Chapter 1

Manifolds and Varieties via Sheaves

In rough terms, a manifold is a topological space along with a distinguished collection of functions, which looks locally like Euclidean space. Although it is rarely presented this way in introductory texts (e. g. [Spv, Wa]), sheaf theory is a natural language in which to make such a notion precise. An algebraic variety can be defined similarly as a space which looks locally like the zero set of a collection of polynomials. The sheaf theoretic approach to varieties was introduced by Serre in the early 1950's, this approach was solidified with the work of Grothendieck shortly thereafter, and algebraic geometry has never been the same since.

1.1 Sheaves of functions

In many parts of mathematics, we encounter spaces with distinguished classes of functions on them. When these classes are closed under restriction, as they often are, then they give rise to presheaves. More precisely, let X be a topological space, and T a set. For each open set $U \subseteq X$, let $Map_T(U)$ be the set of maps from U to T.

Definition 1.1.1. *A collection of subsets $P(U) \subset Map_T(U)$, with $U \subset X$ nonempty open, is called a presheaf of (T-valued) functions on X, if it is closed under restriction, i. e. if $f \in P(U)$ and $V \subset U$ then $f|_V \in P(V)$.*

If the defining conditions for $P(U)$ are local, which means that they can be checked in a neighbourhood of a point, then the presheaf is called sheaf. Or to put it another way:

Definition 1.1.2. *A presheaf of functions P is called a sheaf if $f \in P(U)$ whenever there is an open cover $\{U_i\}$ of U such that $f|_{U_i} \in P(U_i)$.*

Example 1.1.3. *Let $P_T(U)$ be the set of constant functions from U to T. This is a presheaf but not a sheaf in general.*

Example 1.1.4. *A function is locally constant if it is constant in a neighbourhood of a point. The set of locally constant functions, denoted by $T(U)$ or $T_X(U)$, is a sheaf. It is called the constant sheaf.*

Example 1.1.5. *Let T be another topological space, then the set of continuous functions $Cont_{X,T}(U)$ from $U \subseteq X$ to T is a sheaf. When T is discrete, this coincides with the previous example.*

Example 1.1.6. *Let $X = \mathbb{R}^n$, the sets $C^\infty(U)$ of C^∞ real valued functions form a sheaf.*

Example 1.1.7. *Let $X = \mathbb{C}$ (or $\mathbb{C}^n$), the sets $\mathcal{O}(U)$ of holomorphic functions on U form a sheaf.*

Example 1.1.8. *Let L be a linear differential operator on $\mathbb{R}^n$ with C^∞ coefficients (e. g. $\sum \partial^2/\partial x_i^2$). Let $S(U)$ denote the space of C^∞ solutions in U. This is a sheaf.*

Example 1.1.9. *Let $X = \mathbb{R}^n$, the sets $L^1(U)$ of L^1-functions forms a presheaf which is not a sheaf.*

We can always force a presheaf to be a sheaf by the following construction.

Example 1.1.10. *Given a presheaf P of functions to T. Define the*

$$P^s(U) = \{f : U \to T \mid \forall x \in U, \exists \text{ a neighbourhood } U_x \text{ of } x, \text{ such that } f|_{U_x} \in P(U_x)\}$$

This is a sheaf called the sheafification of P.

When P is a presheaf of constant functions, P^s is exactly the sheaf of locally constant functions. When this construction is applied to the presheaf L^1, we obtain the sheaf of locally L^1 functions.

Exercise 1.1.11.

1. *Check that P^s is a sheaf.*
2. *Let $\pi : B \to X$ be a surjective continuous map of topological spaces. Prove that the presheaf of sections*

   $$B(U) = \{\sigma : U \to B \mid \sigma \text{ continuous}, \forall x \in U, \pi \circ \sigma(x) = x\}$$

   is a sheaf.
3. *Let $F : X \to Y$ be surjective continuous map. Suppose that P is a sheaf of T-valued functions on X. Define $f \in Q(U) \subset Map_T(U)$ if and only if its* pullback $F^*f = f \circ F|_{f^{-1}U} \in P(F^{-1}(U))$. *Show that Q is a sheaf on Y.*

4. *Let $Y \subset X$ be a closed subset of a topological space. Let P be a sheaf of T-valued functions on X. For each open $U \subset Y$, let $P_Y(U)$ be the set of functions $f : U \to T$ locally extendible to an element of P, i.e. $f \in P_Y(U)$ if and only there for each $y \in U$, there exists a neighbourhood $V \subset X$ and an element of $P(V)$ restricting to $f|_{V \cap U}$. Show that P_Y is a sheaf.*

1.2 Manifolds

Let k be a field.

Definition 1.2.1. *Let $\mathcal{R}$ be a sheaf of k-valued functions on X. We say that $\mathcal{R}$ is a sheaf of algebras if each $R(U) \subseteq Map_k(U)$ is a subalgebra. We call the pair $(X, \mathcal{R})$ a concrete ringed space over k, or simply a k-space.*

$(\mathbb{R}^n, C_{\mathbb{R}})$, $(\mathbb{R}^n, C^\infty)$ and $(\mathbb{C}^n, \mathcal{O})$ are examples of $\mathbb{R}$ and $\mathbb{C}$-spaces.

Definition 1.2.2. *A morphism of k-spaces $(X, \mathcal{R}) \to (Y, \mathcal{S})$ is a continuous map $F : X \to Y$ such that $f \in \mathcal{S}(U)$ implies $F^* f \in \mathcal{R}(F^{-1}U)$.*

This is good place to introduce, or perhaps remind the reader of, the notion of a *category*. A category $\mathcal{C}$ consists of a set (or class) of objects $Obj\mathcal{C}$ and for each pair $A, B \in \mathcal{C}$, a set $Hom_{\mathcal{C}}(A, B)$ of morphisms from A to B. There is a composition law

$$\circ : Hom_{\mathcal{C}}(B, C) \times Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{C}}(A, C),$$

and distinguished elements $id_A \in Hom_{\mathcal{C}}(A, A)$ which satisfy

1. associativity: $f \circ (g \circ h) = (f \circ g) \circ h$,
2. identity: $f \circ id_A = f$ and $id_A \circ g = g$,

whenever these are defined.

Categories abound in mathematics. A basic example is the category of $Sets$ consisting of the class of all sets, $Hom_{Sets}(A, B)$ is just the set of maps from A to B, and composition and id_A have the usual meanings. Similarly, we can form the category of groups and group homomorphisms, the category of rings and rings homomorphisms, and the category of topological spaces and continuous maps. We have essentially constructed another example. We can take the objects to be k-spaces, and morphisms as above. These can be seen to constitute a category, once we observe that the identity is a morphism and the composition of morphisms is a morphism.

The notion of an isomorphism makes sense in any category, we will spell in the above example.

Definition 1.2.3. *An isomorphism of k-spaces $(X, \mathcal{R}) \cong (Y, \mathcal{S})$ is a homeomorphism $F : X \to Y$ such that $f \in \mathcal{S}(U)$ if and only if $F^* f \in \mathcal{R}(F^{-1}U)$.*

Given a sheaf S on X and open set $U \subset X$, let $S|_U$ denote the sheaf on U defined by $V \mapsto S(V)$ for each $V \subseteq U$.

Definition 1.2.4. *An n-dimensional C^∞ manifold is an $\mathbb{R}$-space (X, C^∞_X) such that*

1. *The topology of X is given by a metric*[1].
2. *X admits an open covering $\{U_i\}$ such that each $(U_i, C^\infty_X|_{U_i})$ is isomorphic to $(B_i, C^\infty|_{B_i})$ for some open ball $B \subset \mathbb{R}^n$.*

The isomorphisms $(U_i, C^\infty|_{U_i}) \cong (B_i, C^\infty|_{B_i})$ correspond to coordinate charts in more conventional treatments. The whole collection of data is called an atlas. There a number of variations on this idea:

Definition 1.2.5. 1. *An n-dimensional topological manifold is defined as above but with $(\mathbb{R}^n, C^\infty)$ replaced by $(\mathbb{R}^n, Cont_{\mathbb{R}^n,\mathbb{R}})$.*

2. *An n-dimensional complex manifold can be defined by replacing $(\mathbb{R}^n, C^\infty)$ by $(\mathbb{C}^n, \mathcal{O})$.*

One dimensional complex manifolds are usually called Riemann surfaces.

Definition 1.2.6. *A C^∞ map from one C^∞ manifold to another is just a morphism of $\mathbb{R}$-spaces. A holomorphic map between complex manifolds is defined as a morphism of $\mathbb{C}$-spaces.*

C^∞ (respectively complex) manifolds and maps form a category; an isomorphism in this category is called a diffeomorphism (respectively biholomorphism). By definition any point of manifold has neighbourhood diffeomorphic or biholomorphic to a ball. Given a complex manifold $(X, \mathcal{O}_X)$, we say that $f : X \to \mathbb{R}$ is C^∞ if and only if $f \circ g$ is C^∞ for each holomorphic map $g : B \to X$ from a ball in $\mathbb{C}^n$. We state for the record:

Lemma 1.2.7. *An n-dimensional complex manifold together with its sheaf of C^∞ functions is a $2n$-dimensional C^∞ manifold.*

Let us consider some examples of manifolds. Certainly any open subset of $\mathbb{R}^n$ ($\mathbb{C}^n$) is a (complex) manifold in an obvious fashion. To get less trivial examples, we need one more definition.

Definition 1.2.8. *Given an n-dimensional manifold X, a closed subset $Y \subset X$ is called a closed m-dimensional closed submanifold if for any point $x \in Y$, there exists a neighbourhood U of x in X and a diffeomorphism of to a ball $B \subset \mathbb{R}^n$ such that $Y \cap U$ maps to the intersection of B with an m-dimensional linear space.*

Given a closed submanifold $Y \subset X$, define C^∞_Y to be the sheaf of functions which are locally extendible to C^∞ functions on X. For a complex submanifold $Y \subset X$, we define $\mathcal{O}_Y$ to be the sheaf of functions which locally extend to holomorphic functions.

[1] It is equivalent and perhaps more standard to require that the topology is Hausdorff and paracompact. (The paracompactness of metric spaces is a theorem of A. Stone. In the opposite direct use a partition of unity to construct a Riemannian metric, then use the Riemannian distance.)

Lemma 1.2.9. *If $Y \subset X$ is a closed submanifold of C^∞ (respectively) manifold, then (Y, C^∞_Y) (respectively $(Y, \mathcal{O}_Y)$ is also a C^∞ (respectively complex) manifold.*

With this lemma in hand, it is possible to produce many interesting examples of manifolds starting from $\mathbb{R}^n$. For example, the unit sphere $S^{n-1} \subset \mathbb{R}^n$, which is the set of solutions to $\sum x_i^2 = 1$, is an $n-1$-dimensional manifold. The following example is of fundamental importance in algbraic geometry.

Example 1.2.10. *Let Let $\mathbb{P}^n_{\mathbb{C}} = \mathbb{CP}^n$ be the set of one dimensional subspaces of $\mathbb{C}^{n+1}$. (We will usually drop the $\mathbb{C}$ and simply write $\mathbb{P}^n$ unless there is danger of confusion.) Let $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ be the natural projection which sends a vector to its span. In the sequel, we usually denote $\pi(x_0, \ldots x_n)$ by $[x_0, \ldots x_n]$. $\mathbb{P}^n$ is given the quotient topology which is defined in so that $U \subset \mathbb{P}^n$ is open if and only if $\pi^{-1}U$ is open. Define a function $f : U \to \mathbb{C}$ to be holomorphic exactly when $f \circ \pi$ is holomorphic. Then the presheaf of holomorphic functions $\mathcal{O}_{\mathbb{P}^n}$ is a sheaf, and the pair $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ is an complex manifold. In fact, if we set*

$$U_i = \{[x_0, \ldots x_n] \mid x_i \neq i\},$$

then the map

$$[x_0, \ldots x_n] \mapsto (x_0/x_i, \ldots \widehat{x_i/x_i} \ldots x_n/x_i)$$

induces an isomomorphism $U_i \cong \mathbb{C}^n$ Here $\ldots \hat{a} \ldots$ means skip a in the list.

Exercise 1.2.11.

1. *Let $T = \mathbb{R}^n/\mathbb{Z}^n$ be a torus. Let $\pi : \mathbb{R}^n \to T$ be the natural projection. Define $f \in C^\infty(U)$ if and only if the pullback $f \circ \pi$ is C^∞ in the usual sense. Show that (T, C^∞) is a C^∞ manifold.*

2. *Let τ be a nonreal complex number. Let $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ and π denote the projection. Define $f \in \mathcal{O}_E(U)$ if and only if the pullback $f \circ \pi$ is holomorphic. Show that E is a Riemann surface. Such a surface is called an elliptic curve.*

3. *Show a map $F : \mathbb{R}^n \to \mathbb{R}^m$ is C^∞ in the usual sense if and only if it induces a morphism $(\mathbb{R}^n, C^\infty) \to (\mathbb{R}^m, C^\infty)$ of $\mathbb{R}$-spaces.*

4. *Prove lemma 1.2.9 .*

5. *Assuming the implicit function theorem [Spv], check that $f^{-1}(0)$ is a closed $n-1$ dimensional submanifold of $\mathbb{R}^n$ provided that $f : \mathbb{R}^n \to \mathbb{R}$ is C^∞ function such that the gradient $(\partial f/\partial x_i)$ does not vanish at 0. In particular, show that the quadric defined by $x_1^2 + x_2^2 + \ldots + x_k^2 - x_{k+1}^2 \ldots - x_n^2 = 1$ is a closed $n-1$ dimensional submanifold of $\mathbb{R}^n$ for $k \geq 1$.*

6. *Let $f_1, \ldots f_r$ be C^∞ functions on $\mathbb{R}^n$, and let X be the set of common zeros of these functions. Suppose that the rank of the Jacobian $(\partial f_i/\partial x_j)$ is $n - m$ at every point of X. Then show that X is an m dimensional submanifold. Apply this to show that the set $O(n)$ of orthogonal matrices $n \times n$ matrices is a submanifold of $\mathbb{R}^{n^2}$.*

7. *The complex Grassmanian $G = \mathbb{G}(2, n)$ is the set of 2 dimensional subspaces of $\mathbb{C}^n$. Let $M \subset \mathbb{C}^{2n}$ be the open set of $2 \times n$ matrices of rank 2. Let $\pi : M \to G$ be the surjective map which sends a matrix to the span of its rows. Give G the quotient topology induced from M, and define $f \in \mathcal{O}_G(U)$ if and only if $\pi \circ f \in \mathcal{O}_M(\pi^{-1}U)$. For $i \neq j$, let $U_{ij} \subset M$ be the set of matrices with $(1, 0)^t$ and $(0, 1)^t$ for the ith and jth columns. Show that*
$$\mathbb{C}^{2n-4} \cong U_{ij} \cong \pi(U_{ij})$$
and conclude that G is a $2n - 4$ dimensional complex manifold.

1.3 Algebraic varieties

Let k be an algebraically closed field. Affine space of dimension n over k is given by $\mathbb{A}^n_k = k^n$. When $k = \mathbb{C}$, we can endow this space with the standard topology induced by the Euclidean metric, and we will refer to this as the *classical topology*. At the other extreme is the *Zariski topology* which makes sense for any k. This topology can be defined to be the weakest topology for which the polynomials are continuous. The closed sets are precisely the sets of zeros
$$V(S) = \{a \in \mathbb{A}^n \mid f(a) = 0 \,\forall f \in S\}$$
of sets of polynomials $S \subset R = k[x_1, \ldots x_n]$. Sets of this form are also called *algebraic*. By Hilbert's nullstellensatz the map $I \mapsto V(I)$ is a bijection between the collection of radical ideals of R and algebraic subsets of $\mathbb{A}^n$. Will call an algebraic set $X \subset \mathbb{A}^n$ an *algebraic subvariety* if it is irreducible, which means that X is not a union of proper closed subsets, or equivalently if $X = V(I)$ with I prime. The Zariski topology of X has a basis given by affine sets of the form $D(g) = X - V(g)$, $g \in R$. At this point, it may be helpful to summarize this by a dictonary between the algebra and geometry:

Algebra	Geometry
maximal ideals of R	points of $\mathbb{A}^n$
radical ideals in R	algebraic subsets of $\mathbb{A}^n$
prime ideals in R	algebraic subvarieties of $\mathbb{A}^n$
localizations $R[1/g]$	basic open sets $D(g)$

An *affine variety* is subvariety of some $\mathbb{A}^n_k$. However, there are some disadvantages to always working with an explicit embedding into $\mathbb{A}^n$ (just as it is not always useful to treat manifolds as subsets of $\mathbb{R}^n$). Sheaf theory provides the tools for formulating this in a more coordinate free fashion. We call a function $F : D(g) \to k$ *regular* if it can be expressed as a rational function with a power of g in the denominator i.e. an element of $k[x_1, \ldots x_n][1/g]$. For a general open set $U \subset X$, $F : U \to k$ is regular if every point has a basic open neighbourhood for which F restricts to a regular function. With this notation, then:

Lemma 1.3.1. *Let X be an affine variety, and let $\mathcal{O}_X(U)$ denote the set of regular functions on U. Then $U \to \mathcal{O}_X(U)$ is a sheaf of k-algebras.*

Thus an affine variety gives rise to a k-space $(X, \mathcal{O}_X)$. The irreducibility of X guarantees that $\mathcal{O}(X) = \mathcal{O}_X(X)$ is an integral domain called the *coordinate ring* of X. Its field of fractions is called the *function field* of X, and it can be identified with the field of rational functions on X. The coordinate ring determines $(X, \mathcal{O}_X)$ completely. The space X is homeomorphic to the maximal ideal spectrum of $\mathcal{O}(X)$, and $\mathcal{O}_X(U)$ is isomormorphic to the intersection of the localizations

$$\bigcap_{m \in U} \mathcal{O}(X)_m$$

inside the function field.

In analogy with manifolds, we define:

Definition 1.3.2. *A prevariety over k is a k-space $(X, \mathcal{O}_X)$ such that X is connected and there exists a finite open cover $\{U_i\}$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic, as a k-space, to an affine variety. A morphism of prevarieties is a morphism of the underlying k-spaces.*

This is a "prevariety" because we are missing a Hausdorff type condition. Before explaining what this means, let us consider the most important nonaffine example.

Example 1.3.3. *Let $\mathbb{P}^n_k$ be the set of one dimensional subspaces of k^{n+1}. Let $\pi : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ be the natural projection. The Zariski topology on this is defined in such a way that $U \subset \mathbb{P}^n$ is open if and only if $\pi^{-1}U$ is open. Equivalently, the closed sets are zeros of sets of homogenous polynomials in $k[x_0, \dots x_n]$. Define a function $f : U \to k$ to be regular exactly when $f \circ \pi$ is regular. Then the presheaf of regular functions $\mathcal{O}_{\mathbb{P}^n}$ is a sheaf, and the pair $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ is easily seen to be a prevariety with affine open cover $\{U_i\}$ as in example 1.2.10.*

Now we can make the separation axiom precise. The Hausdorff condition for a space X is equivalent to the requirement that the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$ with its product topology. In the case of (pre)varieties, we have to be careful about what we mean by products. We expect $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$, but notice that the topology on this space is not the product topology. The safest way to define products is in terms of a universal property. The collection of prevarieties and morphisms forms a category. The following can be found in [M]:

Proposition 1.3.4. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ and be prevarieties. Then the Cartesian product $X \times Y$ carries a topology and a sheaf of functions $\mathcal{O}_{X \times Y}$ such that the projections to X and Y are morphisms. If $(Z, \mathcal{O}_Z)$ is any prevariety which maps via morphisms f and g to X and Y then the map $f \times g : Z \to X \times Y$ is a morphism.*

Thus $(X \times Y, \mathcal{O}_{X \times Y})$ is the product in the categorical sense. If $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are affine, then the prevariety structure associated to $X \times Y \subset \mathbb{A}^{n+m}$ coincides with the one given by the proposition. The product $\mathbb{P}^n \times \mathbb{P}^n$ can be constructed by more classical methods by using the Segre embedding $\mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{(n+1)(n+1)-1}$ [Hrs].

Definition 1.3.5. *A prevariety X is a variety (in the sense of Serre) if the diagonal $\Delta \subset X \times X$ is closed.*

Clearly affine spaces are varieties in this sense. Projective spaces can also be seen to be varieties. Further examples can be obtained by taking open or closed subvarieties of these examples. Let $(X, \mathcal{O}_X)$ be an algebraic variety over k. A closed irreducible subset $Y \subset X$ is called a closed subvariety. Imitating the construction for manifolds, given an open set $U \subset Y$ define $\mathcal{O}_Y(U)$ to be the set functions which are locally extendible to regular functions on X. Then

Proposition 1.3.6. *If $Y \subset X$ is a closed subvariety of an algebraic variety, $(Y, \mathcal{O}_Y)$ is an algebraic variety.*

It is worth making the description of closed subvarieties of projective space more explicit. Let $X \subset \mathbb{P}^n_k$ be an irreducible Zariski closed set. The affine cone of X is the affine variety $CX = \pi^{-1}X \cup \{0\}$. Now let π denote the restriction of the standard projectjion to $CX - \{0\}$. Define a function f on an open set $U \subset X$ to be regular when $f \circ \pi$ is regular. Zariski closed cones and therefore closed subvarieties of $\mathbb{P}^n_k$ can be described explicitly as zeros of homogeneous polynomials in $S = k[x_0, \dots x_n]$. Let $S_+ = (x_0, \dots x_n)$. We have a dictionary analogous to the earlier one:

Algebra	Geometry
homogeneous radical ideals in S containing S_+	algebraic subsets of $\mathbb{P}^n$
homogeneous prime ideals in S containing S_+	algebraic subvarieties of $\mathbb{P}^n$

When $k = \mathbb{C}$, we can use the stronger topology on $\mathbb{P}^n_{\mathbb{C}}$ introduced in 1.2.10. This is inherited by subvarieties, and is called the classical topology. When there is danger of confusion, we write X^{an} to indicate, a variety X with its classical topology.

Exercise 1.3.7.

