{p+1,q-1} \ldots$. In the other direction, $H^{pq} = F^p \cap \overline{F}^{i-p}$. □

Given a complex manifold, we define a filtration on differential forms by

$$F^p\mathcal{E}^i(X) = \mathcal{E}^{pq}(X) \oplus \mathcal{E}^{p+1,q-1}(X) \ldots$$

In others $\alpha \in F^p$ if there are at least p dz's when written in coordinates.

Lemma 6.5.5. *We have $dF^p\mathcal{E}^i(X) \subseteq F^p\mathcal{E}^{i+1}(X)$, or in other words $F^p\mathcal{E}^\bullet(X)$ is a subcomplex of the de Rham complex.*

Proof. If $\alpha \in \mathcal{E}^{ab}(X)$, then $d\alpha \in \mathcal{E}^{a,b+1}(X) \oplus \mathcal{E}^{a+1,b}(X)$. □

We define

$$F^pH^i(X) = \mathrm{im}[H^i(F^p\mathcal{E}^\bullet(X)) \to H^i(X)]$$

We want to show that this filtration defines a Hodge structure when X is compact Kähler. As a first step observe that the complex conjugate

$$\overline{F}^pH^i(X) = \mathrm{im}[H^i(\overline{F}^p\mathcal{E}^\bullet(X)) \to H^i(X)]$$

where

$$\overline{F}^p\mathcal{E}^i(X) = \mathcal{E}^{pq}(X) \oplus \mathcal{E}^{p-1,q+1}(X) \ldots$$

is also a subcomplex. It is clear that

$$\mathcal{E}^i(X) = F^p\mathcal{E}^i(X) \oplus \overline{F}^{i-p+1}\mathcal{E}^i(X)$$

This easily implies

Lemma 6.5.6. *If X is compact complex manifold,*

$$H^i(X) = F^p H^i(X) + \overline{F}^{i-p+1} H^i(X)$$

Note the sum on the right is not necessarily a direct sum. It is possible the two subspaces have a nontrivial intersection. To handle this, we need some homological algebra.

Lemma 6.5.7. *Let $C^\bullet$ be a bounded complex of vector spaces over a field, with a finite filtration $F^\bullet C^\bullet$ by subcomplexes. Suppose that for all i, p*

$$\sum_p \dim H^i(Gr^p C^\bullet) = \dim H^i(C^\bullet)$$

where the dimensions are assumed finite, and $Gr^p C^\bullet = F^p C^\bullet / F^{p+1} C^\bullet$. Then

$$\dim F^p H^i(C^\bullet) = \dim H^i(Gr^p C^\bullet) + \dim H^i(Gr^{p+1} C^\bullet) + \dots$$

The filtration is called *strict* if the above conditions hold. This is an extremely strong condition. It is equivalent to degeneration of the spectral sequence associated to F at the first page.

Lemma 6.5.8. *If X is compact Kähler, $F^\bullet \mathcal{E}^\bullet(X)$ and $\overline{F}^\bullet \mathcal{E}^\bullet(X)$ are strict. We have*

$$H^i(X) = F^p H^i(X) \oplus \overline{F}^{i-p+1} H^i(X)$$

Therefore the $F^\bullet$ defines a Hodge structure.

Proof. Dolbeault's theorem implies that

$$H^i(Gr_F^p \mathcal{E}^\bullet(X)) = H^i(\mathcal{E}^{p,\bullet}(X)[-p]) \cong H^{i-p}(X, \Omega_X^p)$$

The symbol $[-p]$ means shift the complex p places to the right. The Hodge decomposition theorem now implies that $F^\bullet \mathcal{E}^\bullet(X)$ is strict. Strictness of the conjugate filtration is similar. We know that

$$H^i(X) = F^p H^i(X) + \overline{F}^{i-p+1} H^i(X)$$

To prove that this is a direct sum, it suffces to show that the sum of dimensions on the right is the dimension on the left. By the previous lemma, it follows that

$$\dim F^p H^i(X) = h^{pq} + h^{p+1,q-1} + \dots$$

$$\dim \overline{F}^{q+1} H^i(X) = h^{p-1,q+1} + h^{p-2,q+2} + \dots$$

where $q = i - p$ and $h^{pq} = \dim H^q(X, \Omega^p)$. This implies the desired equality. $\square$

This last lemma proves the theorem.