1. *Let X be an affine variety with coordinate ring R and function field K. Show that X is homeomorphic to $Max\, R$, which is the set of maximal ideals of R with closed sets given by $V(I) = \{m \mid m \supset I\}$ for ideals $I \subset R$. Given $m \in Max\, R$, define $R_m = \{g/f \mid f, g \in R, f \notin m\}$. Show that $\mathcal{O}_X(U)$ is isomorphic to $\cap_{m \in U} R_m$.*

2. *Prove that a prevariety is a variety if there exists a finite open cover $\{U_i\}$ such that U_i and the intersections $U_i \cap U_j$ are isomorphic to affine varieties. Use this to check that $\mathbb{P}^n$ is an algebraic variety.*

3. *Given an open subset U of an algebraic variety X. Let $\mathcal{O}_U = \mathcal{O}_X|_U$. Prove that $(U, \mathcal{O}_U)$ is a variety.*

4. *Prove proposition 1.3.6.*

5. *Make the Grassmanian $\mathbb{G}_k(2, n)$, which is the set of 2 dimensional subspaces of k^n, into a prevariety by imitating the constructions of the exercises 1.2.11.*

6. *Check that $\mathbb{G}_k(2, n)$ is a variety.*

7. *After identifying $\mathbb{P}^5_k$ with the space of lines in $\wedge^2 k^4$, $\mathbb{G}_k(2, 4)$ can be embedded in $\mathbb{P}^5_k$, by sending the span of $v, w \in k^4$ to the line spanned by $\omega = v \wedge w$. Check that this is a morphism and that the image is a subvariety given by the Plücker equation $\omega \wedge \omega = 0$. Write this out as a homogeneous quadratic polynomial equation in the coordinates of ω.*

1.4 Stalks and tangent spaces

Given two functions defined in possibly different neighbourhoods of a point $x \in X$, we say they have the same *germ* at x if their restrictions to some common neigbourhood agree. This is is an equivalence relation. The germ at x of a function f defined near X is the equivalence class containing f. We denote this by f_x.

Definition 1.4.1. *Given a presheaf of functions P, its stalk P_x at x is the set of germs of functions contained in some $P(U)$ with $x \in U$.*

From a more abstract point of view, P_x is nothing but the direct limit

$$\varinjlim_{x \in U} P(U).$$

When $\mathcal{R}$ is a sheaf of algebras of functions, then $\mathcal{R}_x$ is a commutative ring. In most of the examples considered earlier, $\mathcal{R}_x$ is a local ring, i. e. it has a unique maximal ideal. This follows from:

Lemma 1.4.2. *$\mathcal{R}_x$ is a local ring if and only if the following property holds: If $f \in \mathcal{R}(U)$ with $f(x) \neq 0$, then $1/f$ is defined and lies in $\mathcal{R}(V)$ for some open set $x \in V \subseteq U$.*

Proof. Let m be the set of germs of functions vanishing at x. Then any $f \in \mathcal{R}_x - m$ is invertible which implies that m is the unique maximal ideal. □

Definition 1.4.3. *We will say that a k-space is* locally ringed *if each of the stalks are local rings.*

C^∞ and complex manifolds and algebraic varieties are locally ringed. When $(X, \mathcal{O}_X)$ is an n-dimensional complex manifold, the local ring $\mathcal{O}_{X,x}$ can be identified with ring of convergent power series in n variables. When X is a variety, the local ring $\mathcal{O}_{X,x}$ is also well understood. We may replace X by an affine variety with coordinate ring R. Consider the maximal ideal

$$m_x = \{f \in R \mid f(x) = 0\}$$

then

Lemma 1.4.4. *$\mathcal{O}_{X,x}$ is isomorphic to the localization R_{m_x}.*

Proof. Let K be the field of fractions of R. A germ in $\mathcal{O}_{X,x}$ is represented by a by regular function in a neighbourhood of x, but this is fraction $f/g \in K$ with $g \notin m_x$. □

In the previous cases the local rings are Noetherian. By contrast, when (X, C^∞) is a C^∞ manifold, the stalks are non Noetherian local rings. This is easy to check by a theorem of Krull [AM, E] which says that a local ring R with maximal ideal m satisfies $\cap_n m^n = 0$ if it is Noetherian. If R is the ring of germs of C^∞ functions, then the intersection $\cap_n m^n$ contains nonzero functions such as

$$\begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(see figure 1.1).

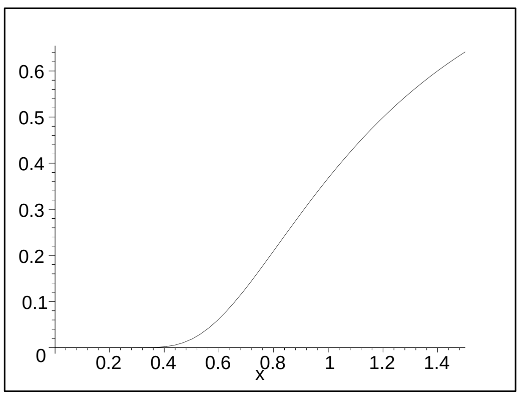


Figure 1.1: function in $\cap_n m^n$

Nevertheless, the maximal ideals are finitely generated.

Proposition 1.4.5. *If R is the ring of germs at 0 of C^∞ functions on $\mathbb{R}^n$. Then its maximal ideal m is generated by the coordinate functions $x_1, \ldots x_n$.*

If R is a local ring with maximal ideal m (which will be indicated by (R, m)), then R/m is a field called the residue field. The *cotangent space* of R is the R/m-vector space $m/m^2 \cong m \otimes_R R/m$, and the *tangent space* is its dual (over R/m). When m is a finitely generated, these spaces are finite dimensional.

Definition 1.4.6. *When X is a C^∞ or complex manifold or an algebraic variety with local ring $(\mathcal{O}_{X,x}, m_x)$, the tangent space at x, $T_x = T_{X,x}$ is the tangent space of R.*

When R is the local ring of a manifold or variety X at x, it is an algebra over its residue field. Therefore R/m^2 splits canonically into $k \oplus T_x^*$ (where $k = \mathbb{R}$ or $\mathbb{C}$ for a C^∞ or complex manifold).

Definition 1.4.7. *Given the germ of a function f at x on variety or a manifold, let df be its projection to T_x^* under the above decomposition. In other words, $df = f - f(x) \mod m^2$.*

Lemma 1.4.8. *$d : R \to T_x^*$ is a k-linear derivation, i. e. it satisfies the Leibnitz rule $d(fg) = f(x)dg + g(x)df$.*

As a corollary, it follows that a tangent vector $v \in T_x = T_x^{**}$ gives rise to a derivation $v \circ d : R \to k$. Conversely, any such derivation corresponds to a tangent vector. In particular,

Lemma 1.4.9. *If (R, m) is the ring of germs at 0 of C^∞ functions on $\mathbb{R}^n$. Then a basis for the tangent space T_0 is given*

$$D_i = \left.\frac{\partial}{\partial x_i}\right|_0 \quad i = 1, \ldots n$$

Manifolds are locally quite simple. By contrast algebraic varieties can be locally very complicated. We want to say that a point of a variety over an algebraically closed field k is nonsingular or smooth if it looks like affine space at a microscopic level. The precise definition requires some commutative algebra.

Theorem 1.4.10. *Let $X \subset \mathbb{A}_k^N$ be a closed subvariety defined by the ideal $(f_1, \ldots f_r)$. Choose $x \in X$ and let $R = \mathcal{O}_{X,x}$. Then the following statements are equivalent*

1. *R is a regular local ring i.e. $\dim T_x$ equals the Krull dimension of R.*
2. *The rank of the Jacobian $(\partial f_i/\partial x_j|_x)$ is $N - \dim X$.*

Proof. [E, 16.6]. □

When $k = \mathbb{C}$, we can apply the holomorphic implicit function theorem [GH, p. 19] to deduce an additional equivalent statement:

3 *There exists a neighbourhood U of $x \in \mathbb{C}^N$ in the usual Euclidean topology, and a biholomorphism (i. e. holomorphic isomorphism) of U to a ball B such that $X \cap U$ maps to the intersection of B and an n-dimensional linear subspace.*

Definition 1.4.11. *A point x on variety is called a nonsingular point if $\mathcal{O}_x$ is regular, otherwise it is singular. X is nonsingular or smooth, if every point is nonsingular.*

Affine and projective spaces and Grassmanians are examples of nonsingular varieties. It follows from (4) that a nonsingular affine or projective variety is a complex submanifold of affine or projective space.

Exercise 1.4.12.

1. *Prove proposition 1.4.5. (Hint: given $f \in m$, let*
$$f_i = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \dots tx_n)\, dt$$
show that $f = \sum f_i x_i$.)

2. *Prove lemma 1.4.8.*

3. *Let $F : (X, R) \to (Y, S)$ be a morphism of k-spaces. If $x \in X$ and $y = F(x)$, check that the homorphism $F^* : S_y \to R_x$ taking a germ of f to the germ of $f \circ F$ is well defined. When X and S are both locally ringed, show that F^* is local, i.e. $F^*(m_y) \subseteq m_x$ where m denotes the maximal ideals.*

4. *When $F : X \to Y$ is a C^∞ map of manifolds, use the previous exercise to construct the induced linear map $dF : T_x \to T_y$. Calculate this for $(X, x) = (\mathbb{R}^n, 0)$ and $(Y, y) = (\mathbb{R}^m, 0)$ and show that this is given by a matrix of partial derivatives.*

5. *Check that with the appropriate identification given a C^∞ function on X viewed as a C^∞ map from $f : X \to \mathbb{R}$. df in the sense of 1.4.8 and in the sense of the previous exercise coincide.*

6. *Suppose either that the characteristic of k is 0, or that it does not divide m. Show that Fermat's hypersurface defined by $x_0^m + \dots x_n^m = 0$ in $\mathbb{P}^n_k$ is nonsingular.*

7. *Show that the set of singular points of a variety form a Zariski closed set.*

1.5 Vector fields and bundles

A C^∞ vector field on a manifold X is a choice $v_x \in T_x$, for each $x \in X$, which varies in a C^∞ fashion. The dual notion is that of 1-form (or covector field). There are number of ways to

Definition 1.5.1. *A C^∞ vector field on X is a collection $v_x \in T_x$ such that the map $x \mapsto \langle v_x, df_x\rangle \in C^\infty(U)$ for each open $U \subseteq X$ and $f \in C^\infty(U)$. A 1-form is a collection $\omega_x \in T^*_x$ such that $x \mapsto \langle v_x, \omega_x\rangle \in C^\infty(X)$ for every C^∞-vector field.*

Given a C^∞-function f on X, we can define $df = x \mapsto df_x$. This is the basic example of C^∞ 1-form. Let $\mathcal{T}(X)$ and $\mathcal{E}^1(X)$ denote the space of C^∞ vector fields and 1-forms on X. These are modules over the ring $C^\infty(X)$ and we have an isomorphism $\mathcal{E}^1(X) \cong Hom_{C^\infty(X)}(\mathcal{T}(X), C^\infty(X))$. The maps $U \mapsto \mathcal{T}(U)$ and $U \mapsto \mathcal{E}^1(U)$ are easily seen to be sheaves (of respectively $\cup T_x$ and $\cup T^*_X$ valued functions) on X denoted by $\mathcal{T}_X$ and $\mathcal{E}^1_X$ respectively. These are prototypes of sheaves of locally free C^∞-modules: Each $\mathcal{T}(U)$ is a $C^\infty(U)$-module, and hence a $C^\infty(V)$-module for any $U \subset V$ and the restriction $\mathcal{T}(V) \to T(U)$ is $C^\infty(V)$-linear. Every point has a neighbourhood U such that $\mathcal{T}(U)$ and $\mathcal{E}^1(U)$ are free $C^\infty(U)$-modules. More specifically, if U is a coordinate neighbourhood with coordinates $x_1, \ldots x_n$, then $\{\partial/\partial x_1, \ldots \partial/\partial x_n\}$ and $\{dx_1, \ldots dx_n\}$ are bases for $\mathcal{T}(U)$ and $\mathcal{E}^1(U)$ respectively. Parallel constructions can be carried out for holomorphic (respectively regular) vector fields and forms on complex manifolds and nonsingular algebraic varieties. The corresponding sheaf of forms will be denoted by Ω^1_X. We will say more about this later on.

These notions are usually phrased in the equivalent language of vector bundles.

Definition 1.5.2. *A rank n (C^∞ real, holomorphic, algebraic) vector bundle is a morphism of C^∞ or complex manifolds or algebraic varieties $\pi : V \to X$ such that there exists an open cover $\{U_i\}$ of X and commutative diagrams*

$$\begin{tikzcd}
\pi^{-1}U_i \arrow[rr, "\cong", "\phi_i"'] \arrow[dr] & & U_i \times k^n \arrow[dl] \\
& U_i &
\end{tikzcd}$$

such that $\phi_i \circ \phi_j^{-1}$ are linear on each fiber. Where $k = \mathbb{R}$ or $\mathbb{C}$ in the C^∞ case, and $\mathbb{C}$ in the holomorphic case.

The data $\{(U_i, \phi_i)\}$is called a local trivialization. Given a vector bundle $\pi : V \to X$, define the presheaf of sections

$$V(U) = \{s : U \to \pi^{-1}U \mid s \text{ is } C^\infty,\, \pi \circ s = id_U\}$$

This is easily seen to be a sheaf of locally free modules. Conversely, we will see in section 6.3 that every such sheaf arises this way. The vector bundle corresponding to $\mathcal{T}_X$ is called the tangent bundle of X.

An explicit example of a nontrivial vector bundle is the tautological bundle L which we will encounter again. Projective space $\mathbb{P}^n_k$ is the set of lines $\{\ell\}$ in k^{n+1} through 0, and we can choose each line as a fiber of L. That is

$$L = \{(x, \ell) \in k^{n+1} \times \mathbb{P}^n_k \mid x \in \ell\}$$

Let $P : L \to \mathbb{P}^n_k$ be given by projection onto the second factor. Then L is rank one algebraic vector bundle, or line bundle, over $\mathbb{P}^n_k$. When $k = \mathbb{C}$ this can also be regarded as holomorphic line bundle or a C^∞ complex line bundle. L is often called the universal line bundle for the following reason:

Theorem 1.5.3. *If X is a compact C^∞ manifold with a C^∞ complex line bundle $\pi : M \to X$. There exists a C^∞ map, called a classifying map, $f : X \to \mathbb{P}^n_C$ with $n >> 0$, such that M is isomorphic to as a bundle to the pullback*

$$f^*L = \{(v, x) \in L \times X \mid \pi(v) = f(x)\} \to X$$

Proof. We sketch the proof. Here we consider the dual line bundle M^*. Sections of this correspond to $\mathbb{C}$-valued functions on M which are linear on the fibers. By compactness, we can find finitely many sections $f_0, \dots f_n \in M^*(X)$ which do not simulataneaously vanish at any point $x \in X$. Thus we get a map $M \to L$ given by $v \mapsto (f_0(x), \dots f_n(x))$. To get a bundle map, we need to map the bases as well. Under a local trivialization $M|_{U_i} \to U_i \times \mathbb{C}$, we can identify the f_i with $\mathbb{C}$-valued functions. X. The maps

$$U_i \to [f_0(x), \dots f_n(x)] \in \mathbb{P}^n$$

are independent of the choice of trivialization, and this gives map from $X \to \mathbb{P}^n$ compatible with previous map $M \to L$. $\square$

If (X, M) is holomorphic, then the map f cannot be chosen holomorphic in general. It is possible if and if the dual M^* has enough holomorphic global sections.

Exercise 1.5.4.

1. *Show that* $v = \sum f_i(x) \frac{\partial}{\partial x_i}$ *is a* C^∞ *vector field in the above sense on* $\mathbb{R}^n$ *if and only if the coefficients* f_i *are* C^∞.

2. *Check that* $\mathcal{T}_X$ *is a locally free sheaf for any manifold* X.

3. *Check that the presheaf of sections of a* C^∞ *vector bundle is a locally free sheaf.*

4. *Let* $X = f^{-1}(0)$ *where* $f : \mathbb{R}^n \to \mathbb{R}$ *has nonvanishing gradient along* X. *Let*
$$T_X = \{((v_1, \dots v_n), p) \in \mathbb{R}^n \times X \mid \sum v_i \frac{\partial f}{\partial x_i}|_p = 0\}$$
Check that the map $T_X \to X$ *given by the second projection makes,* T_X *into a rank* $n-1$ *vector bundle.*

5. *Continuing the notation from the previous problem, show that the space of* C^∞ *sections of* T_X *over* U *can be identified with vector fields on* U. *Thus* T_X *is the tangent bundle of* X.

6. *Check that* L *is an algebraic line bundle.*

7. *Tie up all the loose ends in the proof of theorem 1.5.3.*

8. *Let* $G = \mathbb{G}(2, n)$ *be the Grassmanian of* 2 *dimensional subspaces of* $\mathbb{C}^n$. *This is a complex manifold by the exercises 1.2.11. Let* $S = \{(x, V) \in \mathbb{C}^n \times G \mid x \in V\}$. *Show that the projection* $S \to G$ *is a holomorphic vector bundle of rank* 2.

Chapter 2

Generalities about Sheaves

We introduced sheaves of functions in the previous chapter as a convenient language for defining manifolds and varieties. However there is much more to the story...

2.1 The Category of Sheaves

It will be convenient to define presheaves of things other than functions. For instance, one might consider sheaves of equivalence classes of functions, distributions and so on. For this more general notion of presheaf, the restrictions maps have to be included as part the data:

Definition 2.1.1. *A presheaf P of sets (respectively groups or rings) on a topological space X consists of a set (respectively group or ring) $P(U)$ for each open set U, and maps (respectively homomorphisms) $\rho_{UV} : P(U) \to P(V)$ for each inclusion $V \subseteq U$ such that:*

1. $\rho_{UU} = id_{P(U)}$
2. $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$ *if* $W \subseteq V \subseteq U$.

We will usually write $f|_V = \rho_{UV}(f)$.

Definition 2.1.2. *A sheaf P is a presheaf such that for any open covering $\{U_i\}$ of U and $f_i \in P(U_i)$ satisfying $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, there exists a unique $f \in P(U)$ with $f|_{U_i} = f_i$.*

In English, this says that a collection of local sections can be patched together provided they agree on the intersections.

Definition 2.1.3. *Given presheaves of sets (respectively groups) P, P' on the same topological space X, a morphism $f : P \to P'$ is collection of maps (respectively homomorphisms) $f_U : P(U) \to P'(U)$ which commute with the restrictions. Given morphisms $f : P \to P'$ and $g : P' \to P''$, the compositions $g_U \circ f_U$*

determine a morphism from $P \to P''$. The collection of presheaves of Abelian groups and morphisms with this notion of composition constitutes a category $PAb(X)$.

Definition 2.1.4. *The category $Ab(X)$ is the full subcategory of $PAb(X)$ generated by sheaves of Abelian groups on X. In other words, objects of $Ab(X)$ are sheaves, and morphisms are defined in the same way as for presheaves.*

A special case of a morphism is the notion of a *subsheaf* of a sheaf. This is a morphism of sheaves where each $f_U : P(U) \subseteq P'(U)$ is an inclusion.

Example 2.1.5. *The sheaf of C^∞-funtions on $\mathbb{R}^n$ is a subsheaf of the sheaf of continuous functions.*

Example 2.1.6. *Let Y be a closed subset of a k-space $(X, \mathcal{O}_X)$, the ideal sheaf associated to Y,*

$$\mathcal{I}_Y(U) = \{f \in \mathcal{O}_X(U) \,|\, f|_Y = 0\},$$

is a subsheaf of $\mathcal{O}_X$

Example 2.1.7. *Given a sheaf of rings of functions R over X, and $f \in R(X)$, the map $R(U) \to R(U)$ given multipication by $f|_U$ is a morphism.*

Example 2.1.8. *Let X be a C^∞ manifold, then $d : C^\infty_X \to \mathcal{E}^1_X$ is a morphism of sheaves.*

We now introduce the notion of a *covariant functor* (or simply functor) $F : \mathcal{C}_1 \to \mathcal{C}_2$ between categories. This consists of a map $F : Obj\mathcal{C}_1 \to Obj\mathcal{C}_2$ and maps

$$F : Hom_{\mathcal{C}_1}(A, B) \to Hom_{\mathcal{C}_2}(F(A), F(B))$$

such that

1. $F(f \circ g) = F(f) \circ F(g)$
2. $F(id_A) = id_{F(A)}$

Contravariant functors are defined similary but with

$$F : Hom_{\mathcal{C}_1}(A, B) \to Hom_{\mathcal{C}_2}(F(B), F(A))$$

and the rule for composition adjusted accordingly.

Let *Ab* denote the category of abelian groups. There are number of functors from $PAb(X)$ to *Ab*.

Example 2.1.9. *The global section functor $\Gamma(P) = \Gamma(X, P) = P(X)$. Given a morphism $f : P \to P'$, $\Gamma(f) = f_X$.*

For any $x \in X$ and presheaf P, we define the *stalk* P_x of P at x, as we did earlier, to be the direct limit $\varinjlim P(U)$ over neighbourhoods of x. The elements of P_x are equivalence classes of elements of $P(U)$, with varying U, where two elements are equivalent if their restrictions to a common subset coincide.

Example 2.1.10. *Given a morphism $f : P \to P'$, the maps $f_U : P(U) \to P'(U)$ induce a map on the direct limits $P_x \to P'_x$. Thus $P \mapsto P_x$ determines a functor from $PSh(X) \to Ab$.*

There is a functor generalizing a construction from section 1.1.

Theorem 2.1.11. *There is a functor $P \mapsto P^+$ from $PAb(X) \to Ab(X)$ called sheafication, with the following properties:*

1. *If P is a presheaf of functions, then $P^+ \cong P^s$, where the right side is defined in example 1.1.10.*
2. *There is a canonical morphism $P \to P^+$.*
3. *If P is a sheaf then the morphism $P \to P^+$ is an isomomorphism*
4. *Any morphism from P to a sheaf factors uniquely through $P \to P^+$*
5. *The map $P \to P^+$ induces an isomorphism on stalks.*

Proof. We sketch the construction of P^+. We do this in two steps. First, we construct a presheaf of functions P'. Set $Y = \prod P_x$. We define a sheaf P' of Y-valued functions and a morphism $P \to P'$ as follows. There is a canonical map $\sigma_x : P(U) \to P_x$ if $x \in U$; if $x \notin U$ then send everything to 0. Then $f \in P(U)$ determines a function $f' : U \to Y$ given by $f'(x) = \sigma_x(f)$. Let

$$P'(U) = \{f' \mid f \in P(U)\},$$

this is yields a presheaf. Now apply the construction given earlier in example 1.1.10, to produce a sheaf $P^+ = (P')^s$. The $P(U) \to P'(U) \subset P^+(U)$ given by $f \mapsto f'$, yields the desired morphism $P \to P^+$. □

Exercise 2.1.12.