6.6 Hodge cycles

Let X be a complex nonsingular projective variety of dimension n, and let $Y \subset X$ be a closed nonsingular subvariety of dimension m. The difference $p = n - m$ is the codimension. We define a functional

$$\int_Y \in H^{2m}(X, \mathbb{R})^*$$

which sends a real m-form $\alpha \in \mathcal{E}^m(X)$ representing the class to

$$\int_Y \alpha$$

Since

$$\int_Y (\alpha + d\beta) = \int_Y \alpha$$

by Stokes' theorem, this is well defined. By Poincaré duality

$$H^{2m}(X, \mathbb{R})^* \cong H^{2p}(X, \mathbb{R})$$

this gives an element $[Y] \in H^{2p}(X, \mathbb{R})$ called the fundamental class. The duality isomorphism can be made more explicit, and this leads to a more explicit description of the class.

Lemma 6.6.1. *$[Y]$ can be represented by a closed $2p$-form $\eta_Y \in \mathcal{E}^{2p}(X)$ such that*

$$\int_Y \alpha = \int_X \eta_Y \wedge \alpha$$

for all closed $2m$-forms α.

Lemma 6.6.2. *$[Y]$ can be represented by a closed (p,p)-form.*

Proof. We can choose a form η_Y as above, which is harmonic. We want to show this is of type (p,p). Suppose not. Then the (a,b)-part $\eta^{a,b} \neq 0$, for some $a < p, b = p - a$. Let $\alpha = \bar{*}\eta^{a,b}$. This is of type $(n-b, n-a)$ and satisfies

$$\int_X \eta_Y \wedge \alpha = ||\alpha||^2 \neq 0$$

This implies

$$\int_Y \alpha \neq 0$$

However, this is impossible because $\alpha|_Y = 0$. □

In fact, the construction can be extended to define the fundamental class of possibly *singular* subvariety Y. One way to do this is appeal to a deep theorem of Hironaka that says there is a nonsingular variety $\tilde{Y}$ and a morphism $\pi : \tilde{Y} \to Y$ which is an isomorphism over the nonsingular part of Y. The map $\tilde{Y} \to Y$

is called a *resolution of singularities*. Then $[Y]$ can be defined as the class in $H^{2m}(X,\mathbb{R})$ dual to $\int_{\tilde{Y}}$. The resolution is not unique, but the class $[Y]$ can be seen to be well defined.

Although we won't prove it, there are alternative constructions which show that

Proposition 6.6.3. *The fundamental class of a subvariety lies in the image of $H^{2p}(X,\mathbb{Z})$.*

Putting these facts together, we find that fundamental class lies in

$$H^{2p}(X,\mathbb{Z})\cap H^{p,p}(X)$$

where we take $H^{p,p}(X) = F^p\cap\bar{F}^p$ associated to the functorial Hodge structure. (NB: $H^{2p}(X,\mathbb{Z})$ might have torsion, which should be understood as lying in the intersection.) An element of this intersection will be called an *integral Hodge cycle*. A *rational Hodge cycle* or simply a *Hodge cycle* is an element of

$$H^{2p}(X,\mathbb{Q})\cap H^{p,p}(X)$$

We now come to the famous:

Conjecture 6.6.4 (Hodge conjecture)**.** *A Hodge cycle is a linear combination of fundamental classes of subvarieties. Such a linear combination is called an algebraic cycle.*

Historical Remarks:

1. Hodge didn't actually use the word "conjecture", but he formulated it as a problem in his 1950 ICM talk. In fact, he expected it should hold with integer coefficients. But by 1960 counter-examples were found to the integral version by Atiyah-Hirzebruch, and it has since been formulated as above.

2. The conjecture gained importance in the 1960's partly because of its relation to Grothendieck's theory of motives. In this connection, he wrote a paper in English, famously entitled "Hodge's general conjecture is false for trivial reasons". Although, he quickly points out that he didn't mean the conjecture every thinks of, but instead something related.

3. Although the statement of the Hodge conjecture makes sense for compact Kähler manifolds, it is known to be false (Zucker, Voisin).

At the time Hodge formulated the conjecture, he had one important piece of evidence

Theorem 6.6.5 (Lefschetz $(1,1)$ theorem)**.** *An integral Hodge cycle $H^2(X)$ is an algebraic cycle.*

Before proving it, we give an equivalent formulation due to Kodaira and Spencer, which makes the proof easy. Recall that we have the first Chern class map

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

coming from the exponential sequence. The group on the left is the set of isomorphism classes of line bundles. The image is called the *Neron-Severi* group $NS(X)$.