1. *Let X be a topological space. Construct a category $Open(X)$, whose objects are opens subsets of $X.Hom_{Open(X)}(U, V)$ consists of a single element, say $*$, if $U \subset V$, otherwise it is empty. Show that a presheaf of sets (or groups...) on $Open(X)$ is the same thing as a contravariant functor to the category of sets (or groups...).*
2. *Finish the proof of theorem 2.1.11.*

2.2 Exact Sequences

The categories $PAb(X)$ and $Ab(X)$ are *additive* which means among other things that $Hom(A, B)$ has an Abelian group structure such that composition is bilinear. Actually, more is true. These categories are *Abelian* [GM, Wl] which means, roughly speaking, that they possesses many of the basic constructions and properties of the category of abelian groups. In particular, there is an intrinsic notion of an exactness in this category. We give a nonintrinsic, but equivalent, formulation of this notion for $Ab(X)$.

Definition 2.2.1. *A sequence of sheaves on* X

$$\dots A \to B \to C \dots$$

is called exact in the if and only if

$$\dots A_x \to B_x \to C_x \dots$$

is exact for every $x \in X$.

The definition of exactness of presheaves will be given in the exercises.

Lemma 2.2.2. *Let* $f : A \to B$ *and* $g : B \to C$*, then* $A \to B \to C$ *is exact if and only if for any open* $U \subseteq X$

1. $g_U \circ f_U = 0$.
2. *Given* $b \in B(U)$ *with* $g(b) = 0$*, there exists an open cover* $\{U_i\}$ *of* U *and* $a_i \in A(U_i)$ *such that* $f(a_i) = b|_{U_i}$.

Proof. We will prove one direction. Suppose that $A \to B \to C$ is exact. Given $a \in A(U)$, $g(f(a)) = 0$, since $g(f(a))_x = g(f(a_x)) = 0$ for all $x \in U$. This shows (1).

Given $b \in B(U)$ with $g(b) = 0$, then for each $x \in U$, b_x is the image of a germ in A_x. Choose a representative a for this germ in some $A(U)$ where U is a neighbourhood of x. After shrinking U if necessary, we have $f(a) = b|_U$. This gives an open cover, and a collection of sections as required. □

Corollary 2.2.3. *If* $A(U) \to B(U) \to C(U)$ *is exact for every open set* U*, then* $A \to B \to C$ *is exact.*

The converse is false, but we do have:

Lemma 2.2.4. *If*

$$0 \to A \to B \to C \to 0$$

is an exact sequence of sheaves, then

$$0 \to A(U) \to B(U) \to C(U)$$

is exact for every open set U.

Proof. Let $f : A \to B$ and $g : B \to C$ denote the maps. By lemma 2.2.2, $g \circ f = 0$. Suppose $a \in A(U)$ maps to 0 under f, then $f(a_x) = f(a)_x = 0$ for each $x \in U$. Therefore $a_x = 0$ for each $x \in U$, and this implies that $a = 0$.

Suppose $b \in B(U)$ satisfies $g(b) = 0$. Then by lemma 2.2.2, there exists an open cover $\{U_i\}$ of U and $a_i \in A(U_i)$ such that $f(a_i) = b|_{U_i}$. Then $f(a_i|_{U_i \cap U_j} - a_j|_{U_i \cap U_j}) = 0$, which implies $a_i|_{U_i \cap U_j} - a_j|_{U_i \cap U_j}$ by the first paragraph. Therefore $\{a_i\}$ patch together to yield an element of $A(U)$.

□

We give some *natural* examples to show that $B(X) \to C(X)$ is not usually surjective.

Example 2.2.5. *Let X denote the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Then*

$$0 \to \mathbb{R}_X \to C_X^\infty \xrightarrow{d} \mathcal{E}_X^1 \to 0$$

is exact. However $C^\infty(X) \to \mathcal{E}^1(X)$ is not surjective.

To see the first statement, let $U \subset X$ be an open set diffeomorphic to an open interval. Then the sequence

$$0 \to \mathbb{R} \to C^\infty(U) \xrightarrow{f \to f'} C^\infty(U)dx \to 0$$

is exact by calculus. Thus one gets exactness on stalks. For the second, note that the constant form dx is not the differential of a periodic function.

Example 2.2.6. *Let $(X, \mathcal{O}_X)$ be a C^∞ or complex manifold or algebraic variety and $Y \subset X$ a submanifold or subvariety. Let*

$$\mathcal{I}_Y(U) = \{f \in \mathcal{O}_X(U) \mid f|_Y = 0\}$$

then this is a sheaf called the ideal sheaf of Y, and

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

is exact. The map $\mathcal{O}_X(X) \to \mathcal{O}_Y(X)$ need not be surjective. For example, let $X = \mathbb{P}^1_{\mathbb{C}}$ with $\mathcal{O}_X$ the sheaf of holomorphic functions. Let $Y = \{p_1, p_2\} \subset \mathbb{P}^1$ be a set of distinct points. Then the function $f \in O_Y(X)$ which takes the value 1 on p_1 and 0 on p_2 cannot be extended to a global holomorphic function on $\mathbb{P}^1$ since all such functions are constant by Liouville's theorem.

Given a sheaf S and a subsheaf $S' \subseteq S$, we can define a new presheaf with $Q(U) = S(U)/S'(U)$ and restriction maps induced from S. In general, this is not a sheaf. We define $S/S' = Q^+$.

Exercise 2.2.7.

1. *Finish the proof of lemma 2.2.2.*

2. *Give an example of a subsheaf $S' \subseteq S$, where $Q(U) = S(U)/S'(U)$ fails to be a sheaf. Check that*

   $$0 \to S' \to S \to S/S' \to 0$$

   is an exact sequence of sheaves.

3. *Given a morphism of sheaves $f : S \to S'$, define $ker\, f$ to be the subpresheaf of S with $ker\, f(U) = ker[f_U : S(U) \to S'(U)]$. Check that $ker\, f$ is a sheaf, and that $(ker\, f)_x \cong ker[S_x \to S'_x]$.*

4. *Define a sequence of presheaves $A \to B \to C$ to be exact in $PAb(X)$ if $A(U) \to B(U) \to C(U)$. Show that if a sequence of sheaves is exact in $PAb(X)$ then it is exact in the sense of 2.2.1, but that converse can fail.*

2.3 Direct and Inverse images

Sometimes it is useful to transfer a sheaf from one space to another. Let $f : X \to Y$ be a continuous map of topological spaces.

Definition 2.3.1. *Given a presheaf A on X, the direct image f_*A is a presheaf on Y given by $f_*A(U) = A(f^{-1}U)$ with restrictions*

$$\rho_{f^{-1}Uf^{-1}V} : A(f^{-1}U) \to A(f^{-1}V)$$

Given any subset $S \subset X$ of a topological space and a presheaf $\mathcal{F}$, define

$$\mathcal{F}(S) = \lim_{\rightarrow} \mathcal{F}(U)$$

as U ranges over all open neigbourhoods of S. When S is a point, $\mathcal{F}(S)$ is just the stalk. If $S' \subset S$, there is a restriction map $\mathcal{F}(S) \to \mathcal{F}(S')$. An element of $\mathcal{F}(S)$ can be viewed as germ of section defined in a neighbourhood of S, where two sections define the same germ if there restrictions agree in a common neighbourhood .

Definition 2.3.2. *If B is a presheaf on Y, the inverse image $f^{-1}B$ is a presheaf on X given by $f^{-1}B(U) = B(f(U))$ with restrictions as above.*

If $f : X \to Y$ is the inclusion of a closed set, we also call F_*A extension of A by 0 and $f^{-1}B$ restriction of B.

Lemma 2.3.3. *Direct and inverse images of sheaves are sheaves.*

These operations extend to functors $f_* : Ab(X) \to Ab(Y)$ and $f^{-1} : Ab(Y) \to Ab(X)$ in an obvious way. While these operations are generally not inverses, their is a relationship, which is given by the adjointness property:

Lemma 2.3.4. *There is a natural isomorphism*

$$Hom_{Ab(X)}(f^{-1}A, B) \cong Hom_{Ab(Y)}(A, f_*B)$$

Corollary 2.3.5. *There are canonical morphisms $A \to f_*f^{-1}A$ and $f^{-1}f_*B \to B$ corresponding to the identity under the isomorphisms*

$$Hom_{Ab(X)}(f^{-1}A, f^{-1}A) \cong Hom_{Ab(Y)}(A, f_*f^{-1}A)$$

$$Hom_{Ab(X)}(f^{-1}f_*B, B) \cong Hom_{Ab(Y)}(f_*B, f_*B)$$

Definition 2.3.6. *Given $\mathcal{R}$ be a sheaf of commutative rings (respectively k-algebras) over a space X, the pair $(X, \mathcal{R})$ is called a ringed (respectively k-ringed) space.*

For example, any k-space (section 1.2) is a k-ringed space. The collection of ringed spaces form a category. To motivate the definition of morphism, observe that from a morphism F of k-spaces $(X, \mathcal{R}) \to (Y, \mathcal{S})$, we get a morphism of sheaves of rings $\mathcal{S} \to F_*\mathcal{R}$ given by $f \mapsto f \circ F$.

Definition 2.3.7. *A morphism of (k-) ringed spaces $(X, \mathcal{R}) \to (Y, \mathcal{S})$ is a continuous map $F : X \to Y$ together a morphism of sheaves of rings (or algebras) $\mathcal{S} \to F_*\mathcal{R}$.*

By the above lemma, this is equivalent to giving adjoint map $F^{-1}\mathcal{S} \to \mathcal{R}$.

Exercise 2.3.8.

1. *Prove lemma 2.3.3.*

2. *Prove lemma 2.3.4.*

3. *Give examples where $A \to f_*f^{-1}A$ and $f^{-1}f_*B \to B$ are not isomorphisms.*

4. *Generalize lemma 2.2.4 to show that an exact sequence $0 \to A \to B \to C \to 0$ of sheaves gives rise to an exact sequence $0 \to f_*A \to f_*B \to f_*C$.*

2.4 The notion of a scheme

Returning to the table of section 1.3, suppose we added a new entry consisting of "all ideals" in the Algebra column. What would go in the Geometry column? The answer is "closed subschemes of $\mathbb{A}_k^n$". A scheme is a massive generalization of the notion of an algebraic variety due to Grothendieck. We will give only a small taste of the subject. The canonical reference is [EGA]. Hartshorne's book [Har] has become the standard introduction to these ideas for most people.

Let R be a commutative ring. Let $Spec\, R$ denote the set of prime ideals of R. For any ideal, $I \subset R$, let

$$V(I) = \{p \in Spec\, R \,|\, I \subseteq p\}.$$

Lemma 2.4.1.

1. $V(IJ) = V(I) \cup V(J)$.

2. $V(\sum I_i) = \cap_i V(I_i)$,

As a corollary, it follows that the sets of the form $V(I)$ form the closed sets of a topology on $Spec\, R$ called the Zariski topology. Note that when R is the coordinate ring of an affine variety Y over an algebraically closed field k, the Hilbert Nullstellensatz shows that any maximal ideal of R is of the form $m_y = \{f \in R \,|\, f(y) = 0\}$ for a unique $y \in Y$. Thus we can embed Y into $Spec\, R$ by sending y to m_y. Under this embedding $V(I)$ pulls back to the algebraic subset

$$\{y \in Y \,|\, f(y) = 0, \forall f \in I\}$$

Thus this notion of Zariski topology is an extension of the classical one.

A basis of the Zariski topology on $X = Spec\,R$ is given by $D(f) = X - V(f)$, This means that any open set $U \subset X$ is a union of $D(f)$'s. Define

$$\mathcal{O}_X(U) = \lim_{\rightarrow} R\left[\frac{1}{f}\right], \text{ as } D(f) \text{ ranges inside } U$$

When R is an integral domain with fraction field K, $\mathcal{O}_X(U) \subset K$ consists of the elements r such that for any $p \in U$, $r = g/f$ with $f \notin p$. This remark applies, in particular, to the case where R is the coordinate ring of an algebraic variety Y. In this case, $\mathcal{O}_X(U)$ can be identified with the ring of regular functions on $U \cap Y$ under the above embedding.

Lemma 2.4.2. *$\mathcal{O}_X$ is a sheaf of commutative rings such that $\mathcal{O}_{X,p} \cong R_p$ for any $p \in X$.*

Proof. We give the proof in the special case where R is a domain. This implies that X is irreducible, i. e. any two nonempty open sets intersect, because

$$D(g_i) \cap D(g_j) = D(g_i g_j) \neq \emptyset$$

if $g_i \neq 0$. Consequently the constant presheaf K_X with values in K is already a sheaf. $\mathcal{O}_X$ is a subpresheaf of K_X. Let $U = \cup U_i$ be a union of nonempty open sets, and let $f_i \in \mathcal{O}_X(U_i)$ be collection of sections agreeing on the intersections. Then $f_i = f_j$ as elements of K. Call the common value f. Since $p \in U$ lies in some U_i, f can be written as a fraction with denominator in $R - p$. Thus $f \in \mathcal{O}_X(U)$, and this shows that $\mathcal{O}_X$ is a sheaf.

One sees readily that the stalk $\mathcal{O}_{X,p}$ is the subring of K of fractions where the denominator can be chosen in $R - p$. Thus $\mathcal{O}_{X,p} \cong R_p$. ☐

The ringed space $(Spec\,R, \mathcal{O}_{SpecR})$ is called the *affine scheme* associated to R. If R is k-algebra, then this is a k-ringed space.

Definition 2.4.3. *A (k-) scheme is a (k-) ringed space which is locally isomorphic to an affine scheme.*

A morphism of (k-) schemes is simply a morphism of (k-) ringed spaces. For example, if $f : R \to S$ is a homomorphism of rings, then there is morphism of schemes $(Spec\,S, \mathcal{O}_{SpecS}) \to (Spec\,R, \mathcal{O}_{SpecR})$ such that the map on spaces is $p \mapsto f^{-1}p$. Given an affine variety Y with coordinate ring R, we have seen how to embed $Y \hookrightarrow Spec\,R$ so that $\mathcal{O}_{SpecR}$ restricts to the sheaf of regular functions on Y. More generally:

Theorem 2.4.4. *Given a prevariety Y over an algebraically closed field k. There is a k-scheme Y^{sch}, and an embedding of spaces $\iota : Y \hookrightarrow Y^{sch}$ such that*

1. *$y \in Y$ if and only if y is closed i.e. $\overline{\{y\}} = \{y\}$*
2. *$i^{-1}\mathcal{O}_{Y^{sch}} = \mathcal{O}_Y$.*

The operation $Y \mapsto Y^{sch}$ is functorial, and it gives a full faithful embedding of the category of k-varieties into the category of k-schemes.

Thus, we can redefine varieties as schemes of the form Y^{sch} (although we will frequently return to the original viewpoint). Not all k-schemes are varieties. For example, the "fat point" $Spec\, k[x]/(x^n)$ has no classical analogue.

Exercise 2.4.5.

1. *Prove lemma 2.4.1.*
2. *For any commutative R ring, check that $m \in Spec\, R$ is closed if and only if it is a maximal ideal.*
3. *Let k be an algebraically closed field. Define $(\mathbb{P}^n_k)^{sch}$ to be the set of homogeneous nonzero prime ideals in $S = k[x_0, \dots x_n]$. Give it the topology induced from $(\mathbb{P}^n_k)^{sch} \subset Spec\, S$. Define a map $\mathbb{P}^n_k \to (\mathbb{P}^n_k)^{sch}$ by sending $a = [a_0, \dots a_n]$ to the ideal generated by $\{a_i x_j - a_j x_i \mid \forall i, j\}$. Check that this is injective and that the image is exactly the set of closed points.*

2.5 Gluing schemes and toric varieties

Schemes can constructed explicitly by gluing a collection affine schemes together. This is similar to giving a manifold, by specifying an atlas for it. Let us describe the process explicitly for a pair of affine schemes. Let $X_1 = Spec\, R_1$ and $X_2 = Spec\, R_2$, and suppose we have an isomorphism

$$\phi : R_1[\frac{1}{r_1}] \cong R_2[\frac{1}{r_2}]$$

for some $r_i \in R_i$. Then, we can define the set $X = X_1 \cup_\phi X_2$ as the disjoint union modulo the equivalence relation $x \sim Spec(\phi)(x)$ for $x \in Spec\, R_1[1/r_1] \subset X_1$. We can equip X with the quotient topology, and then define

$$\mathcal{O}_X(U) = \{(s_1, s_2) \in \mathcal{O}(U \cap U_1) \times \mathcal{O}(U \cap U_2) \mid \phi(s_1) = s_2\}$$

Then X becomes a scheme with $\{X_i\}$ as a open cover. For example,

$$R_1 = k[x],\ r_1 = x,\ R_2 = k[y],\ r_2 = y,\ \phi(x) = y^{-1}$$

yields $X = \mathbb{P}^1_k$. More than two schemes can be handled in a similar way. The data consists of rings R_i and isomorphisms

$$\phi_{ij} : R_i[\frac{1}{r_{ij}}] \cong R_j[\frac{1}{r_{ji}}].$$

These are subject to a compatibility condition that the isomorphisms

$$R_i[\frac{1}{r_{ij} r_{ik}}] \cong R_k[\frac{1}{r_{ki} r_{kj}}]$$

induced by ϕ_{ik} and $\phi_{ij}\phi_{jk}$ coincide, and $\phi_{ii} = id$, $\phi_{ji} = \phi_{ij}^{-1}$. The spectra can then be glued as above, see the exercises.

Toric varieties are an interesting class of varieties that are explicitly constructed by gluing of affine schemes. The beauty of the subject stems from the interplay between the algebraic geometry and the combinatorics. See [F2] for further information (including an explanation of the name).

To simplify our discussion, we will consider only two dimensional examples. Let $\langle , \rangle$ denote the standard inner product on $\mathbb{R}^2$. A cone in $\mathbb{R}^2$ is subset of the form

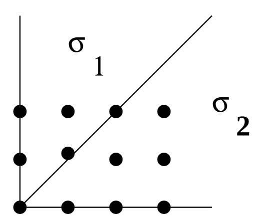
$$\sigma = \{t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \mid t_i \in \mathbb{R}, t_i \geq 0\}$$

for vectors $\mathbf{v}_i \in \mathbb{R}^2$ called generators. It is called rational if the generators can be chosen in $\mathbb{Z}^2$, and strongly convex if the angle between the generators is less than 180^o. If the generators are nonzero and coincide, σ is called a ray. The dual cone can be defined by

$$\sigma^\vee = \{\mathbf{v} \mid \langle \mathbf{v}, \mathbf{w}\rangle \geq 0, \forall \mathbf{w} \in \sigma\}$$

This is rational and spans $\mathbb{R}^2$ if σ is rational and strongly convex. Fix a field k. For each rational strongly convex cone σ, define S_σ to be the subspace of $k[x, x^{-1}, y, y^{-1}]$ spanned by $x^m y^n$ for all $(m, n) \in \sigma^\vee \cap \mathbb{Z}^2$. This easily seen to be a finitely generated subring. The affine toric variety associated to σ is $X(\sigma) = Spec\, S_\sigma$.

A fan Δ in $\mathbb{R}^2$ is a finite collection of nonoverlapping rational strongly convex cones.



Let $\{\sigma_i\}$ be the collection of cones of Δ. Any two cones interest in a ray or in the cone $\{0\}$. The maps $S_{\sigma_i} \to S_{\sigma_i \cap \sigma_j}$ are localizations at single elements, say s_{ij}. This provides us with gluing data

$$S_{\sigma_i}[\frac{1}{s_{ij}}] \cong S_{\sigma_j}[\frac{1}{s_{ji}}]$$

We define the toric variety $X = X(\Delta)$ by gluing the above schemes together.

In the example pictured above σ_1 and σ_2 are generated by $(0, 1), (1, 1)$ and $(1, 0), (1, 1)$ respectively. The varieties

$$X(\sigma_1) = Spec\, k[x, x^{-1}y] = Spec\, k[x, t]$$

$$X(\sigma_2) = Spec\, k[y, xy^{-1}] = Spec\, k[y, s]$$

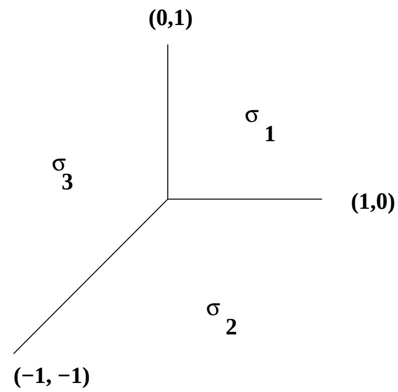
are both isomorphic to the affine plane. These can be glued by identifying (x, t) in the first plane with $(y, s) = (xt, t^{-1})$ in the second. We will see this example again in a different way. It is the *blow up* of $\mathbb{A}^2$ at $(0, 0)$.

Exercise 2.5.1.

1. *Suppose we are given gluing data as described in the first paragraph. Let $\sim$ be the equivalence relation on $\coprod \mathrm{Spec}\, R_i$ generated by $x \sim \phi_{ij}(x)$. Show $X = \coprod \mathrm{Spec}\, R_i / \sim$ can be made into a scheme with $\mathrm{Spec}\, R_i$ as an open affine covering.*

2. *Describe $\mathbb{P}^n$ by a gluing construction.*

3. *Show that the toric variety corresponding to the fan:*



is $\mathbb{P}^2$

2.6 Sheaves of Modules

Let $(X, \mathcal{R})$ be a ringed space.

Definition 2.6.1. *A sheaf of $\mathcal{R}$-modules or simply an $\mathcal{R}$-module is a sheaf M such that each $M(U)$ is an $\mathcal{R}(U)$-module and the restrictions $M(U) \to M(V)$ are $\mathcal{R}(U)$-linear.*

These sheaves form a category $\mathcal{R}$-Mod where the morphisms are morphisms of sheaves $A \to B$ such that each map $A(U) \to B(U)$ is $\mathcal{R}(U)$-linear. This is an fact an abelian category. The notion of exactness in this category coincides with the notion introduced in section 2.2.

We have already seen a number of examples in section 1.5.

Example 2.6.2. *The sheaf $\mathcal{I}_Y$ introduced in example 2.2.6 is an $\mathcal{O}_X$-module. It is called an ideal sheaf.*

Before giving more examples, we recall the tensor product and related constructions.

Theorem 2.6.3. *Given modules M_1, M_2 over a commutative ring R, there exists an R-module $M_1 \otimes_R M_2$ and map $(m_1, m_2) \mapsto m_1 \otimes m_2$ of $M_1 \times M_2 \to M_1 \otimes_R M_2$ which is bilinear and universal:*

1. $(rm + r'm') \otimes m_2 = r(m \otimes m_2) + r'(m' \otimes m_2)$
2. $m_1 \otimes (rm + r'm') = r(m_1 \otimes m) + r'(m_1 \otimes m')$
3. *Any map $M_1 \times M_2 \to N$ satisfying the first two properties is given by $\phi(m_1 \otimes m_2)$ for a unique homomorphism $\phi : M_1 \otimes M_2 \to N$.*

Given a module M over a commutative ring R, the module

$$T^*(M) = R \oplus M \oplus (M \otimes_R M) \oplus \dots$$

becomes an noncommutative associative R-algebra called the tensor algebra with product induced by $\otimes$. The exterior algebra $\wedge^* M$ (respectively symmetric algebra S^*M) is the quotient of $T^*(M)$ by the two sided ideal generated $m \otimes m$ (resp. $(m_1 \otimes m_2 - m_2 \otimes m_1)$). The product in $\wedge^* M$ is denoted by $\wedge$. $\wedge^k M$ ($S^k M$) is the submodule generated by products of k elements. If V is a finite dimensional vector space, $\wedge^k V^*$ ($S^k V^*$) can be identified with the set of alternating (symmetric) multilinear forms on V in k-variables. After choosing a basis for V, one sees that $S^k V^*$ are degree k polynomials in the coordinates.

Finally if $f : R \to S$ is a homomorphism of commutative rings, then any R-modules M gives rise to an S-modules $S \otimes_R M$, with S acting by $s(s_1 \otimes m) = ss_1 \otimes m$. This constructions is called extension of scalars.

Example 2.6.4. *Let R be a commutative ring, and M an R-module. Let $X = Spec\, R$. Let $\tilde{M}(U) = \mathcal{O}_X(U) \otimes_R M$. This is an $\mathcal{O}_X$-module. Such a module is called quasi-coherent.*

It will useful to observe:

Lemma 2.6.5. *The functor $M \to \tilde{M}$ is exact.*

Proof. This follows from the fact that $\mathcal{O}_X(U)$ is a flat R-module. We will defer the discussion of this notion until section 16.3. $\square$

Let us specialize this to the case where R is the coordinate ring of an affine algebraic variety $X \subset \mathbb{A}^n_k$ over an algebraically closed field k. As we saw earlier, X embeds into $Spec\, R$ as the set of closed points (maximal ideals). We will usually write $\tilde{M}$ instead of $\tilde{M}|_X$ when no confusion is likely. In fact, the original sheaf can be recovered from its restriction.

Most standard linear algebra operations can be carried over to modules.

Definition 2.6.6. *Given a two $\mathcal{R}$-modules M and N, their direct sum is the sheaf $U \mapsto M(U) \oplus N(U)$. The dual M^* of a $\mathcal{R}$-module M is the sheafification of the presheaf $U \mapsto Hom_{\mathcal{R}(U)}(M(U), \mathcal{R}(U))$. The tensor product $M \otimes N$ is the sheafification of the presheaf $U \mapsto M(U) \otimes_{\mathcal{R}(U)} N(U)$.*

For example the sheaf of 1-forms on a manifold is the dual of the tangent sheaf.

Definition 2.6.7. *When M is an $\mathcal{R}$-module, the kth exterior power $\wedge^k M$ and kth symmetric power $S^k M$ is sheafification of $U \mapsto \wedge^k M(U)$ and $U \mapsto S^k M(U)$. When X is manifold the sheaf of k-forms is $\mathcal{E}_X^k = \wedge^k \mathcal{E}_X^1$.*

Definition 2.6.8. *A module M is locally free (of rank n) if for every point has a neighbourhood U, such that $M|_U$ is isomorphic to a finite (n-fold) direct sum $\mathcal{R}|_U \oplus \ldots \oplus \mathcal{R}|_U$*

Given an $\mathcal{R}$-module M over X, the stalk M_x is an $\mathcal{R}_x$-module for any $x \in X$. If M is locally free, then each stalk is free of finite rank. Note that the converse may fail.

Example 2.6.9. *A variety X is nonsingular if and only if Ω_X^1 is locally free by theorem 1.4.10.*

As noted in section 1.5, locally free sheaves arise from vector bundles. Let L be the tautological line bundle on projective space $\mathbb{P} = \mathbb{P}_k^n$ over an algebraically closed field k. The sheaf of regular sections is denoted by $\mathcal{O}_{\mathbb{P}}(-1) = \mathcal{O}_{\mathbb{P}_k^n}(-1)$. $\mathcal{O}_{\mathbb{P}}(1)$ is the dual and

$$\mathcal{O}_{\mathbb{P}}(m) = \begin{cases} S^m\mathcal{O}(1) \cong \mathcal{O}(1) \otimes \ldots \mathcal{O}(1) \ (m \text{ times}) \text{ if } m > 0 \\ \mathcal{O}_{\mathbb{P}} \text{ if } m = 0 \\ S^{-m}\mathcal{O}(-1) = \mathcal{O}(-m)^* \text{ otherwise} \end{cases}$$

Let $V = k^{n+1}$. By construction $T \subset V \times \mathbb{P}^n$, so $\mathcal{O}_{\mathbb{P}}(-1)$ is a subsheaf of the $n+1$-fold sum $\mathcal{O}_{\mathbb{P}} \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}}$ which can also be expressed as $V \otimes_k \mathcal{O}_{\mathbb{P}}$. Dualizing, gives

$$V^* \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(1) \to 0$$

Taking symmetric powers gives a map, in fact an epimorphism

$$S^m V^* \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(m) \to 0$$

when $m \geq 0$. Taking global sections gives maps

$$S^m V^* \to S^m V^* \otimes \Gamma(\mathcal{O}_{\mathbb{P}}) \to \Gamma(\mathcal{O}_{\mathbb{P}}(m))$$

We will see later that these maps are isomorphisms. Thus the global sections of $\mathcal{O}_{\mathbb{P}}(m)$ are homogenous degree m polynomials in the homogeneous coordinates of m.

Suppose that $f : (X, \mathcal{R}) \to (Y, \mathcal{S})$ is a morphism of ringed spaces. Given an $\mathcal{R}$-module M, f_*M is naturally an $f_*\mathcal{R}$-module, and hence an $\mathcal{S}$-module by restriction of scalars. Similarly given an $\mathcal{S}$-module N, $f^{-1}N$ is naturally an $f^{-1}S$-module. We define the $\mathcal{R}$-module

$$f^*N = \mathcal{R} \otimes_{f^{-1}S} f^{-1}N$$

where the $\mathcal{R}$ is regarded as a $f^{-1}S$-module under the adjoint map $f^{-1}\mathcal{S} \to \mathcal{R}$. When f is injective, we often write $N|_X$ instead of f^*N.

The inverse image of a locally sheaf is locally free. This has an interpretation in the context of vector bundles, section 1.5. If $\pi : V \to Y$ is a vector bundle, then the pullback $f^*V \to X$ is given set theoretically as the projection

$$f^*V = \{(v, x) \mid \pi(v) = f(x)\} \to X$$

Then

$$f^*(\text{sheaf of sections of } V) = (\text{sheaf of sections of } f^*V)$$

Exercise 2.6.10.

1. *Show that the stalk of $\tilde{M}$ at p is precisely the localization M_p.*
2. *Show that direct sums, tensor products, exterior, and symmetric powers of locally free sheaves are locally free.*

2.7 Differentials

With basic sheaf theory in hand, we can now construct sheaves of differential forms on manifolds and varieties in a unified way. In order to motivate things, let us start with a calculation. Suppose that $X = \mathbb{R}^n$ with coordinate $x_1, \ldots x_n$. Let us complete this to coordinate system for $X \times X = \mathbb{R}^{2n}$ by adding $y_1, \ldots y_n$. Given a C^∞ function f on X, we can develop a Taylor expansion about $(y_1, \ldots y_n)$:

$$f(x_1, \ldots x_n) = f(y_1, \ldots y_n) + \sum \frac{\partial f}{\partial x_i}(y_1, \ldots y_n)(x_i - y_i) + O((x_i - y_i)^2)$$

Thus the differential is given by

$$df = f(x_1, \ldots x_n) - f(y_1, \ldots y_n) \mod (x_i - y_i)^2.$$

Let X be C^∞ or complex manifold or an algebraic variety over a field k. We take $k = \mathbb{R}$ or $\mathbb{C}$ in the first two cases. We have diagonal map $\delta : X \to X \times X$ given by $x \mapsto (x, x)$, and projections $p_i : X \times X \to X$. Let I_Δ be the ideal sheaf of the image, and let I_Δ^2 be the submodule locally generated by products of pairs of sections of I_Δ. Then we define the sheaf of 1-forms by

$$\Omega_X^1 = I_\Delta / I_\Delta^2.$$

This has two different $\mathcal{O}_X$-module structures, we pick the first one where $\mathcal{O}_X$ acts on I_Δ through $p_1^{-1}\mathcal{O}_X \to \mathcal{O}_{X\times X}$. We define the sheaf of p-forms by $\Omega_X^p = \wedge^p \Omega_X^1$ We define a morphism $d : \mathcal{O}_X \to \Omega_X^1$ of sheaves (but not $\mathcal{O}_X$-modules) by $df = p_1^*(f) - p_2^*(f)$. This has the right formal properties because of the following:

Lemma 2.7.1. *d is an k-linear derivation*

Proof. By direct calculation

$$d(fg) - fdg - gdf = [p_1^*(f) - p_2^*(f)][p_2^*(g) - p_2^*(f)] \in I_\Delta^2.$$

$\square$

The calculations at the beginning basically shows that:

Lemma 2.7.2. *If X is C^∞ manifold, then $\Omega_X^1 \cong \mathcal{E}_X^1$ and d coincides the derivative constructed in section 1.5.*

Now suppose that $X \subset \mathbb{A}_k^N$ be a closed subvariety defined by the ideal $(f_1, \dots f_r)$. Let $R = k[x_1, \dots x_n]/(f_1, \dots f_r)$ be the coordinate ring. Then Ω_X^1 is quasi-coherent. In fact, it is given by $\tilde{\Omega}_{R/k}$ where the module of Kähler differentials

$$\Omega_{R/k} = \frac{ker[R \otimes_k R \to R]}{ker[R \otimes_k R \to R]^2}$$

Using standard exact sequences [Har, II.8], we arrive at more congenial description of this module.

$$\Omega_{R/k} \cong \frac{\bigoplus_\ell R dx_\ell}{(\text{span of } \sum_j \partial f_i/\partial x_j \, dx_j)} \cong coker(\partial f_i/\partial x_j)$$

It follows from theorem 1.4.10 that if X is nonsingular, then Ω_X^1 is locally free of rank equal to $\dim X$.

Differentials behave contravariantly with respect to morphisms. Given a morphism of $f : Y \to X$, we get a morphism $Y \times Y \to X \times X$ preserving the diagonal. This induces a morphisms of sheaves

$$g^* : g^*\Omega_X^1 \to \Omega_Y^1.$$

Exercise 2.7.3.

1. *The tangent sheaf $\mathcal{T}_X = (\Omega_X^1)^*$. Show that a section $D \in \mathcal{T}_X(X)$ determines an $\mathcal{O}_X(X)$-linear derivation from $\mathcal{O}_X(X)$ to $\mathcal{O}_X(X)$.*

Chapter 3

Sheaf Cohomology

In this chapter, we give a rapid introduction to sheaf cohomology. It lies at the heart of everything else in these notes.

3.1 Flasque Sheaves

Definition 3.1.1. *A sheaf $\mathcal{F}$ on X is called flasque (or flabby) if the restriction maps $\mathcal{F}(X) \to \mathcal{F}(U)$ are surjective for any nonempty open set.*

The importance of flasque sheaves stems from the following:

Lemma 3.1.2. *If $0 \to A \to B \to C$ is an exact sequence of sheaves with A flasque, then $B(X) \to C(X)$ is surjective.*

Proof. We will prove this by the no longer fashionable method of transfinite induction[1]. Let $\gamma \in C(X)$. By assumption, there is an open cover $\{U_i\}_{i\in I}$, such that $\gamma|_{U_i}$ lifts to a section $\beta_i \in B(U_i)$. By the well ordering theorem, we can assume that the index set I is the set of ordinal numbers less than a given ordinal κ. We will define

$$\sigma_i \in B(\bigcup_{j<i} U_j)$$

inductively, so that it maps to the restriction of γ. Set $\sigma_1 = \beta_0$. Now suppose that σ_i exists. Let $U = U_i \cap (\cup_{j<i} U_j)$. Then $\beta_i|_U - \sigma_i|_U$ is the image of section $\alpha_i' \in A(U)$. By hypothesis α_i' extends to a global section $\alpha_i \in A(X)$. Then set σ_{i+1} to be σ_i on $\cup_{j<i} U_j$, and $\beta_i - \alpha_i|_{U_i}$ on U_i. If i is a limit (non-successor) ordinal, then the previous σ_j's patch to define σ_i. Then σ_κ is a global section of B mapping to γ. □

Corollary 3.1.3. *The sequence $0 \to A(X) \to B(X) \to C(X) \to 0$ is exact if A is flasque.*

[1] For most cases of interest to us, X will have a countable basis, so ordinary induction will suffice

Example 3.1.4. *Let X be a space with the property that any open set is connected (e.g. if X is irreducible). Then any constant sheaf is flasque.*

Let $\mathcal{F}$ be a presheaf, define the presheaf $G(\mathcal{F})$ of discontinuous sections of $\mathcal{F}$, by

$$U \mapsto \prod_{x\in U} \mathcal{F}_x$$

with the obvious restrictions. There is a canonical morphism $\mathcal{F} \to G(\mathcal{F})$.

Lemma 3.1.5. *$G(\mathcal{F})$ is a flasque sheaf, and the morphism $\mathcal{F} \to G(\mathcal{F})$ is a monomorphism if $\mathcal{F}$ is a sheaf.*

G is clearly a functor from $Ab(X)$ to itself.

Lemma 3.1.6. *G is an exact functor i.e. it preserves exactness.*

Proof. Given an exact sequence $0 \to A \to B \to C \to 0$, the sequence $0 \to A_x \to B_x \to C_x \to 0$ is exact by definition. Therefore

$$0 \to \prod A_x \to \prod B_x \to \prod C_x \to 0$$

is exact. □

Lemma 3.1.7. *Let $\Gamma : Ab(X) \to Ab$ denote the functor of global sections. Then $\Gamma \circ G : Ab(X) \to Ab$ is exact.*

Proof. Follows from corollary 3.1.3 and lemma 3.1.6. □

Exercise 3.1.8.

1. *Find a proof of lemma 3.1.2 which uses Zorn's lemma.*
2. *Prove lemma 3.1.5.*
3. *Prove that the sheaf of bounded continuous real valued functions on $\mathbb{R}$ is flasque*
4. *Prove the same thing for the sheaf of bounded C^∞ functions on $\mathbb{R}$.*
5. *Prove that if $0 \to A \to B \to C$ is exact and A is flasque, then $0 \to f_*A \to f_*B \to f_*C \to 0$ is exact for any continuous map f.*

3.2 Cohomology

Define $C^0(\mathcal{F}) = \mathcal{F}$, $C^1(\mathcal{F}) = coker[\mathcal{F} \to G(\mathcal{F})]$ and $C^{n+1}(\mathcal{F}) = C^1C^n(\mathcal{F})$. Now sheaf cohomology can be defined inductively:

Definition 3.2.1.

$$\begin{aligned} H^0(X,\mathcal{F}) &= \Gamma(X,\mathcal{F}) \\ H^1(X,\mathcal{F}) &= coker[\Gamma(X,G(\mathcal{F})) \to \Gamma(X,C^1(\mathcal{F}))] \\ H^{n+1}(X,\mathcal{F}) &= H^1(X,C^n(\mathcal{F})) \end{aligned}$$

$H^i(X,-)$ is clearly a functor from $Ab(X) \to Ab$. The main result is the following which says in effect that the $H^i(X,-)$ form a "delta functor" [Gr, Har].

Theorem 3.2.2. *Given an exact sequence of sheaves*

$$0 \to A \to B \to C \to 0,$$

there is a long exact sequence

$$0 \to H^0(X,A) \to H^0(X,B) \to H^0(X,C) \to H^1(X,A) \to H^1(X,B) \to \dots$$

Before giving the proof, we need:

Lemma 3.2.3. *There is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & G(A) & \to & G(B) & \to & G(C) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^1(A) & \to & C^1(B) & \to & C^1(C) & &
\end{array}$$

Proof. By lemma 3.1.6, there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & G(A) & \to & G(B) & \to & G(C) & \to & 0
\end{array}$$

The snake lemma [GM, Wl] gives the rest. $\square$

Proof. From the previous lemma and lemmas 2.2.4 and 3.1.7, we get a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & \Gamma(G(A)) & \to & \Gamma(G(B)) & \to & \Gamma(G(C)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Gamma(C^1(A)) & \to & \Gamma(C^1(B)) & \to & \Gamma(C^1(C)) & &
\end{array}$$

From the snake lemma, we obtain a 6 term exact sequence

$$0 \to H^0(X,A) \to H^0(X,B) \to H^0(X,C)$$
$$\to H^1(X,A) \to H^1(X,B) \to H^1(X,C)$$

Repeating this with A replaced by $C^1(A), C^2(A)\dots$ allows us to continue this sequence indefinitely. $\square$

Corollary 3.2.4. $B(X) \to C(X)$ *is surjective if* $H^1(X,A) = 0$.

Exercise 3.2.5.

1. *If $\mathcal{F}$ is flasque prove that $H^i(X,\mathcal{F}) = 0$ for $i > 0$. (Prove this for $i = 1$, and that $\mathcal{F}$ flasque implies that $C^1(\mathcal{F})$ is flasque.)*

3.3 Soft sheaves

In order to do some computations, we introduce the class of soft sheaves. The definition is similar to the flasque condition. We assume through out this section that X is a metric space although the results hold under the weaker assumption of paracompactness.

Definition 3.3.1. *A sheaf $\mathcal{F}$ is called soft if the map $\mathcal{F}(X) \to \mathcal{F}(S)$ is surjective for all closed sets.*

Lemma 3.3.2. *If $0 \to A \to B \to C$ is an exact sequence of sheaves with A soft, then $B(X) \to C(X)$ is surjective.*

Proof. The proof is very similar to the proof of 3.1.2. We just indicate the modifications. We can assume that the open cover $\{U_i\}$ consists of open balls. Let $\{V_i\}$ be a new open cover where we shrink the radii of each ball, so that $\bar{V}_i \subset U_i$. Define

$$\sigma_i \in B(\cup_{j<i}\bar{V}_j)$$

inductively as before. □

Corollary 3.3.3. *If A and B are soft then so is C.*

Proof. $B(X) \to B(S)$ is surjective, and the lemma shows that $B(S) \to C(S)$ is surjective. Therefore $B(X) \to C(S)$ is surjective, and this implies the same for $C(X) \to C(S)$. □

One trivially has:

Lemma 3.3.4. *A flasque sheaf is soft.*

Lemma 3.3.5. *If $\mathcal{F}$ is soft then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.*

Proof. Lemma 3.3.2 implies that $H^1(\mathcal{F}) = 0$. Corollary 3.3.3 implies that $C^1(\mathcal{F})$ is soft, and therefore inductively that all the $C^i(\mathcal{F})$ are soft. Hence $H^i(\mathcal{F}) = 0$. □

Theorem 3.3.6. *The sheaf $Cont_{X,\mathbb{R}}$ of continuous real valued functions on a metric space X is soft.*

Proof. Suppose that S is a closed subset and $f : U \to \mathbb{R}$ a real valued continuous function defined in a neighbourhood of S. We have to extend the germ of f to X. Let $d(,)$ denote the metric. We extend this to a function $d(A, B)$ for subsets $A, B \subseteq X$ by taking the minimum distance between pairs of points, i.e. $d(A, B) = \inf d(a, b)$ with $a \in A$ and $b \in B$. Let $S' = X - U$ and let

$$\rho(x) = \begin{cases} d(x, S')/\epsilon \text{ if } d(x, S') < \epsilon \\ 1 \text{ otherwise} \end{cases}$$

where $\epsilon = d(S, S')/2$. Then ρf extends by 0 to a continuous function on X. □

We get many more examples of soft sheaves with the following.

Lemma 3.3.7. *Let $\mathcal{R}$ be a soft sheaf of rings, then any $\mathcal{R}$-module is soft.*

Proof. The basic strategy is the same as above. Let f be section of an $\mathcal{R}$-module defined in the neighbourhood of a closed set S, and let S' be the complement of this neighbourhood . Since $\mathcal{R}$ is soft, the section which is 1 on S and 0 on S' extends to a global section ρ. Then ρf extends to a global section of the module. □

$U \subset \mathbb{C}$ denote the unit circle, and let $e : \mathbb{R} \to U$ denote the normalized exponential $e(x) = exp(2\pi i x)$. Let us say that X is locally simply connected if every neighbourhood of every point contains a simply connected neighbourhood.

Lemma 3.3.8. *If X is locally simply connected, then the sequence*

$$0 \to \mathbb{Z}_X \to Cont_{X,\mathbb{R}} \xrightarrow{e} Cont_{X,U} \to 1$$

is exact.

Lemma 3.3.9. *If X is simply connected and locally simply connected, then $H^1(X, \mathbb{Z}_X) = 0$.*

Proof. Since X is simply connected, any continuous map from X to U can be lifted to a continuous map to its universal cover $\mathbb{R}$. In other words, $C_{\mathbb{R}}(X)$ surjects onto $C_U(X)$. Since $\mathbb{C}_{\mathbb{R}}$ is soft, lemma 3.3.8 implies that $H^1(X, \mathbb{Z}_X) = 0$. □

Corollary 3.3.10. $H^1(\mathbb{R}^n, \mathbb{Z}) = 0$.

Exercise 3.3.11.

1. *Show that $Cont_{\mathbb{R},\mathbb{R}}$ is not flasque.*

3.4 C^∞-modules are soft

We want prove that the sheaf of C^∞ functions on a manifold is soft. We start with a few lemmas.