Theorem 6.6.6 (Lefschetz $(1,1)$ theorem, version 2)**.** *An integral Hodge cycle in $H^2(X)$ is the first Chern class of a line bundle.*

Proof. Consider the exact sequence

$$H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O})$$

The last map can be interpreted as the composition

$$H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C}) \xrightarrow{\pi} H^2(X, \mathbb{C})/F^1 \cong H^2(X, \mathcal{O})$$

An integral Hodge cycle α is an element of $H^2(X, \mathbb{Z}) \cap H^{11}(X)$, so it would map to zero under π . Therefore α lies in the image of $H^1(X, \mathcal{O}^*)$. $\square$

We still have to prove version 1 of the theorem. A divisor is a finite linear combination $\sum n_i D_i$, where $D_i \subset X$ are irreducible varieties of codimension one. The line bundle $\mathcal{O}_X(D)$ constructed earlier for Riemann surfaces can be generalized to divisors in this sense. The first version follows from the next lemma.

Lemma 6.6.7.

1. *Any line bundle is isomorphic to $\mathcal{O}_X(D)$ for some divisor D.*
2. *If $D = \sum n_i D_i$, then $c_1(\mathcal{O}_X(D)) = \sum n_i [D_i]$.*

As a corollary, we give a useful criterion to the check the conjecture in some examples. First, observe that cohomology $H^*(X, \mathbb{C})$ forms a ring under cup product, and the space of Hodge cycles forms a subring.

Corollary 6.6.8. *If the ring Hodge cycles on X is generated by divisors, then the Hodge conjecture holds for X.*

6.7 Hodge cycles on self products of elliptic curves

Recall that an elliptic curve is a quotient of $\mathbb{C}$ by a lattice L. One can always normal the lattice to the form $L = \mathbb{Z} + \mathbb{Z}\tau$, where $Im\, \tau > 0$. A useful fact is

Theorem 6.7.1. *Any elliptic curve can be embedded into $\mathbb{P}^2$ as cubic curve.*

A proof can be found in Silverman's book on elliptic curves. A consequence of this and some basic algebraic geometry is that if E is an elliptic curve, then $E^n = E \times E \ldots$ is nonsingular and projective.

Theorem 6.7.2 (Tate)**.** *If E is an elliptic curve, the Hodge conjecture holds for E^n.*

We will prove this under an extra assumption. We say $\mathbb{C}/L$ has *complex multiplication* or is CM if there $\alpha \in \mathbb{C} - \mathbb{Z}$ such that $\alpha L \subseteq L$.

Example 6.7.3. *The curve corresponding to $L = \mathbb{Z}+\mathbb{Z}i$ is CM because $iL \subseteq L$.*

Lemma 6.7.4. *If $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ is CM then τ is algebraic, and in fact it is contained in an imaginary quadratic extension of $\mathbb{Q}$ (an extension of the form $\mathbb{Q}(\sqrt{-d}), d > 0$).*

Proof. The CM condition implies that $\alpha = a + b\tau$ and $\alpha\tau = c + d\tau$ for integers a, b, c, d. Therefore τ satisfies a quadratic equation with integer coefficients. Since $\tau \notin \mathbb{R}$, it lies in an imaginary quadratic extension of $\mathbb{Q}$. □

This shows that the CM curves are atypical, but they are nevertheless very important. We will outline to proof of Tate's theorem for CM curves. We start by analyzing E^2. We need on fact from topology which we state without proof.

Theorem 6.7.5 (Künneth formula)**.** *If X and Y are manifolds, de Rham cohomology satisfies*

$$H^i(X \times Y) \cong \bigoplus_{j+k=i} H^j(X) \otimes H^k(X)$$

where the isomorphism is induced by

$$\mathcal{E}^j(X) \otimes \mathcal{E}^k(Y) \to \mathcal{E}^k(X \times Y), \quad \alpha \otimes \beta \mapsto p_1^*\alpha \wedge p_2^*\beta$$

with p_i denoting projections.