Lemma 3.4.1. *Given $\epsilon > 0$, there exists an $\mathbb{R}$-linear operator $\Sigma_\epsilon : C(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ such that if $f \equiv 1$ for all points in an ϵ ball about x_0, then $\Sigma_\epsilon(f)(x_0) = 1$.*

Proof. Let ψ be a C^∞ function on $\mathbb{R}^n$ with support in $||x|| < 1$ such that $\int \psi = 1$. Rescale this by setting $\phi(x) = \epsilon^n \psi(x/\epsilon)$. Then

$$\Sigma_\epsilon(f)(x) = \int_{\mathbb{R}^n} f(y)\phi(x-y)dy$$

will have the desired properties. □

Lemma 3.4.2. *Let $S \subset \mathbb{R}^n$ be a closed subset, and let U be an open neighbourhood of S, then there exists a C^∞ function $\rho : \mathbb{R}^n \to \mathbb{R}$ which is 1 on S and 0 outside U.*

Proof. We constructed a continuous function $\rho_1 : \mathbb{R}^n \to \mathbb{R}$ with these properties in the previous section. Then set $\rho = \Sigma_\epsilon(\rho_1)$ with ϵ sufficiently small. $\square$

We want to extend this to a manifold X. For this, we need the following construction. Let $\{U_i\}$ be a locally finite open cover of X, which means that every point of X is contained in a finite number of U_i's. A *partition of unity* subordinate to $\{U_i\}$ is a collection C^∞ functions $\phi_i : X \to [0,1]$ such that

1. The support of ϕ_i lies in U_i.
2. $\sum \phi_i = 1$ (the sum is meaningful by local finiteness).

Partitions of unity always exist for any locally finite cover, see [Spv, Wa] and exercises.

Lemma 3.4.3. *Let $S \subset X$ be a closed subset, and let U be an open neighbourhood of S, then there exists a C^∞ function $\rho : X \to \mathbb{R}$ which is 1 on S and 0 outside U.*

Proof. Let $\{U_i\}$ be a locally finite open cover of X such that each U_i is diffeomorphic to $\mathbb{R}^n$. Then we have functions $\rho_i \in C^\infty(U_i)$ which are 1 on $S \cap U_i$ and 0 outside $U \cap U_i$ by the previous lemma. Choose a partition of unity $\{\phi_i\}$. Then $\rho = \sum \phi_i \rho_i$ will give the desired function. $\square$

Theorem 3.4.4. *Given a C^∞ manifold X, C^∞_X is soft.*

Proof. Given a function f defined a neighbourhood of a closed set $S \subset X$. Let ρ be a given as in the lemma 3.4.3, then ρf gives a global C^∞ function extending f. $\square$

Corollary 3.4.5. *Any C^∞-module is soft.*

This arguments can be extended by introducing the notion of a *fine* sheaf.

Exercise 3.4.6.

1. *Let $\{U_i\}$ be a locally finite open cover of X such that each U_i is diffeomorphic to $\mathbb{R}^n$. Construct a partition of unity in this case, by first constructing a family of continuous functions satisfying these conditions, and then applying Σ_ϵ.*

3.5 Mayer-Vietoris sequence

We will introduce a basic tool for computing cohomology groups which is a prelude to Čech cohomology. Let $U \subset X$ be open. For any sheaf, we want to define natural restriction maps $H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F})$. If $i = 0$, this is just the usual restriction. For $i = 1$, we have a commutative square

$$\begin{array}{ccc} \Gamma(X, G(\mathcal{F})) & \to & \Gamma(X, C^1(\mathcal{F})) \\ \downarrow & & \downarrow \\ \Gamma(U, G(\mathcal{F})) & \to & \Gamma(U, C^1(\mathcal{F})) \end{array}$$

which induces a map on the cokernels. In general, we use induction.

Theorem 3.5.1. *Let X be a union to two open sets $U \cup V$, then for any sheaf there is a long exact sequence*

$$\ldots H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}) \oplus H^j(V, \mathcal{F}) \to H^i(U \cap V, \mathcal{F}) \to H^{i+1}(X, \mathcal{F}) \ldots$$

where the first indicated arrow is the sum of the restrictions, and the second is the difference.

Proof. The proof is very similar to the proof of theorem 3.2.2, so we will just sketch it. Construct a diagram

$$\begin{array}{ccccccccc} 0 & \to & \Gamma(X, G(\mathcal{F})) & \to & \Gamma(U, G(\mathcal{F})) \oplus \Gamma(V, G(\mathcal{F})) & \to & \Gamma(U \cap V, G(\mathcal{F})) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \Gamma(X, C^1(\mathcal{F})) & \to & \Gamma(U, C^1(\mathcal{F})) \oplus \Gamma(V, C^1(\mathcal{F})) & \to & \Gamma(U \cap V, C^1(\mathcal{F})) & & \end{array}$$

and apply the snake lemma to get the sequence of the first 6 terms. Then repeat with $C^i(\mathcal{F})$ in place of $\mathcal{F}$. □

Exercise 3.5.2.

1. *Use Mayer-Vietoris to prove that $H^1(S^1, \mathbb{Z}) \cong \mathbb{Z}$.*
2. *Show that $H^1(S^n, \mathbb{Z}) = 0$ if $n \geq 2$.*

Chapter 4

De Rham cohomology

In this chapter, we apply sheaf theory to study C^∞ manifolds.

4.1 Acyclic Resolutions

A complex of abelian groups (or sheaves or more generally elements in an abelian category) is a possibly infinite sequence

$$\dots F^i \xrightarrow{d^i} F^{i+1} \xrightarrow{d^{i+1}} \dots$$

of groups and homomorphisms satisfying $d^{i+1}d^i = 0$. These condtions guarantee that $image(d^i) \subseteq ker(d^{i+1})$. We denote a complex by $F^\bullet$ and we often suppress the indices on d. The cohomology groups of $F^\bullet$ are defined by

$$\mathcal{H}^i(F^\bullet) = \frac{ker(d^i)}{image(d^{i-1})}.$$

These groups are zero precisely when the complex is exact.

Definition 4.1.1. *A sheaf $\mathcal{F}$ is called acyclic if $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

For example, flasque and soft sheaves are acyclic.

Definition 4.1.2. *An acyclic resolution of $\mathcal{F}$ is an exact sequence*

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \dots$$

of sheaves such that each $\mathcal{F}^i$ is acyclic.

Given a complex of sheaves $\mathcal{F}^\bullet$, the sequence

$$\Gamma(X, \mathcal{F}^0) \to \Gamma(X, \mathcal{F}^1) \to \dots$$

need not be exact, however it is necessarily a complex by functoriallity. Thus we can form its cohomology groups.

Theorem 4.1.3. *Given an acyclic resolution of $\mathcal{F}$ as above,*

$$H^i(X, \mathcal{F}) \cong \mathcal{H}^i(\Gamma(X, \mathcal{F}^\bullet))$$

Proof. Let $K^{-1} = \mathcal{F}$ and $K^i = ker(\mathcal{F}^{i+1} \to \mathcal{F}^{i+2})$. Then there are exact sequences

$$0 \to K^{i-1} \to \mathcal{F}^i \to K^i \to 0$$

Since $\mathcal{F}^i$ are acyclic, theorem 3.2.2 implies that

$$0 \to H^0(K^{i-1}) \to H^0(\mathcal{F}^i) \to H^0(K^i) \to H^1(K^{i-1}) \to 0 \tag{4.1}$$

is exact, and

$$H^j(K^i) \cong H^{j+1}(K^{i-1}) \tag{4.2}$$

for $j > 0$. The sequences (4.1) leads to a commutative diagram

$$\begin{tikzcd}
H^0(K^{i-1}) \arrow[r, hook] & H^0(\mathcal{F}^i) \arrow[rr] \arrow[dr] & & H^0(\mathcal{F}^{i+1}) \\
& & H^0(K^i) \arrow[ur, hook] &
\end{tikzcd}$$

where the hooked arrows are injective. Therefore

$$H^0(K^{i-1}) \cong ker[H^0(\mathcal{F}^i) \to H^0(\mathcal{F}^{i+1})]$$

This already implies the first case of the theorem when $i = 0$. This isomorphism together with the sequence (4.1) implies that

$$H^1(K^{i-1}) \cong \frac{ker[H^0(\mathcal{F}^{i+1}) \to H^0(\mathcal{F}^{i+2})]}{image[H^0(\mathcal{F}^i)]}$$

Combining this with the isomorphisms

$$H^{i+1}(K^{-1}) \cong H^i(K^0) \cong \ldots H^1(K^{i-1})$$

of (4.2) finishes the proof. □

4.2 De Rham's theorem

Let X be a C^∞ manifold and $\mathcal{E}^k = \mathcal{E}^k_X$ the sheaf of k-forms. Note that $\mathcal{E}^0 = C^\infty$.

Theorem 4.2.1. *There exists canonical $\mathbb{R}$-linear maps $d : \mathcal{E}^k(X) \to \mathcal{E}^{k+1}(X)$, called exterior derivatives, satisfying the following*

1. *$d : \mathcal{E}^0(X) \to \mathcal{E}^1(X)$ is the operation introduced in section 1.5.*
2. *$d^2 = 0$.*
3. *$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^i \alpha \wedge d\beta$ for all $\alpha \in \mathcal{E}^i(X)$, $\beta \in \mathcal{E}^j(X)$.*

4. *If $g : Y \to X$ is C^∞ map, $g^* \circ d = d \circ g^*$.*

Proof. A complete proof can be found in almost any book on manifolds (e.g. [Wa]). We will only sketch the idea. When X is a ball in $\mathbb{R}^n$ with coordinates x_i, one sees that there is a unique operation satisfying the above rules given by

$$d(f dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = \sum_j \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots dx_{i_k}$$

This applies to any coordinate chart. By uniqueness, these local d's patch. □

When $X = \mathbb{R}^3$, d can be realized as the $div, grad, curl$ of vector calculus. The theorem tells that $\mathcal{E}^\bullet(X)$ forms a complex. We define the De Rham cohomology groups (actually vector spaces) as

$$H^k_{dR}(X) = \mathcal{H}^k(\mathcal{E}^\bullet(X))$$

Notice that the exterior derivative is really a map of sheaves $d : \mathcal{E}^k_X \to \mathcal{E}^{k+1}_X$ satisfying $d^2 = 0$. Thus we have complex. Moreover, $\mathbb{R}_X$ is precisely the kernel of $d : \mathcal{E}^0_X \to \mathcal{E}^1_X$.

We compute the de Rham cohomology of Euclidean space.

Theorem 4.2.2 (Poincaré's lemma). *For all n and $k > 0$,*

$$H^k_{dR}(\mathbb{R}^n) = 0.$$

Proof. Assume, by induction, that the theorem holds for $n-1$. Identify $\mathbb{R}^{n-1}$ with the hyperplane $x_1 = 0$. Let I be the identity transformation and $R : \mathcal{E}^k(\mathbb{R}^n) \to \mathcal{E}^k(\mathbb{R}^n)$ the operator which drops terms containing x_1 and dx_1. The image of R can be identified with $\mathcal{E}^k(\mathbb{R}^{n-1})$. Note that R commutes with d. So if $\alpha \in \mathcal{E}^k(\mathbb{R}^n)$ is closed, which means that $d\alpha = 0$, then $dR\alpha = Rd\alpha = 0$. By the induction assumption, $R\alpha$ is exact which means that it lies in the image of d.

For each k, define a map $h : \mathcal{E}^k(\mathbb{R}^n) \to \mathcal{E}^{k-1}(\mathbb{R}^n)$ by

$$h(f(x_1, \ldots x_n) dx_1 \wedge dx_{i_2} \wedge \ldots) = (\int_0^{x_1} f dx_1) dx_{i_2} \wedge \ldots$$

and

$$h(f dx_{i_1} \wedge dx_{i_2} \wedge \ldots) = 0$$

if $1 \notin \{i_1, i_2, \ldots\}$. Then one checks that $dh + hd = I - R$ (in other words, h is homotopy from I to R). Given $\alpha \in \mathcal{E}^k(\mathbb{R}^n)$ satisfying $d\alpha = 0$. We have $\alpha = dh\alpha + R\alpha$, which is exact. □

We will prove that the sheaves $\mathcal{E}^k$ are acyclic. It is enough, by lemma 3.3.7, to establish the following.

Theorem 4.2.3. *The sequence*

$$0 \to \mathbb{R}_X \to \mathcal{E}^0_X \to \mathcal{E}^1_X \dots$$

is an acyclic resolution of $\mathbb{R}_X$.

Proof. Any ball is diffeomorphic to Euclidean space, and any point on a manifold has a fundamental system of such neigbourhoods. Therefore the above sequence is exact on stalks, and hence exact.

The acyclicity of $\mathcal{E}^k_X$ follows from corollary 3.4.5. □

Corollary 4.2.4 (De Rham's theorem).

$$H^k_{dR}(X) \cong H^k(X, \mathbb{R})$$

Later on, we will work with complex valued differential forms. Essentially the same argument shows that $H^*(X, \mathbb{C})$ can be computed using such forms.

Exercise 4.2.5.

1. *We will say that a manifold is of finite type if it has a finite open cover* $\{U_i\}$ *such that any nonempty intersection of the* U_i *are diffeomorphic to the ball. Compact manifolds are known to have finite type [Spv, pp 595-596]. Using Mayer-Vietoris and De Rham's theorem, prove that if* X *is an* n*-dimensional manifold of finite type, then* $H^k(X, \mathbb{R})$ *vanishes for* $k > n$*, and is finite dimensional otherwise.*

4.3 Poincaré duality

Let X be an C^∞ manifold. Let $\mathcal{E}^k_c(X)$ denote the set of C^∞ k-forms with compact support. Since $d\mathcal{E}^k_c(X) \subset \mathcal{E}^{k+1}_c(X)$, this forms a complex.

Definition 4.3.1. *Compactly supported de Rham cohomology is defined by* $H^k_{cdR}(X) = \mathcal{H}^k(\mathcal{E}^\bullet_c(X))$.

Lemma 4.3.2. *For all* n,

$$H^k_{cdR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} \text{ if } k = n \\ 0 \text{ otherwise} \end{cases}$$

Proof. [Spv]. □

This computation suggests that these groups are roughly opposite to the usual de Rham groups. The precise statement requires the notion of orientation. An orientation on an n dimensional real vector space V is a connected component of $\wedge^n V - \{0\}$. An ordered basis $v_1, \dots v_n$ is positively oriented if $v_1 \wedge \dots \wedge v_n$ lies in this component. An orientation on an n dimensional manifold X is a choice of a connected component of $\wedge^n T_X$ minus its zero section.

Theorem 4.3.3. *Let X be an oriented n-dimensional manifold. Then*

$$H^k_{cdR}(X) \cong H^{n-k}(X, \mathbb{R})^*$$

Given an open set $U \subset X$, define the space of poor man's *currents*[1] of degree k to be $\mathcal{C}^k(U) = \mathcal{C}^k_X(U) = \mathcal{E}^{n-k}_c(U)^*$. Given $V \subset U$, $\alpha \in \mathcal{C}^k_X(U)$, $\beta \in \mathcal{E}^k(V)$, let $\alpha|_V(\beta) = \alpha(\tilde{\beta})$ where $\tilde{\beta}$ is the extension of β by 0. This makes $\mathcal{C}^k_X$ a presheaf.

Lemma 4.3.4. *$\mathcal{C}^k_X$ is a sheaf.*

Proof. Let $\{U_i\}$ be an open cover of U, and $\alpha_i \in \mathcal{C}^k_X(U_i)$. Let $\{\rho_i\}$ be a C^∞ partition of unity subordinate to $\{U_i\}$. Then define $\alpha \in \mathcal{C}^k_X(U)$ by

$$\alpha(\beta) = \sum_i \alpha_i(\rho_i\beta)$$

Suppose that $\beta \in \mathcal{E}^k_c(U_j)$ is extended by 0 to U. Then $\rho_i\beta$ will be supported in $U_i \cap U_j$. Consequently, $\alpha_i(\rho_i\beta) = \alpha_j(\rho_i\beta)$. Therefore

$$\alpha(\beta) = \alpha_j(\sum_i \rho_i\beta) = \alpha_j(\beta).$$

∎

Define a map $\delta : \mathcal{C}^k_X(U) \to \mathcal{C}^{k+1}_X(U)$ by $\delta(\alpha)(\beta) = \alpha(d\beta)$. One automatically has $\delta^2 = 0$. Thus one has a complex of sheaves.

Let X be an oriented n-dimensional manifold. Suppose that α is an n-form supported in a coordinate neighbourhood U_i. We can write $\alpha = f(x_1, \ldots x_n)dx_1 \wedge \ldots dx_n$, where the order of the coordinates are chosen so that $\partial/\partial x_1 \ldots$ gives a positive orientation of T_X. Then

$$\int_X \alpha = \int_{\mathbb{R}^n} f(x_1, \ldots x_n)dx_1 \ldots dx_n$$

The standard change of variables formulas from calculus shows that this satifies the patching condition on $U_i \cap U_j$, therefore $\int_X$ defines a canonical global section of $\mathcal{C}^0_X$.

Theorem 4.3.5 (Stokes). *Let X be an oriented n-dimensional manifold, then $\int_X d\beta = 0$.*

Proof. See [Spv, Wa]. ∎

Corollary 4.3.6. $\int_X \in ker[\delta]$.

We define a map $\mathbb{R}_X \to \mathcal{C}^0_X$ induced by the map from the constant presheaf sending $r \to r\int_X$. Then theorem 4.3.3 follows from

[1] True currents are required to be continuous for an appropriate topology on $\mathcal{E}^{n-k}_c(U)$

Lemma 4.3.7.
$$0 \to \mathbb{R}_X \to \mathcal{C}^0_X \to \mathcal{C}^1_X \to \dots$$
is an acyclic resolution.

Proof. Lemma 4.3.2 implies that this complex is exact. The sheaves $\mathcal{C}^k_X$ are soft since they are C^∞_X-modules. $\square$

Proof of theorem 4.3.3. We can now use the complex $\mathcal{C}^\bullet_X$ to compute the cohomology of $\mathbb{R}_X$ to obtain

$$H^i(X,\mathbb{R}) \cong H^i(\mathcal{C}^\bullet_X(X)) = H^i(\mathcal{E}^\bullet_c(X)^*).$$

The right hand space is easily seen to be isomorphic to $H^i_{cdR}(X,\mathbb{R})^*$. This completes the proof of the theorem. $\square$

Corollary 4.3.8. *If X is a compact oriented n-dimensional manifold. Then*

$$H^k(X,\mathbb{R}) \cong H^{n-k}(X,\mathbb{R})^*$$

The following is really a corollary of the proof.

Corollary 4.3.9. *If X is a connected oriented n-dimensional manifold. Then the map $\alpha \mapsto \int_X \alpha$ induces an isomorphism (denoted by same symbol)*

$$\int_X : H^n_{cdR}(X,\mathbb{R}) \cong \mathbb{R}$$

Let us suppose that X is a compact connected oriented n dimensional manifold. We can make the duality isomorphism much more explicit. If α and β are closed forms (i.e. lie in the kernel of d), then so is $\alpha \wedge \beta$ by theorem 4.2.1. If $[\alpha]$ and $[\beta]$ denote the classes in $H^*_{dR}(X)$ represented by these forms, then define $[\alpha] \cup [\beta] = [\alpha \wedge \beta]$. This is a well defined operation which makes de Rham cohomology into a graded ring. The following will be proved later on (cor. 7.2.2): *If $f \in H^{n-i}(X,\mathbb{R})^*$, then there exists a unique $\alpha \in H^i(X,\mathbb{R})$ such that $f(\beta) = \int_X \alpha \cup \beta$.*

4.4 Fundamental class

Let $Y \subset X$ be a closed connected oriented m dimensional submanifold of an n dimensional oriented manifold. Denote the inclusion by i. There is a natural restriction map
$$i^* : H^a(X,\mathbb{R}) \to H^a(Y,\mathbb{R})$$
induced by restriction of forms. Using Poincaré duality we get a map going in the opposite direction

$$i_! : H^a(Y,\mathbb{R}) \to H^{a+n-m}(X,\mathbb{R})$$

called the *Gysin map*. Let 1_Y denote constant function 1 on Y. This is the natural generator for $H^0(Y,\mathbb{R})$.

Definition 4.4.1. *The fundamental class of Y in X is $[Y] = i_! 1_Y \in H^{n-m}(X, \mathbb{R})$.*

The basic relation is given by

$$\int_Y i^* \alpha = \int_X [Y] \cup \alpha \tag{4.3}$$

We want to make this more explicit. But first, we need:

Theorem 4.4.2. *There exists an open neigbourhood T, called a tubular neigbourhood, of Y in X and a $\pi : T \to Y$ which makes T a locally trivial rank $(n-m)$ real vector bundle over Y.*

Proof. See [Spv, p. 465]. ☐

We can factor $i_!$ as a composition

$$H^a(Y) \to H^{a+n-m}_{cdR}(T) \to H^{a+n-m}(X)$$

where the first map is the Gysin map for the inclusion $j : Y \hookrightarrow T$ and the second is induced by extension by zero. $j_!$ is an isomorphism since it is dual to

$$H^{m-a}(T) \xrightarrow{\cong} H^{m-a}_{cdR}(Y)$$

The *Thom class* of T is $\tau_Y = j_! 1_Y$. The image of τ_Y in $H^{n-m}(X, \mathbb{R})$ is precisely the fundamental class $[Y]$. τ_Y can be represented by any closed compactly supported $n-m$ form on T whose integral along any fiber is 1. It is possible to choose a neighbourhood U of a point of Y with local coordinates x_i, such that Y is given by $x_{m+1} = \dots x_n = 0$ and π is given by $(x_1, \dots x_n) \mapsto (x_1, \dots x_m)$. The restriction map

$$H^i_{cdR}(T) \to H^{i-n-m}(U) \otimes H^{n-m}_{cdR}(\mathbb{R}^{n-m})$$

is an isomorphism. Therefore the Thom class can be represented by an expression

$$f(x_{m+1}, \dots x_n) dx_{m+1} \wedge \dots dx_n$$

where f is compactly supported in $\mathbb{R}^{n-m}$.

Let $Y, Z \subset X$ be oriented submanifolds such that $dimY + dimZ = n$. Then $[Y] \cup [Z] \in H^n(X, \mathbb{R}) \cong \mathbb{R}$ corresponds to a number $Y \cdot Z$. This has a geometric interpretation. We say that Y and Z are transverse if $Y \cap Z$ is finite and if $T_{Y,p} \oplus T_{Z,p} = T_{X,p}$ for each p in the intersection. Choose ordered bases $v_1(p), \dots v_m(p) \in T_{Y,p}$ and $v_{m+1}(p) \dots v_n(p) \in T_{Z,p}$ which are positively oriented with respect to the orientations of Y and Z. We define the intersection number

$$i_p(Y, Z) = \begin{cases} +1 \text{ if } v_1(p) \dots v_m(p), v_{m+1}(p), \dots v_n(p) \text{ is positively oriented} \\ -1 \text{ otherwise} \end{cases}$$

This is easily seen to be independent of the choice of bases.

Proposition 4.4.3. *If Y and Z are transverse, $Y \cdot Z = \sum_p i_p(Y, Z)$.*

Proof. Choose tubular neighbourhoods T of Y and T' of Z. These can be chosen "small enough" so that $T \cap T'$ is a union of disjoint neighbourhoods around U_p each $p \in Y \cap Z$ diffeomorphic to $\mathbb{R}^n = \mathbb{R}^{\dim Y} \times \mathbb{R}^{\dim Z}$ Then

$$Y \cdot Z = \int_X \tau_Y \wedge \tau_Z = \sum_p \int_{U_p} \tau_Y \wedge \tau_Z$$

Choose coordinates $x_1, \ldots x_n$ around p so that Y is given by $x_{m+1} = \ldots x_n = 0$ and Z by $x_1 = \ldots x_m = 0$. Then as above, the Thom classes of T and T' can be written as

$$\tau_Y = f(x_{m+1}, \ldots x_n) dx_{m+1} \wedge \ldots dx_n$$

$$\tau_Z = g(x_1, \ldots x_m) dx_1 \wedge \ldots dx_m$$

with

$$\int_{\mathbb{R}^{n-m}} f(x_{m+1}, \ldots x_n) dx_{m+1} \ldots dx_n = \int_{\mathbb{R}^m} g(x_1, \ldots x_m) dx_1 \ldots dx_m = 1.$$

Fubini's theorem gives

$$\int_{U_p} \tau_Y \wedge \tau_Z = i_p(Y, Z)$$

□

4.5 Lefschetz trace formula

Let X be a compact n dimensional oriented manifold with a C^∞ map $f : X \to X$. The Lefschetz formula is a formula for the number of fixed points counted appropriately. First, we have to explain what that number means. Let

$$\Gamma_f = \{(x, f(x)) \mid x \in X\}$$

$$\Delta = \{(x, x) \mid x \in X\}$$

be the graph of f and the diagonal respectively. These are both n dimensional submanifolds of $X \times X$ which intersect precisely at points (x, x) with $x = f(x)$. We define the "number of fixed points" as $\Gamma_f \cdot \Delta$, which makes sense if it interesection is infinite. If these manifolds are transverse, we see that this can be evaluated as the sum over fixed points

$$\sum_x i_x(\Gamma_f, \Delta)$$

Given a C^∞ map of manifolds $g : Y \to X$, we get map $g^* : \mathcal{E}^*(X) \to \mathcal{E}^\bullet(Y)$ of the de Rham complexes which induces a map g^* on cohomology. It follows easily that $X \mapsto H^i_{dR}(X, \mathbb{R})$ is a contravariant functor.

Theorem 4.5.1. *The number* $\Gamma_f \cdot \Delta$ *is given by*

$$L(f) = \sum_p (-1)^p trace[f^* : H^p(X, \mathbb{R}) \to H^p(X, \mathbb{R})]$$

Proof. Let $n = \dim X$. By Poincaré duality, we have isomorphisms $H^{n-p}(X) \cong H^p(X)^*$. For each p, choose a basis $\alpha_{p,i}$ of $H^p(X)$, and let $\alpha^*_{p,i}$ denote the dual basis of $H^{n-p}(X)$. Let $p_i : X \times X \to X$ denote the projections. Then $\{p_1^*\alpha_{p,i} \cup p_2^*\alpha^*_{p,j}\}_{p,i,j}$ gives a basis for $H^n(X \times X)$, and $\{(-1)^{n-p} p_1^*\alpha^*_{p,i} \cup p_2^*\alpha_{p,j}\}$ is seen to give the dual basis. Thus we can express

$$[\Delta] = \sum c_{p,i,j} p_1^*\alpha_{p,i} \cup p_2^*\alpha^*_{p,j}$$

The coefficients can be computed by integrating the dual basis

$$c_{p,i,j} = \int_\Delta (-1)^{n-p} p_1^*\alpha^*_{p,i} \cup p_2^*\alpha_{p,j} = (-1)^{n-p} \int_X \alpha_{p,i} \cup \alpha_{p,j} = (-1)^p \delta_{ij}$$

Therefore

$$[\Delta] = \sum_{i,p} (-1)^{n-p} p_1^*\alpha_{p,i} \cup p_2^*\alpha^*_{p,i}$$

Consequently,

$$\begin{aligned}
\Gamma_f \cdot \Delta &= \int_{\Gamma_f} [\Delta] \\
&= \sum_p (-1)^p \sum_i \int_{\Gamma_f} p_1^*\alpha_{p,i} \cup p_2^*\alpha^*_{p,i} \\
&= \sum_p (-1)^{n-p} \sum_i \int_X \alpha_{p,i} \cup f^*\alpha^*_{p,i} \\
&= \sum_p (-1)^{n-p} trace[f^* : H^{n-p}(X, \mathbb{R}) \to H^{n-p}(X, \mathbb{R})] \\
&= L(f)
\end{aligned}$$

□

Corollary 4.5.2. *If* $L(f) \neq 0$*, then* f *has a fixed point.*

Proof. If $\Gamma_f \cap \Delta = \emptyset$, then $\Gamma_f \cdot \Delta = 0$. □

4.6 Examples

We look as some basic examples to illustrate the previous ideas. Let $T = \mathbb{R}^n/\mathbb{Z}^n$. Let $\{e_i\}$ be the standard basis, and let x_i be coordinates on $\mathbb{R}^n$. Then

Proposition 4.6.1. *Every de Rham cohomology class on* T *contains a unique form with constant coefficients.*

We will postpone the proof until section 7.2.

Corollary 4.6.2. *There is an algebra isomorphism* $H^*(T,\mathbb{R}) \cong \wedge^*\mathbb{R}^n$

Since T is a product of circles, the corollary also follows from repeated application of the Künneth formula:

Theorem 4.6.3. *Let* X *and* Y *be* C^∞ *manifolds, and let* $p : X \times Y \to X$ *and* $q : X \times Y \to Y$ *be the the projections. Then the map*

$$\sum \alpha_i \otimes \beta_j \mapsto \sum p^*\alpha_i \wedge q^*\beta_j$$

induces an isomorphism

$$\bigoplus_{i+j=k} H^i(X,\mathbb{R}) \otimes H^j(Y,\mathbb{R}) \cong H^k(X \times Y,\mathbb{R})$$

On the torus, Poincaré duality becomes the standard isomorphism

$$\wedge^k\mathbb{R}^n \cong \wedge^{n-k}\mathbb{R}^n.$$

If $V_I \subset \mathbb{R}^n$ is the span of $\{e_i \mid i \in I\}$, then $T_I = V_I/(\mathbb{Z}^n \cap V_I)$ is a submanifold of T. Its fundamental class is $dx_{i_1} \wedge \ldots \wedge dx_{i_d}$, where $i_1 < \ldots < i_d$ are the elements of I in increasing order. If J is the complement of I, then $T_I \cdot T_J = \pm 1$.

Next consider, complex projective space $\mathbb{P}^n_{\mathbb{C}}$. Then

$$H^i(\mathbb{P}^n,\mathbb{R}) = \begin{cases} \mathbb{R} \text{ if } 0 \leq i \leq 2n \text{ is even} \\ 0 \text{ otherwise} \end{cases}$$

This is the basic example for us, and it will be studied further in section 6.2.

Chapter 5

Riemann Surfaces

Recall that Riemann surfaces are the same thing as one dimensional complex manifolds. Later on we will call these objects complex curves.

5.1 Topological Classification

A Riemann surface can be regarded as a manifold of real dimension 2. It has a canonical orientation: if we identify the real tangent space at any point with the complex tangent space, then for any nonzero vector v, we declare the ordered basis (v, iv) to be positively oriented. Let us now forget the complex structure and consider the purely topological problem of classifying these surfaces up to homeomorphism.

Given two 2 dimensional topological manifolds X and Y with points $x \in X$ and $y \in Y$, we can form new topological manifold $X\#Y$ called the connected sum. To construct this, choose open disks $D_1 \subset X$ and $D_2 \subset Y$. Then $X\#Y$ is obtained by gluing $X - D_1 \cup S^1 \times [0,1] \cup Y - D_2$ appropriately. Figure (5.1) depicts the connected sum of two tori.

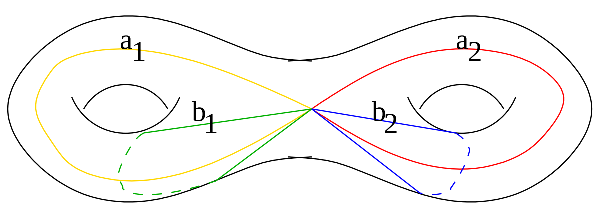


Figure 5.1: Genus 2 Surface

Theorem 5.1.1. *A compact connected orientable 2 dimensional topological manifold is classified, up to homeomorphism, by a nonnegative integer called the genus. A genus 0 surface is manifold is homeomorphic to the 2-sphere S^2.*

A manifold of genus $g > 0$ is homeomorphic to a connected sum of the 2-torus and a surface of genus $g - 1$.

Proof. See [Ms]. $\square$

There is another standard model for these surfaces which is also quite useful (for instance for computing the fundamental group). A genus g surface can constructed by gluing the sides of a $2g$-gon. It is probably easier to visualize this in reverse. After cutting the genus 2 surface of (5.1) along the indicated curves, it can be opened up to an octagon (5.2).

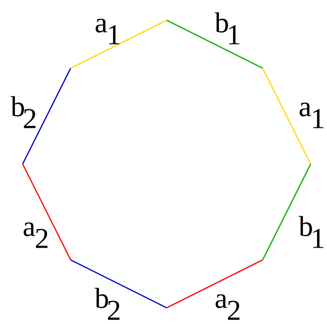


Figure 5.2: Genus 2 surface cut open

It is easy to produce Riemann surfaces of every genus, and this will be done in the next section.

The topological Euler characteristic of space X is

$$e(X) = \sum (-1)^i dim\, H^i(X, \mathbb{R}).$$

We need to assume that these dimensions are finite and that all but finitely many of them are zero for this to make sense.

Lemma 5.1.2. *If X is a union of two open sets U and V, then $e(X) = e(U) + e(V) - e(U \cap V)$.*

Proof. This follows from the Mayer-Vietoris sequence. $\square$

Corollary 5.1.3. *If X is a manifold of genus g, then $e(X) = 2 - 2g$.*

Consider the pairing

$$(\alpha, \beta) \mapsto \int \alpha \wedge \beta$$

on $H^1(X, \mathbb{C})$. This is skew symmetric and nondegenerate by Poincaré duality (7.2.2). This can be represented by the matrix pf intersection numbers for basis of $H_1(X, \mathbb{Z})$. For example, after orientating the curves a_1, a_2, b_1, b_2 in figure (5.1) properly, we get the intersection matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Exercise 5.1.4.

1. *Prove corollary 5.1.3.*
2. *Let S be the toplogical space associated to a finite simplicial complex (jump ahead to the chapter 6 for the definition if necessary). Prove that $e(S)$ is the alternating sum of the number of simplices.*

5.2 Examples

Many examples of compact Riemann surfaces can be constructed explicitly nonsingular smooth projective curves. In fact, all compact Riemann surfaces are known to be algebraic curves.

Example 5.2.1. *Let $f(x, y, z)$ be a homogeneous polynomial of degree d. Suppose that the partials of f have no common zeros in $\mathbb{C}^3$ except $(0, 0, 0)$. Then the $V(f) = \{f(x, y, z) = 0\}$ in $\mathbb{P}^2_{\mathbb{C}}$ is smooth. We will see later that the genus is $(d - 1)(d - 2)/2$. In particular, not every genus occurs for these examples.*

Example 5.2.2. *Given a collection of homogeneous polynomials $f_i \in \mathbb{C}[x_0, \dots x_n]$ such that $X = V(f_1, f_2 \dots) \subset \mathbb{P}^n_{\mathbb{C}}$ is a nonsingular algebraic curve. Then X will a complex submanifold of $\mathbb{P}^n_{\mathbb{C}}$ and hence a Riemann surface. By a generic projection argument, $n = 3$ suffices to give all such examples.*

Example 5.2.3. *Choose $2g + 2$ distinct points in $a_i \in \mathbb{C}$. Consider the affine curve $X_1 \subset \mathbb{C}^2$ defined by*

$$y^2 = \prod(x - a_i)$$

This can be compactified by passing to the projective algebraic curve defined by

$$z^{2g}y^2 - \prod(x - a_i z) = 0$$

where x, y, z are the homogeneous coordinates. This will have a singularity at $[0, 1, 0]$, which can be resolved by normalizing the curve to obtain a smooth projective curve X. Instead of carrying this out, we construct X directly by gluing another affine curve X_2:

$$Y^2 = \prod(a_i X - 1)$$

to X_1 by identifying $X = x^{-1}$ and $Y = yx^{-g-1}$. Since X is nonsingular, it can be viewed as a Riemann surface. By construction, X comes equipped with *a morphism $f : X \to \mathbb{P}^1_{\mathbb{C}}$ which is 2 to 1 except at the branch points $\{a_i\}$. In the exercises, it will be shown that the genus of X is g. Curves which can realized as two sheeted ramified coverings of $\mathbb{P}^1_{\mathbb{C}}$ are rather special, and are called hyperelliptic if $g > 1$*

From a more analytic point of view, we can construct many examples as quotients of $\mathbb{C}$ or the upper half plane. In fact, the uniformization theorem tells us that all examples other than $\mathbb{P}^1$ arise this way.

Example 5.2.4. *Let $L \subset \mathbb{C}$ be a lattice, i. e. an abelian subgroup generated by two $\mathbb{R}$-linearly independent numbers. The quotient $E = \mathbb{C}/L$ can be made into a Riemann surface (exercise 1.2.11) called an elliptic curve. Since this topologically a torus, the genus is 1. Conversely, any genus 1 curve is of this form. (There is a distinction between these two notions, an elliptic curve is a genus 1 curve with distinguished point which serves as the origin for the group law.)*

This is not an ellipse at all of course. It gets its name because of its relation to elliptic integrals and functions. An elliptic function is a meromorphic function on $\mathbb{C}$ which is periodic with respect to a lattice L. A basic example is the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in L,\, \lambda \neq 0} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

This induces a map on the quotient $E \to \mathbb{P}^1$ which is two sheeted and branched at 4 points. One of the branch points will include ∞. We can construct an algebraic curve E' and a 2 to 1 map $E' \to \mathbb{P}^1$ with the same branch points. For example, if the branch points are $0, 1, \infty, t$ (and we can assume this without loss of generality), E' is given by Legendre's equation

$$y^2 z = x(x-z)(x-tz) \tag{5.1}$$

It can be checked that $E \cong E'$, hence E is algebraic. Alternatively, we can prove algebraicity of E by explicitly embedding it into $\mathbb{P}^2$ by

$$z \mapsto \begin{cases} [\wp(z), \wp'(z), 1] & \text{if } z \notin L \\ [0,1,0] & \text{otherwise} \end{cases}$$

The image is an algebraic curve defined explicitly by the Weierstrass equation

$$zy^2 = 4x^3 - g_2(L)xz^2 - g_3(L)z^3 \tag{5.2}$$

See [Si] for futher details.

The group $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I\}$ acts on $H = \{z \,|\, Im(z) > 0\}$ by fractional linear transformations:

$$z \mapsto \frac{az+b}{cz+d}$$

The action of subgroup $\Gamma \subset PSL_2(\mathbb{R})$ on H is *properly discontinuous* if every point has a neighbourhood V such that $gV \cap V \neq \emptyset$ for all but finitely many $g \in \Gamma$. A point $x \in V$ is called a *fixed point* if its stabilizer is nontrivial. The action is *free* if it has no fixed points.

Proposition 5.2.5. *If Γ acts properly discontinuously on H, the quotient $X = H/\Gamma$ becomes a Riemann surface. If $\pi : H \to X$ denotes the projection, define the structure sheaf $f \in \mathcal{O}_X(U)$ if and only if $f \circ \pi \in \mathcal{O}_H(\pi^{-1}U)$.*

For free actions, the proof is completely straightforward. When Γ acts freely with compact quotient, then it has genus $g > 1$. The quickest way to see this is by applying the Gauss-Bonnet theorem to the hyperbolic metric. A fundamental domain for this action is a region $R \subset H$ such that $\cup_g gR = H$ and two translates of R can only meet at their boundaries. The fundamental domain in this case can be chosen to be the interior of a geodesic $2g$-gon.

The modular group, $SL_2(\mathbb{Z})$, is a particularly interesting example, since the quotient classifies elliptic curves. Two elliptic curves $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ and $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau'$, with $\tau, \tau' \in H$, are isomorphic if and only if τ and τ' lie in the same orbit of $SL_2(\mathbb{Z})$. A fundamental domain for the action is given in figure 5.3 [Si, S4]

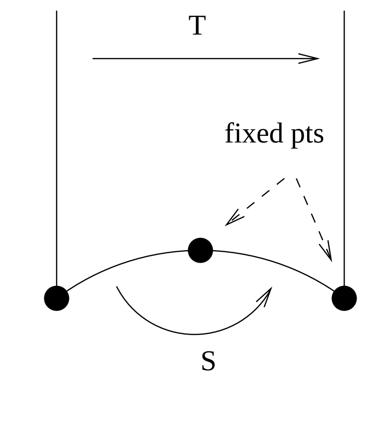


Figure 5.3: fundamental domain of $SL_2(\mathbb{Z})$

After identifying the sides of the domain by the transformations

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

as indicated in the diagram, one sees that that the quotient $H/SL_2(\mathbb{Z})$ is homeomorphic to $\mathbb{C}$. By the Riemann mapping theorem, $H/SL_2(\mathbb{Z})$ isomorphic to either $\mathbb{C}$ or H. To see that it is $\mathbb{C}$, we show that the one point compactification $H/SL_2(\mathbb{Z}) \cup \{*\}$ can also be made into a Riemann surface, necessarily isomorphic to $\mathbb{P}^1$. Any continuous function f defined in a neighbourhood of $*$ can be pulled back to an invariant function $\tilde{f}$ on H. Invariance under T implies that $\tilde{f}$ is periodic, hence it can expanded in a Fourier series

$$\tilde{f}(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z}$$

We declare f to be holomorphic if all the $a_n = 0$ for $n < 0$. To put this another way, $q = e^{2\pi i z}$ is set as a local analytic coordinate at $*$.

Exercise 5.2.6.

1. *Check that the genus of the hyperelliptic curve constructed above is g by triangulating in such way that the $\{a_i\}$ are included in the set of vertices.*

5.3 The $\bar{\partial}$-Poincaré lemma

Let $U \subset \mathbb{C}$ be an open set. Let x and y be real coordinates on $\mathbb{C}$, and $z = x+iy$. Given a complex valued C^∞ function $f : U \to \mathbb{C}$, let

$$\frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}).$$

With this notation, the Cauchy-Riemann equation is simply $\frac{\partial f}{\partial \bar{z}} = 0$. Define the complex valued 1-forms $dz = dx + idy$ and $d\overline{z} = dx - idy$. With this notation, we can formulate Cauchy's formula for C^∞ functions.

Theorem 5.3.1. *Let $D \subset \mathbb{C}$ be a disk. If $f \in C^\infty(\bar{D})$, then*

$$f(\zeta) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z-\zeta}\, dz + \frac{1}{2\pi i}\int_D \frac{\partial f(z)}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z-\zeta}$$

Proof. This follows from Stokes' theorem, see [GH, pp. 2-3]. □

The following as analogue of the Poincaré lemma.

Theorem 5.3.2. *Let $D \subset \mathbb{C}$ be an open disk. Given $f \in C^\infty(\bar{D})$, the function $g \in C^\infty(D)$ given by*

$$g(\zeta) = \frac{1}{2\pi i}\int_D \frac{f(z)}{z-\zeta}\, dz \wedge d\bar{z}$$

satisfies $\frac{\partial g}{\partial \bar{z}} = f$.

Proof. Decompose $f(z) = f_1(z) + f_2(z)$ into a sum of C^∞ functions, where $f_1(z) \equiv f(z)$ in a small neighbourhood of $z_0 \in D$ and vanishes near the boundary of D. In particular, f_2 is zero in neighbourhood of z_0. Let g_1 and g_2 be the functions obtained by substituting f_1 and f_2 for f in the integral of the theorem. Differentiating under the integral sign yields

$$\frac{\partial g_2(\zeta)}{\partial \bar\zeta} = \frac{1}{2\pi i}\int_D \frac{\partial}{\partial\bar\zeta}\left(\frac{f_2(z)}{z-\zeta}\right) dz\wedge d\bar z = \frac{1}{2\pi i}\int_D d\left(\frac{f_2(z)dz}{z-\zeta}\right) = 0$$

for ζ close to z_0. Since, f_1 is compactly supported,

$$g_1(\zeta) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f_1(z)}{z-\zeta}\, dz\wedge d\bar z$$

Then doing a change of variables $w = z-\zeta$,

$$g_1(\zeta) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f_1(w+\zeta)}{w}\, dw\wedge d\bar w$$

Thus for ζ close to z_0

$$\begin{aligned}
\frac{\partial g(\zeta)}{\partial\bar\zeta} &= \frac{\partial g_1(\zeta)}{\partial\bar\zeta}\\
&= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial f_1(w+\zeta)}{\partial\bar\zeta}\,\frac{dw\wedge d\bar w}{w}\\
&= \frac{1}{2\pi i}\int_D \frac{\partial f_1(z)}{\partial\bar z}\,\frac{dz\wedge d\bar z}{z-\zeta}
\end{aligned}$$

Since f_1 vanishes on the boundary, the last integral equals $f_1(\zeta) = f(\zeta)$ by Cauchy's formula 5.3.1. □

In order to make it easier to globalize the above operators to Riemann surfaces, we reinterpret them in terms of differential forms. In this chapter, $C^\infty(U)$ and $\mathcal{E}^n(U)$ will denote the space of complex valued C^∞ functions and n-forms. The exterior derivative extends to a $\mathbb{C}$-linear operator between these spaces. Set

$$\partial f = \frac{\partial f}{\partial z}dz$$

$$\bar\partial f = \frac{\partial f}{\partial\bar z}d\bar z$$

We extend this to 1-forms, by

$$\begin{aligned}
\partial(fd\bar z) &= \frac{\partial f}{\partial z}dz\wedge d\bar z\\
\partial(fdz) &= 0\\
\bar\partial(fdz) &= \frac{\partial f}{\partial\bar z}d\bar z\wedge dz\\
\bar\partial(fd\bar z) &= 0
\end{aligned}$$

A 1-form α is holomorphic $\alpha = f dz$ with f holomorphic. This is equivalent to the condition $\bar{\partial}\alpha = 0$. The following identities can be easity verified:

$$\begin{aligned} d &= \partial + \bar{\partial} \\ \partial^2 &= \bar{\partial}^2 = 0 \\ \partial\bar{\partial} + \bar{\partial}\partial &= 0 \end{aligned} \tag{5.3}$$

Exercise 5.3.3.