Therefore

$$H^2(E^2) = [H^2(E) \otimes H^0(E)] \oplus [H^1(E) \otimes H^1(E)] \oplus [H^0(E) \otimes H^2(E)] \quad (6.2)$$

This is compatible with the Hodge decomposition. In terms of the Hodge numbers, this means

$$h^{11}(E^2) = h^{11}(E)h^{00}(E) + [h^{10}(E)h^{01}(E) + h^{01}(E)h^{10}(E)] + h^{00}(E)h^{11}(E)$$
$$= 1 + [1 + 1] + 1 = 4$$

In fact, we don't need anything fancy to see this. E^2 is a quotient ot $\mathbb{C}^2$. Letting z_1, z_2 denote the coordinates on $\mathbb{C}^2$. Then $H^{11}(E^2)$ has a basis given the 4 elements $dz_i \wedge d\bar{z}_j$. It's convenient to define a new basis

$$\beta_1 = cdz_1 \wedge d\bar{z}_1, \beta_2 = cdz_2 \wedge d\bar{z}_2, \beta_3 = cdz_1 \wedge d\bar{z}_2, \beta_4 = cdz_2 \wedge d\bar{z}_1$$

where c is chosen so that β_1 integrates to 1 over $E \times o$ with $o \in E$.

Lemma 6.7.6. *The rank of $NS(E^2)$ is at least 3, and it is 4 if E is CM.*

Proof. There are 3 divisors D_1, D_2, D_3 given by $E \times o$, $o \times E$ and the diagonal. In terms of the above basis, $[D_1] = \beta_1$. To see this, note that D_1 is the pullback of the divisor o under the first projection $E \times E \to E$, and $[o]$ is given by this formula. For similar reasons, $[D_2] = \beta_2$. Write

$$[D_3] = \sum a_i \beta_i$$

Define an involution by $\beta_1' = \beta_2, \beta_3' = \beta_4$. Then

$$a_i = \int_{E^2} [D_3] \wedge \beta_i' = \int_{D_3} \beta_i' = \int_E c dz \wedge d\bar{z} = 1$$

So that $[D_3] = \beta_1 + \beta_2 + \beta_3 + \beta_4$. It is then clear that the 3 divisors are linearly independent. Therefore $rank NS \geq 3$.

Suppose that E is CM. Then multiplication by α gives a morphism $\alpha : E \to E$. The graph D_4 of α gives a fourth divisor D_4. We omit the details, but we can expand D_4 in the basis as above, and check independence of all 4 divisors. $\square$

In the CM case, we find that $H^{11}(E^2)$ is generated by divisors. The key point is that this generalizes.

Lemma 6.7.7. *If E is CM then $H^{pp}(E^n)$ is spanned by algebraic cycles.*

Sketch. Using the Künneth formula, one checks that $h^{11}(E^n) = n^2$, and $H^{pp}(X) = \wedge^p H^{11}(X)$. If $p_{ij} : E^n \to E^2$ denote the various projections, one checks that $\{p_{ij}^* D_k\}$ contains n^2 linearly independent elements of H^{11}. This proves the lemma for $p = 1$. The general case follows from the equation $H^{pp}(X) = \wedge^p H^{11}(X)$ by taking products. $\square$

2nd proof. We give an alternative argument for $NS = H^{11}$, which is cleaner but less elementary in that it uses some facts about abelian varieties. An abelian variety is a complex torus $A = V/L$ with an isogeny (an almost isomorphism) between A and the dual V^*/L^*. This isomorphism induces an involution $*$ on the algebra $End_{\mathbb{Q}}(A) = End(A) \otimes \mathbb{Q}$. On page 190 of Mumford's Abelian Varieties, one finds an isomorphism

$$NS(A) \otimes \mathbb{Q} \cong \{M \in End_{\mathbb{Q}}(A) \mid M = M^*\}$$

where the map from left to right can be understood as form of c_1. When $A = E^n$ with E CM, $End_{\mathbb{Q}}(E^n)$ is the algebra of $n \times n$ matrices over the imaginary quadratic field K containing τ. The involution is the conjugate transpose. A matrix satisfying $M^* = M$ is determined by $n(n-1)/2$ entries above the diagonal in K, and n entries along the diagonal in $\mathbb{Q}$, giving

$$\dim NS(E^n) = [K : \mathbb{Q}]\frac{n(n-1)}{2} + n = n^2$$

as required. $\square$

Corollary 6.7.8. *Tate's theorem is true for CM elliptic curves.*

In the non CM case $H^{pp}(E^n)$ is not generated by algebraic cycles. So in this case, the proof requires a completely different strategy, which we discuss later.