1. *Check the identities (5.3).*

2. *Show that a form $f dz$ is closed if and only if it is holomorphic.*

5.4 $\bar{\partial}$-cohomology

Let X be a Riemann surface manifold with $\mathcal{O}_X$ its sheaf of holomorphic functions. We write C_X^∞ and $\mathcal{E}_X^n$ for the sheaves of complex valued C^∞ functions and n-forms. We define a C_X^∞-submodule $\mathcal{E}_X^{(1,0)} \subset \mathcal{E}_X^1$ (respectively $\mathcal{E}_X^{(0,1)} \subset \mathcal{E}_X^1$), by $\mathcal{E}_X^{(1,0)}(U) = C^\infty(U)dz$ (resp. $\mathcal{E}_X^{(0,1)}(U) = C^\infty(U)d\bar{z}$) for any coordinate neighbourhood U with holomorphic coordinate z. We have a decomposition

$$\mathcal{E}_X^1 = \mathcal{E}_X^{(1,0)} \oplus \mathcal{E}_X^{(0,1)}$$

We set $\mathcal{E}_X^{(1,1)} = \mathcal{E}_X^2$ as this is locally generated by $dz \wedge d\bar{z}$.

Lemma 5.4.1. *There exists $\mathbb{C}$-linear maps $\partial, \bar{\partial}$ on the sheaves $\mathcal{E}_X^\bullet$ which coincide with the previous expressions in local coordinates.*

It follows that the identities (5.3) hold globally. The kernel of $\bar{\partial}$ on C_X^∞ is $\mathcal{O}_X$, and we define the sheaf of holomorphic 1-forms Ω_X^1 to be the kernel if $\bar{\partial}$ on $\mathcal{E}_X^{(0,1)}$. It is a locally free $\mathcal{O}_X$-module.

Lemma 5.4.2. *The sequences of sheaves*

$$0 \to \mathcal{O}_X \to C_X^\infty \xrightarrow{\bar{\partial}} \mathcal{E}_X^{(0,1)} \to 0$$

$$0 \to \Omega_X^1 \to \mathcal{E}_X^{(1,0)} \xrightarrow{\bar{\partial}} \mathcal{E}_X^{(1,1)} \to 0$$

are acyclic resolutions.

Proof. Any C^∞-module is soft, hence acyclic, and the exactness follows from theorem 5.3.2. ∎

Corollary 5.4.3.

$$H^1(X, \mathcal{O}_X) = \frac{\mathcal{E}^{(0,1)}(X)}{\bar{\partial}C^\infty(X)}$$

$$H^1(X, \Omega^1_X) = \frac{\mathcal{E}^{(1,1)}(X)}{\bar{\partial}\mathcal{E}^{(1,0)}_X(X)}$$

and

$$H^i(X, \mathcal{O}_X) = H^i(X, \Omega^1_X) = 0$$

if $i > 1$.

Next, we want a holomorphic analogue of the de Rham complex.

Proposition 5.4.4. *There is an exact sequence of sheaves*

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \to 0$$

Proof. The only nontrivial part of the assertion is that $\mathcal{O}_X \to \Omega^1_X \to 0$ is exact. We can check this by replacing X by a disk D. A holomorphic 1-form on D is automatically closed, therefore exact by the usual Poincaré lemma. If df is holomorphic then $\bar{\partial} f = 0$, so f is holomorphic. □

Corollary 5.4.5. *There is a long exact sequence*

$$0 \to H^0(X, \mathbb{C}) \to H^0(X, \mathcal{O}_X) \to H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C}) \ldots$$

Holomorphic 1-forms are closed, and

$$H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C})$$

is the map which sends a holomorphic form to its class in de Rham cohomology.

Lemma 5.4.6. *When X is compact and connected, this map is an injection.*

Proof. It is equivalent to proving that

$$H^0(X, \mathbb{C}) \to H^0(X, \mathcal{O}_X)$$

is surjective i.e. that global holomorphic functions are constant. Let f be a holomorphic function on X. f must attain a maximum at some point, say $x \in X$. Choose a coordinate disk $D \subset X$ centered at x. The we can apply the maximum modulus principle to conclude that f is constant on D. Since $f - f(x)$ has a nonisolated 0, it follows by complex analysis that f is globally constant. □

We will postpone the following proposition to section 8.1.

Proposition 5.4.7. *The dimensions of $H^0(X, \Omega^1_X)$ and $H^1(X, \mathcal{O}_X)$ both coincide with the genus.*

This implies that the map

$$H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X)$$

is surjective, and therefore that that $H^1(X, \Omega^1_X)$ is isomorphic to $H^2(X, \mathbb{C}) = \mathbb{C}$.

Exercise 5.4.8.

1. *Let V be a vector space with a nondegenerate skew symmetric pairing $\langle,\rangle$. A subspace $W \subset V$ is called isotropic if $\langle w, w'\rangle = 0$ for all $w, w' \in W$. Prove that* $\dim W \leq \dim V/2$ *if W is isotropic.*

2. *Show that $H^0(X, \Omega^1_X)$ is isotropic for the pairing $\int \alpha \wedge \beta$. Use this to conclude that dim $H^0(X, \Omega^1_X) \leq g$.*

3. *Show that the differentials $x^i dx/y$, with $0 \leq i < g$, are holomorphic on the hyperelliptic curve 5.2.3. Conclude directly that $H^0(X, \Omega^1_X) = g$ in this case.*

5.5 Projective embeddings

Fix a compact Riemann surface X. We introduce some standard shorthand: $h^i = dim\, H^i$, and $\omega_X = \Omega^1_X$. A divisor D on X is a finite integer linear combination $\sum n_i p_i$ where $p_i \in X$. One says that D is effective if all the coefficients are nonnegative. The degree $deg\, D = \sum n_i$. For every meromorphic function defined in a neighbourhood of $p \in X$, let $ord_p(f)$ be the order of vanishing (or minus the order of the pole) of f at p. If D is a divisor define $ord_p(D)$ to be the coefficient of p in D (or 0 if p is absent). Define the sheaf $\mathcal{O}_X(D)$ by

$$\mathcal{O}_X(D)(U) = \{f : U \to \mathbb{C} \cup \{\infty\} \text{ meromorphic} \mid ord_p(f) + ord_p(D) \geq 0, \forall p \in U\}$$

Lemma 5.5.1. *This is a holomorphic line bundle (i.e. a locally free $\mathcal{O}_X$-module of rank one).*

Proof. Let z be a local coordinate defined in a neighbourhood U. Let $D = \sum n_i p_i + D'$ where $p_i \in U$ and D' is a sum of points not in U. Then it can be checked that

$$\mathcal{O}_X(D)(U) = \mathcal{O}_X(U) \frac{1}{(z - p_1)^{n_1} (z - p_2)^{n_2} \ldots},$$

and this is free of rank one. $\square$

Divisors form an abelian group $Div(X)$ in the obvious way.

Lemma 5.5.2. $\mathcal{O}_X(D + D') \cong \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')$.

In later terminology, this says that $D \mapsto \mathcal{O}_X(D)$ is a homomorphism from $Div(X) \to Pic(X)$. If D is effective $O(-D)$ is a sheaf of ideals. In particular, $\mathcal{O}_X(-p)$ is exactly the maximal ideal sheaf at p. We have an exact sequence

$$0 \to \mathcal{O}_X(-p) \to \mathcal{O}_X \to \mathbb{C}_p \to 0 \tag{5.4}$$

where

$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C} \text{ if } p \in U \\ 0 \text{ otherwise} \end{cases}$$

is a so called *skyscraper sheaf*. Tensoring (5.4) by $\mathcal{O}_X(D)$ and observing that $\mathbb{C}_p \otimes L \cong \mathbb{C}_p$ for any line bundle, yields

$$0 \to \mathcal{O}_X(D-p) \to \mathcal{O}_X(D) \to \mathbb{C}_p \to 0 \tag{5.5}$$

In the same way, we get a sequence

$$0 \to \omega_X(D-p) \to \omega_X(D) \to \mathbb{C}_p \to 0 \tag{5.6}$$

where $\omega_X(D) = \omega_X \otimes \mathcal{O}_X(D)$.

Lemma 5.5.3. *For all D, $H^i(X, \mathcal{O}_X(D))$ and $H^i(X, \omega_X(D))$ are finite dimensional and 0 if $i > 1$.*

Proof. Observe that $\mathbb{C}_p$ is flasque, and thus has no higher cohomology. (5.5) yields

$$0 \to H^0(\mathcal{O}_X(D-p)) \to H^0(\mathcal{O}_X(D)) \to \mathbb{C} \to H^1(\mathcal{O}_X(D-p)) \to H^1(\mathcal{O}_X(D)) \to 0$$

and isomorphisms

$$H^i(\mathcal{O}_X(D-p)) \cong H^i(\mathcal{O}_X(D))\ i > 0$$

By adding or subtracting points, we can reduce this to the case of $\mathcal{O}_X(D) = \mathcal{O}_X$. The argument for $\omega_X(D)$ is the same. □

A meromorphic 1-form is an element of $\cup H^0(\omega_X(D))$. The residue of a meromorphic 1-form α at p is

$$res_p(\alpha) = \frac{1}{2\pi i}\int_C \alpha$$

where C is any loop "going once counterclockwise" around p and containing no singularities other than p. Alternatively, if $\alpha = f(z)dz$ locally for some local coordinate z at p, $res_p(\alpha)$ is the coefficient of $\frac{1}{z}$ in the Laurent expansion of $f(z)$.

Lemma 5.5.4 (Residue Theorem). *If α is a meromorphic 1-form, the sum of its residues is 0.*

Proof. $\{p_1, \dots p_n\}$. For each i, choose an open disk D_i containing p_i and no other singularity. Then by Stokes' theorem

$$\sum res_{p_i}\alpha = \frac{1}{2\pi i}\int_{X-\cup D_i} d\alpha = 0.$$

□

Theorem 5.5.5. *Suppose that D is a nonzero effective divisor then*

(A) $H^1(\omega_X(D)) = 0$.

(B) $h^0(\omega_X(D)) = deg\, D + g - 1$.

(A) is due to Serre. It is a special case of the Kodaira vanishing theorem. (B) is a weak form of the Riemann-Roch theorem.

Proof. (A) can be proved by induction on the degree of D. We carry out the initial step, the rest is left as an exercise. Suppose $D = p$. Then $H^0(\omega_X(p))$ consists of the space of meromorphic 1-forms α with at worst a simple pole at p and no other singularities. The residue theorem implies that such an α must be holomorphic. Therefore $H^0(\omega_X(p)) = H^0(\omega_X)$. By the long exact sequence of cohomology groups associated to (5.6), we have

$$0 \to \mathbb{C} \to H^1(\omega_X) \to H^1(\omega_X(p)) \to 0$$

Since the space in the middle is one dimensional, this proves (A) in this case.

(B) will again be proved by induction. As already noted when $D = p$, $h^0(\omega_X(D)) = h^0(\omega_X) = g$. In general, (5.6) and (A) shows

$$h^0(\omega_X(D)) = 1 + h^0(\omega_X(D - p)) = deg\, D + g - 1$$

by induction. □

Corollary 5.5.6. *There exists a divisor (called a canonical divisor) such that* $\omega_X \cong \mathcal{O}_X(K)$

Proof. Choose D so that $H^0(\omega_X(D))$ possesses a nonzero section α. Locally $\alpha = f dz$, and we define $ord_p(\alpha) = ord_p(f)$ (this is independent of the coordinate z). Then

$$K = (\alpha) - D = \sum ord_p(\alpha)p - D$$

satisfies the required properties. □

Although, we won't prove it here, the degree of K is known.

Proposition 5.5.7. $deg\, K = 2g - 2$

We say that a line bundle L on X is*globally generated* if for any point $x \in X$, there exists a section $f \in H^0(X, L)$ such that $f(x) \neq 0$. Suppose that this is the case, Choose a basis $f_0, \ldots f_N$ for $H^0(X, L)$. If fix an isomorphism $\tau : L|_U \cong \mathcal{O}_U$, $\tau(f_i)$ are holomorphic functions on U. Thus we get a holomorphic map $U \to \mathbb{C}^{N+1}$ given by $x \mapsto (\tau(f_i(x)))$. By our assumption, the image lies in the complement of 0, and thus descends to a map to projective space. The image is independent of τ, hence we get a well defined holomorphic map

$$\phi_L : X \to \mathbb{P}^N$$

This map has the property that $\phi_L^* \mathcal{O}_{\mathbb{P}^N}(1) = L$. L is called *very ample* if ϕ_L is an embedding.

Proposition 5.5.8. *A sufficient condition for L to be globally generated is that $H^1(X, L(-p)) = 0$ for all $p \in X$ A sufficient condition for L to be very ample is that $H^1(X, L(-p-q)) = 0$ for all $p, q \in X$*

Corollary 5.5.9. *$\omega_X(D)$ is very ample if D is nonzero effective with $degD > 2$. In particular, any compact Riemann surface can be embedded into a projective space.*

Using Chow's theorem 14.5.3, we obtain

Corollary 5.5.10. *Every compact Riemann surface is isomorphic to an non-singular projective algebraic curve.*

Exercise 5.5.11.

1. *Finish the proof of theorem 5.5.5 (A), by writing $D = p + D'$ and applying (5.6).*

5.6 Automorphic forms

Let $\Gamma \subset SL_2(\mathbb{R})$ be a subgroup acting properly discontinuously on H such that H/Γ is compact. Let k a positive integer. An *automorphic form* of weight $2k$ is a holomorphic function $f : H \to \mathbb{C}$ on the upper half plane satisfying

$$f(z) = (cz + d)^{-2k} f(\frac{az + b}{cz + d}) \tag{5.7}$$

for each

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

Choose a weight $2k$ automorphic form f. Then $f(z)(dz)^{\otimes k}$ is invariant under the group precisely when f is automorphic of weight $2k$. Since $-I$ acts trivially on H, the action of $SL_2(\mathbb{R})$ factors through $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I\}$. Let us suppose that the group $\Gamma/\{\pm I\}/$ acts freely, then the quotient $X = H/\Gamma$ is a Riemann surface, and an automorphic form of weight $2k$ descends to a section of the sheaf $\omega_X^{\otimes k}$. We can apply theorem 5.5.5 to the calculate the dimensions of these spaces.

Proposition 5.6.1. *Suppose that $\Gamma/\{\pm I\}$ acts freely on H and that the quotient $X = H/\Gamma$ is compact of genus g. Then the dimension of the space of automorphic forms of weight $2k$ is*

$$\begin{cases} g & \text{if } k = 1 \\ (g-1)(2k-1) & \text{if } k > 1 \end{cases}$$

Proof. When $k = 1$, this is clear. When $k > 1$, we have

$$h^0(\omega^{\otimes k}) = h^0(\omega((k-1)K)) = (k-1)(deg\, K) + g - 1 = (2k-1)(g-1)$$

$\square$

The above conditions are a bit too stringent, since they don't allow examples such as the modular group $SL_2(\mathbb{Z})$, or its subgroups such as the nth principle congruence subgroup

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a - 1 \equiv d - 1 \equiv b \equiv c \equiv 0 \mod n \right\}$$

The quotient $H/SL_2(\mathbb{Z})$ can be identified with $\mathbb{C}$ as we saw. The natural compactification $\mathbb{P}^1_{\mathbb{C}}$ can be constructed as a quotient as follows. H corresponds to the upper hemisphere of $\mathbb{P}^1_{\mathbb{C}} = S^2$, and $\mathbb{R} \cup \{\infty\}$ corresponds to the equator. We take H and add rational points on the boundary $H^* = H \cup \mathbb{Q} \cup \{\infty\}$. These are called cusps. Then $SL_2(\mathbb{Z})$ acts on this, and the cusps form a unique orbit corresponding the point at infinity in $\mathbb{P}^1$. We can apply the same technique to any finite index subgroup of the modular group.

Theorem 5.6.2. *Given a finite index subgroup $\Gamma \subset SL_2(\mathbb{Z})$, H^*/Γ can be made into a compact Riemann surface, called a modular curve, such that $H^*/\Gamma \to H^*/SL_2(\mathbb{Z}) \cong \mathbb{P}^1$ is holomorphic.*

A *modular form* of weight $2k$ for $\Gamma \subset SL_2(\mathbb{Z})$ is a holomorphic function $f : H \to \mathbb{C}$ satisfying (5.7) and certain growth conditions which forces it to correspond to a holomorphic (or possibly meromorphic) object on H^*/Γ. For example, the Eisentein series

$$G_{2k}(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 - \{(0,0)\}} \frac{1}{(m + n\tau)^{2k}}$$

are modular of weight $2k$ for $SL_2(\mathbb{Z})$ if $k > 1$. Their signifance for elliptic curves is that $g_2 = 60G_4$, $g_3 = 140G_6$ give coefficients of the Weirstrass equation (5.2) for the lattice $\mathbb{Z} + \mathbb{Z}\tau$.

The points of the quotient $H/SL_2(\mathbb{Z})$ correspond to isomorphism classes of elliptic curves. More generally the points of the quotient $H/\Gamma(n)$ correpond to elliptic curves with some extra structure called a level n structure. Let us spell this out for $n = 2$. An elliptic curve is an abelian group, and a level two structure is a minimal set of generators for its subgroup of 2-torsion elements. Given $\tau \in H$, we get an elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ with the level two structure $(1/2, \tau/2) \mod \mathbb{Z} + \mathbb{Z}\tau$. If τ' lies in the orbit of τ under $\Gamma(2)$, then there is an isomorphism $E_\tau \cong E_{\tau'}$ taking $(1/2, \tau/2) \mod \mathbb{Z} + \mathbb{Z}\tau$ to $(1/2, \tau'/2) \mod \mathbb{Z} + \mathbb{Z}\tau'$. Furthermore any elliptic curve is isomorphic to an E_τ, and the isomorphism can be chosen so that a given level two structure goes over to the standard one. Thus $H/\Gamma(2)$ classifies elliptic curves with level two structure as claimed. We can describe the set of such curves in another way. Given an elliptic curve E defined by Legendre's equation (5.1), with $t \in \mathbb{P}^1 - \{0, 1, \infty\}$, the ramification points (the points on E lying over $0, 1, \infty, t$) are precisely the 2-torsion points. We can take the ramification point at ∞ to be the origin, then any other pair of branch points determines a level two structure Conversely, given an elliptic curve E, with origin o and a level two structure (p, q), we have

$h^0(\mathcal{O}_E(2o)) = 2$. This means that there is a meromorphic function $f : E \to \mathbb{P}^1_{\mathbb{C}}$ with a double pole at o. It is not hard to see that f is ramified precisely at the two torsion points $o, p, q, p+q$ (+ refers group law on E). By composing f with a (unique) automorphism of $\mathbb{P}^1_{\mathbb{C}}$, we can put E in Legendre form such that f is projection to the x-axis, and $f(o) = \infty$, $f(p) = t \in \mathbb{P}^1 - \{0, 1, \infty\}$, $f(q) = 0$, $f(p+q) = 1$. Thus $H/\Gamma(2)$ isomorphic to $\mathbb{P}^1 - \{0, 1, \infty\}$. Furthermore, $\Gamma(2)/\{\pm I\}$ acts freely on H. A pretty consequence of all of this is:

Theorem 5.6.3 (Picard's little theorem). *An entire function omitting two or more points must be constant.*

Proof. The universal cover of $\mathbb{P}^1 - \{0, 1, \infty\}$ is H which is isomorphic to the unit disk D. Let f be an entire function omitting two points, which we can assume are 0 and 1. Then f lifts to holomorphic map $\mathbb{C} \to D$ which is bounded and therefore constant by Liouville's theorem. □

Chapter 6

Simplicial Methods

In this chapter, we will develop some tools for actually computing cohomology groups in practice. All of these are based on simplicial methods.

6.1 Simplicial and Singular Cohomology

A systematic development of the ideas in this section can be found in [Sp]. The standard n-simplex is

$$\Delta^n = \{(t_1, \dots t_{n+1}) \in \mathbb{R}^{n+1} \mid \sum t_i = 1,\, t_i \geq 0\}$$

The ith face Δ^n_i is the intersection of Δ^n with the hyperplane $t_i = 0$ (see (6.1)). Each face is homeomorphic to an $n-1$ simplex by an explicit affine map $\delta_i : \Delta^{n-1} \to \Delta^n_i$. More generally, we refer to the intersection of Δ with the linear space $t_{i_1} = \dots t_{i_k} = 0$ as a face.

Some fairly complicated topological spaces, called *polyhedra or triangulable* spaces, can be built up by gluing simplices. It is known, although by no means obvious, that manifolds and algebraic varieties (with classical topology) can be triangulated. The combinatorics of the gluing is governed by a simplicial complex.

Definition 6.1.1. *A simplicial complex (V, Σ) consists of a set V, called the set of vertices, and collection of finite nonempty subsets Σ of V containing all the singletons and closed under taking nonempty subsets.*

We can construct a topological space $|(V, \Sigma)|$ out of a simplicial complex roughly as follows. To each maximal element $S \in \Sigma$, choose an n-simplex $\Delta(S)$, where $n+1$ is the cardinality of S. Glue $\Delta(S)$ to $\Delta(S')$ along the face labeled by $S \cap S'$ whenever this is nonempty. (When V is infinite, this gluing process requires some care, see [Sp, chap. 3].)

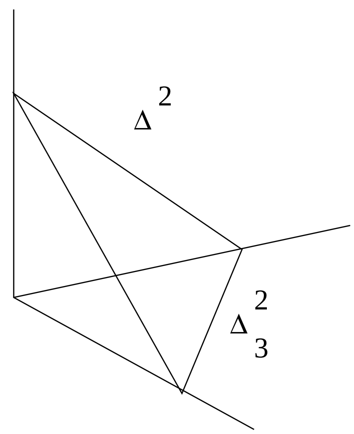


Figure 6.1: 2 simplex

Let $K = (V, \Sigma)$ be a simplical complex, and assume that V is linearly ordered. We will refer to element of Σ as a n-simplex if it has cardinality $n+1$. We define an n-*chain* on a simplicial complex to be a finite formal integer linear combination $\sum_i n_i \Delta_i$ where the Δ_i are n-simplices. In other words, the set of n-chains $C_n(K)$ is the free Abelian group generated by K. Given an abelian group, let $C_n(K, A) = C_n(K) \otimes_{\mathbb{Z}} A$. Dually, we define an n-*cochain* on a simplicial complex with values in A to be function which assigns an element of A to every n-simplex. The set of n-cochains $C^n(K, A) = Hom(C_n(K), A)$. One can think of a n-cochain some sort of combinatorial analogue of an integral of n-form, where the domain of integration is specified by a chain. As in integration theory we need to worry about orientations, and this is where the ordering comes in. An alternative, which is probably more standard, is to use oriented simplices; the complexes one gets this way are bigger, but the resulting cohomology theory is the same.

We define the ith face of a simplex

$$\delta_i(\{v_0, \ldots v_n\}) = \{v_0, \ldots \widehat{v_i} \ldots v_n\}$$

$v_0 < v_1 \ldots < v_n$. (The notation $\widehat{x}$ means omit x.) Given an n-chain $C = \sum_j n_j \Delta_j$, we define a $(n-1)$-chain $\partial(C)$. called the boundary of C by

$$\delta(C) = \sum \sum (-1)^i n_j \delta_i(\Delta_j)$$

This operation can be extended by scalars to $C_\bullet(K, A)$ and it induces a dual operation $\partial : C^n(K, A) \to C^{n+1}(K, A)$ by $\partial(F)(C) = F(\delta C)$. The key relation is

Lemma 6.1.2. $\delta_j \delta_i = \delta_i \delta_{j-1}$ *for* $i < j$.

An immediate corollary is:

Corollary 6.1.3. $\delta^2 = 0$ *and* $\partial^2 = 0$.

Thus we have a complex. The *simplicial homology* of K is defined by

$$H_n(K,A) = \frac{ker[\delta : C_n(K,A) \to C_{n-1}(K,A)]}{im[\delta : C_{n+1}(K,A) \to C_n(K,A)]}$$

Elements of the numerator are called cycles, and elements of the denominator are called boundaries. The simplical cohomology are defined likewise by

$$H^n(K,A) = \mathcal{H}^n(C^\bullet(K,A)) = \frac{ker[\partial : C^n(K,A) \to C^{n+1}(K,A)]}{im[\partial : C^{n-1}(K,A) \to C^n(K,A)]}.$$

Note that when V is finite, these groups are automatically finitely generated and computable. The choices of A of interest to us are $\mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$. The relationships are given by:

Theorem 6.1.4 (Universal coefficient theorem). *If A is torsion free, then there are isomorphisms*

$$H_i(K,A) \cong H_i(K,\mathbb{Z}) \otimes A$$

$$H^i(K,A) \cong H^i(K,\mathbb{Z}) \otimes A \cong Hom(H_i(K,\mathbb{Z}), A)$$

Proof. [Sp]. □

One advantage of cohomology over homology is that it has a mutiplicative structure. When A is a commutative ring R, there is a product on cohomology analogous to the product in De Rham induced by wedging forms. Given two cochains $\alpha \in C^n(K,R)$, $\beta \in C^m(K,R)$, their cup product $\alpha \cup \beta \in C^{n+m}(K,R)$ is given by

$$\alpha \cup \beta(\{v_0, \ldots v_{n+m}\}) = \alpha(\{v_0, \ldots v_n\})\beta(\{v_n, \ldots v_{n+m}) \qquad (6.1)$$

where $v_0 < v_1 < \ldots$. Then:

Lemma 6.1.5. $\partial(\alpha \cup \beta) = \partial(\alpha) \cup \beta + (-1)^n \alpha \cup \partial(\beta)$

Corollary 6.1.6. $\cup$ *induces an operation on cohomology that makes $H^*(K,R)$ into a graded ring.*

Singular (co)homology was introduced partly in order to give conceptual proof of the fact that $H_i(K)$ and $H^i(K)$ depends only on $|K|$, i.e. that this is independant of the triangulation. Here we concentrate on singular cohomology. A singular n-simplex on a topological space X is simply a continuous map from $f : \Delta^n \to X$. When X is a manifold, we can require the maps to be C^∞. We define a singular n-cochain on a X be a map which assigns an element of A to any n-simplex on X. Let $S^n(X,A)$ ($S^n_\infty(X,A)$) denote the group of (C^∞) n-cochains with values in A. When F is an n-cochain, its coboundary is the $(n+1)$-cochain

$$\partial(F)(f) = \sum (-1)^i F(f \circ \delta_i).$$

The following has more or less the same proof as corollary 6.1.3.

Lemma 6.1.7. $\partial^2 = 0$.

The *singular cohomology* groups of X are

$$H^i_{sing}(X, A) = \mathcal{H}^i(S^*(X, A)).$$

Singular cohomolgy is clearly contravariant in X. A basic property of this cohomology theory is its homotopy invariance. We state this in the form that we will need. A subspace $Y \subset X$ is called a deformation retraction, if there exists a a continuous map $F : [0, 1] \times X \to X$ such that $F(0, x) = x$, $F(1, X) = Y$ and $F(1, y) = y$ for $y \in Y$. X is called contractible if it deformation retracts to a point.

Proposition 6.1.8. *If $Y \subset X$ is a deformation retraction, then*

$$H^i_{sing}(X, A) \to H^i_{sing}(Y, A)$$

is an isomorphism for any A.

In particular, the higher cohomology vanishes on a contractible space. This is an analogue of Poincaré's lemma. We call a space *locally contractible* if every point has a contractible neighbourhood . Manifolds and varieties with classical topology are examples of such spaces.

Theorem 6.1.9. *If X is paracompact Hausdorff space (e. g. a metric space) which is locally contractible, then $H^i(X, A_X) \cong H^i_{sing}(X, A)$ for any abelian group A.*

A complete proof can be found in [Sp, chap. 6] (note that Spanier uses Čech approach discussed in the next chapter). In the case of manifolds, a proof which is more natural from our point of view can be found in [Wa]. The key step is to consider the sheaves $\mathcal{S}^n$ associated to the presheaves $U \mapsto S^n(U, A)$. These sheaves are soft since they are modules over the sheaf of real valued continuous functions. The local contractability guarantees that

$$0 \to A_X \to \mathcal{S}^0 \to \mathcal{S}^1 \to \ldots$$

is a fine resolution. Thus one gets

$$H^i(X, A_X) \cong \mathcal{H}^i(\mathcal{S}^*(X))$$

It remains to check that the natural map

$$S^*(X, A) \to \mathcal{S}^*(X)$$

induces an isomorphism on cohomology. We the reader refer to [Wa, pp 196-197] for this.

As a corollary of this (and the universal coefficient theorem), we obtain the form of De Rham's theorem that most people think of.

Corollary 6.1.10 (De Rham's theorem, version 2). *If X is a manifold,*

$$H^i_{dR}(X, \mathbb{R}) \cong H^i_{sing}(X, \mathbb{R}) \cong H^i_{sing}(X, \mathbb{Z}) \otimes \mathbb{R}$$

The theorem holds with C^∞ cochains. The map in the corollary can be defined directly on the level of complexes by

$$\alpha \mapsto (f \mapsto \int_\Delta f^*\alpha)$$

Singular cohomology carries a cup product given by formula (6.1). A stronger form of De Rham's theorem shows that the above map is a ring isomorphism [Wa]. Fundamental classes of oriented submanifolds can be constructed in $H^*(X, \mathbb{Z})$. This can be used to show that the intersection numbers $Y \cdot Z$ are always integers. This can also be deduced from proposition 4.4.3 and transervsality theory.

Exercise 6.1.11.

1. *Prove lemma 6.1.2 and it corollary.*

2. *Calculate the simplicial cohomology with $\mathbb{Z}$ coefficients for the "tetrahedron" which is the powerset of $V = \{1, 2, 3, 4\}$ with $\emptyset$ and V removed.*

3. *Let S^n be the n-sphere realized as the unit sphere in $\mathbb{R}^{n+1}$. Let $U_0 = S^n - \{(0, \ldots 0, 1)\}$ and $U_1 = S^n - \{(0, \ldots 0, -1)\}$. Prove that U_i are contractible, and that $U_0 \cap U_1$ deformation retracts on to the "equatorial" $(n-1)$-sphere.*

4. *Prove that*

$$H^i(S^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } i = 0, n \\ 0 \text{ otherwise} \end{cases}$$

using Mayer-Vietoris.

6.2 $H^*(\mathbb{P}^n, \mathbb{Z})$

Let $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$ with its classical topology.

Theorem 6.2.1.

$$H^i(\mathbb{P}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } 0 \le i \le 2n \text{ is even} \\ 0 \text{ otherwise} \end{cases}$$

Before giving the proof, we will need to develop a few more tools. Let X be a space satisfying the assumptions of theorem 6.1.9, and $Y \subset X$ a closed subspace satisfying the same assumptions. We will insert the restriction map

$$H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z})$$

into a long exact sequence. This can be done in a number of ways, by defining cohomology of the pair (X, Y), by using sheaf theory, or by using a mapping cone. We will choose the last option. Let C be obtained by first gluing the base of the cylinder $\{1\} \times Y \subset [0,1] \times Y$ to X along Y, and then collapsing the top to a point P (figure (6.2)).

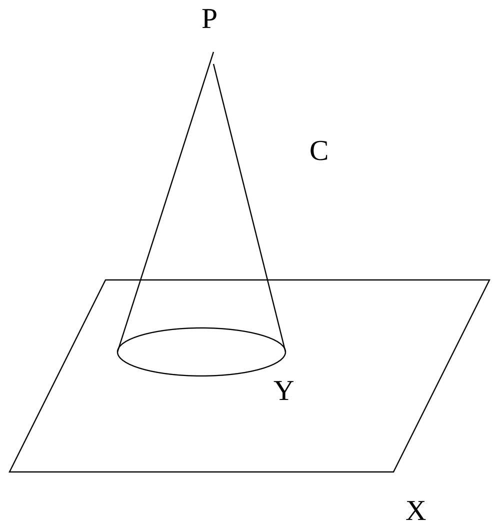


Figure 6.2: Mapping cone

Let $U_1 = C - P$, and $U_2 \subset C$ the open cone $[0,1) \times Y/\{0\} \times Y$ (the notation A/B means collapse B to a point). One sees that U_1 deformation retracts to X, U_2 is contractible, and $U_1 \cap U_2$ deformation retracts to Y. The Mayer-Vietoris sequence, together with these facts, yields a long exact sequence

$$\ldots H^i(C, \mathbb{Z}) \to H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z}) \to H^{i+1}(C, \mathbb{Z}) \ldots$$

when $i > 0$. To make this really useful, note that the map $C \to C/\overline{U_2}$ which collapses the closed cone to a point is a homotopy equivalence. Therefore it induces an isomorphism on cohomology. Since we can identify $C/\overline{U_2}$ with X/Y, we obtain a sequence

$$\ldots H^i(X/Y, \mathbb{Z}) \to H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z}) \to \ldots \tag{6.2}$$

We apply this when $X = \mathbb{P}^n$ and $Y = \mathbb{P}^{n-1}$ embedded as a hyperplane. The complement $X - Y = \mathbb{C}^n$. Collapsing Y to a point amounts to adding a point at infinity to $\mathbb{C}^n$, thus $X/Y = S^{2n}$. Since projective spaces are connected

$$H^0(\mathbb{P}^n, \mathbb{Z}) \cong H^0(\mathbb{P}^{n-1}, \mathbb{Z}) \cong \mathbb{Z}.$$

For $i > 0$, (6.2) and the previous exercise yields isomorphisms

$$H^i(\mathbb{P}^n, \mathbb{Z}) \cong H^i(\mathbb{P}^{n-1}, \mathbb{Z}), \text{ when } i < 2n \tag{6.3}$$

$$H^{2n}(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$$

The theorem follows by induction.

Exercise 6.2.2.

1. *Let $L \subset \mathbb{P}^n$ be a linear subspace of codimension i. Prove that its fundamental class $[L]$ generated $H^{2i}(\mathbb{P}^n, \mathbb{Z})$.*

2. *Let $X \subset \mathbb{P}^n$ be a smooth projective variety. Then $[X] = d[L]$ for some d, where L is a linear subspace of the same dimension. d is called the degree of X. Bertini's theorem, which you can assume, implies that there exists a linear space L' of complementary dimension transverse to X. Check that $X \cdot L' = \#(X \cap L') = d$.*

6.3 Čech cohomology

We return to sheaf theory proper. We will introduce the Čech approach to cohomology which has the advantage of being quite explicit and computable (and the disadvantage of not always giving the "right" answer). Roughly speaking, Čech bears the same relation to sheaf cohomology, as simplicial does to singular cohomology.

One starts with an open covering $\{U_i \,|\, i \in I\}$ of a space X indexed by a totally ordered set I. If $J \subseteq I$, let U_J be the intersection of U_j with $j \in J$. Let $\mathcal{F}$ be a sheaf on X. The group of *Čech n-cochains* is

$$C^n = C^n(\{U_i\}, \mathcal{F}) = \prod_{i_0 < \ldots < i_n} \mathcal{F}(U_{i_0 \ldots i_n})$$

The coboundary map $\partial : C^n \to C^{n+1}$is defined by

$$\partial(f)_{i_0 \ldots i_{n+1}} = \sum_k (-1)^k f_{i_0 \ldots \hat{i}_k \ldots i_{n+1}} |_{U_{i_0 \cdots i_{n+1}}}.$$

By an argument similar to the proof of corollary 6.1.3, we have:

Lemma 6.3.1. $\partial^2 = 0$

Definition 6.3.2. *The nth Cech cohomology group is*

$$\check{H}^n(\{U_i\}, \mathcal{F}) = \mathcal{H}^n(C^\bullet(\{U_i\}, \mathcal{F})) = \frac{ker(\partial : C^n \to C^{n+1})}{im(\partial : C^{n-1} \to C^n)}$$

To get a feeling for this, let us write out the first couple of groups explicitly:

$$\begin{aligned} \check{H}^0(\{U_i\}, \mathcal{F}) &= \{(f_i) \in \prod \mathcal{F}(U_i) \,|\, f_i = f_j \text{ on } U_{ij} \} \\ &= \mathcal{F}(X) \end{aligned}$$

$$\check{H}^1(\{U_i\},\mathcal{F}) = \frac{\{(f_{ij}) \in \prod\ \mathcal{F}(U_{ij}) \,|\, f_{ik} = f_{ij} + f_{jk} \text{ on } U_{ijk}\ \}}{\{(f_{ij} \,|\, \exists(\phi_i),\, f_{ij} = \phi_i - \phi_j\}} \tag{6.4}$$

There is a strong similarity with simplicial cohomology. This can be made precise by introducing a simplicial complex called the *nerve* of the covering. For the set of vertices, we take the index set I. The set of simplices is given by

$$\Sigma = \{\{i_0,\ldots i_n\} \,|\, U_{i_0,\ldots i_n} \neq \emptyset\}$$

If we assume that each $U_{i_0,\ldots i_n}$ is connected, then we see that the Čech complex $C^n(\{U_i\}, A_X)$ coincides with the simplicial complex of the nerve with coefficients in A.

Even though, we are primarily interested in sheaves of abelian groups. It will be convenient to extend (6.4) to a sheaf of arbitrary groups $\mathcal{G}$.

$$\check{H}^1(\{U_i\},\mathcal{G}) = \{(g_{ij}) \in \prod_{i<j} \mathcal{G}(U_{ij}) \,|\, g_{ik} = g_{ij}g_{jk} \text{ on } U_{ijk}\ \}/\sim$$

where $(g_{ij}) \sim (\bar{g}_{ij})$ if there exists $(\gamma_i) \in \prod \mathcal{G}(U_i)$ such that $g_{ij} = \gamma_i \bar{g}_{ij}\gamma_j^{-1}$. Note that this is just a set in general. The (g_{ij}) are called 1-cocycles with values in $\mathcal{G}$. It will be useful to drop the requirement that $i < j$ by setting $g_{ji} = g_{ij}^{-1}$ and $g_{ii} = 1$.

As an example of a sheaf of nonabelian groups, take $U \mapsto GL_n(\mathcal{R}(U))$, where $(X,\mathcal{R})$ is a ringed space (i.e. space with a sheaf of commutative rings).

Theorem 6.3.3. *Let $(X,\mathcal{R})$ be a manifold or a variety over k, and $\{U_i\}$ and open cover of X. There is a bijection between the following sets:*

1. *The set of isomorphism classes of rank n vector bundles over $(X,\mathcal{R})$ triviallizable over $\{U_i\}$.*

2. *The set of isomorphism classes of locally free $\mathcal{R}$-modules M of rank n such $M|_{U_i}$ is free.*

3. *$\check{H}^1(\{U_i\}, Gl_n(\mathcal{R}))$.*

Proof. We merely describe the correspondences.

$1 \to 2$: Take the sheaf of sections.

$2 \to 3$: Given M as above. Choose isomorphisms $F_i : \mathcal{R}^n_{U_i} \to M|_{U_i}$. Set $g_{ij} = F_i \circ F_j^{-1}$. This determines a well defined element of $\check{H}^1$.

$3 \to 1$: Define an equivalence relation $\equiv$ on the disjoint union $W = \coprod U_i \times k^n$ as follows. Given $(x_i, v_i) \in U_i \times k^n$ and $(x_j, v_j) \in U_j \times k^n$, $(x_i, v_i) \equiv (x_j, v_j)$ if and only if $x_i = x_j$ and $v_i = g_{ij}(x)v_j$. Let $V = W/\equiv$ with quotient topology. Given an open set $U'/\equiv = U \subset V$. Define $f : U \to k$ to be regular, C^∞ or holomorphic (as the case may be) if its pullback to U' has this property. ∎

Implicit above, is a construction which associates to a 1-cocycle $\gamma = (g_{ij})$, the locally free sheaf

$$M_\gamma(U) = \{(v_i) \in \prod \mathcal{R}(U \cap U_i)^n \mid v_i = g_{ij} v_j\}.$$

Consider the case of projective space $\mathbb{P} = \mathbb{P}^n_k$. Suppose $x_0, \dots x_n$ are homogeneous coordinates. Let U_i be the complement of the hyperplane $x_i = 0$. Then U_i is isomorphic to $\mathbb{A}^n_k$ by

$$[x_0, \dots x_n] \to (\frac{x_0}{x_i}, \dots \frac{\widehat{x_i}}{x_i} \dots)$$

Define $g_{ij} = x_j/x_i \in \mathcal{O}(U_{ij})^*$. This is a 1-cocycle, and $M_{g_{ij}} \cong \mathcal{O}_{\mathbb{P}}(1)$. Likewise $(x_j/x_i)^d$ is the 1-cocyle for $\mathcal{O}(d)$.

We get rid of the dependence on coverings by taking direct limits. If $\{V_j\}$ is refinement of $\{U_i\}$, there is a natural restriction map

$$\check{H}^i(\{U_i\}, \mathcal{F}) \to \check{H}^i(\{V_j\}, \mathcal{F})$$

Definition 6.3.4. *The nth Čhech cohomology group*

$$\check{H}^i(X, \mathcal{F}) = \lim_{\to} \check{H}^i(\{U_i\}, \mathcal{F})$$

Corollary 6.3.5. *There is a bijection between the following sets:*

1. *The set of isomorphism classes of rank n vector bundles over $(X, \mathcal{R})$.*
2. *The set of isomorphism classes of locally free $\mathcal{R}$-modules M of rank n.*
3. $\check{H}^1(X, Gl_n(\mathcal{R}))$.

A *line bundle* is a rank one vector bundle. We won't distinguish between lines bundles and rank one locally free sheaves. The set of isomorphism classes of line bundles carries the stucture of a group namely $\check{H}^1(X, \mathcal{R}^*)$. This group is called the Picard group, and is denoted by $Pic(X)$.

Exercise 6.3.6.

1. *Check that the Čech coboundary satisfies $\partial^2 = 0$.*
2. *Check the description of $\mathcal{O}_{\mathbb{P}}(1)$ given above.*
3. *Show that multiplication in $Pic(X)$ can be interpreted as tensor product of line bundles.*

6.4 Čech versus sheaf cohomology

We define a sheafified version of the Čech complex. Given a sheaf $\mathcal{F}$ on a space X, let

$$\mathcal{C}^n(\{U_i\}, \mathcal{F}) = \prod_{i_0<\ldots<i_n} \iota_*\mathcal{F}|_{U_{i_0\cdots i_n}}$$

where ι denotes the inclusion $U_{i_0\ldots i_n} \subset X$. We construct a differential ∂ using the same formula as before. Taking global sections yields the usual Čech complex.

Lemma 6.4.1. *These sheaves fit into a resolution*

$$\mathcal{F} \to \mathcal{C}^0(\{U_i\}, \mathcal{F}) \to \mathcal{C}^1(\{U_i\}, \mathcal{F}) \ldots$$

Proof. [Har, III, lemma 4.2]. □

Lemma 6.4.2. *Suppose that $\mathcal{F}$ is flasque, then $\check{H}^n(X, \mathcal{F}) = 0$ for all $n > 0$.*

Proof. If $\mathcal{F}$ is flasque, then the sheaves $\mathcal{C}^n(\{U_i\}, \mathcal{F})$ are also seen to be flasque. Then this gives an acyclic resolution of $\mathcal{F}$. Therefore

$$\check{H}^n(\mathcal{F}) = \mathcal{H}^n(\mathcal{C}^\bullet(\{U_i\}, \mathcal{F})) = H^n(X, \mathcal{F}) = 0.$$

□

Lemma 6.4.3. *Suppose that $H^1(U_J, A) = 0$ for all nonempty finite sets J. Then given an exact sequence*

$$0 \to A \to B \to C \to 0$$

of sheaves, there is a long exact sequence:

$$0 \to \check{H}^0(\{U_i\}, A) \to \check{H}^0(\{U_i\}, B) \to \check{H}^0(\{U_i\}, C) \to \check{H}^1(\{U_i\}, A) \ldots$$

Proof. The hypothesis guarantees that there is a short exact sequence of complexes:

$$0 \to C^\bullet(\{U_i\}, A) \to C^\bullet(\{U_i\}, B) \to C^\bullet(\{U_i\}, C) \to 0.$$

The long exact now follows from a standard result in homological algebra [Wl, theorem 13.1]. □

Definition 6.4.4. *An open covering $\{U_i\}$ is called a Leray covering for a sheaf $\mathcal{F}$ if $H^n(U_J, \mathcal{F}) = 0$ for all nonempty finite sets J and all $n > 0$.*

Theorem 6.4.5. *If $\{U_i\}$ is a Leray covering for the sheaf $\mathcal{F}$, then*

$$\check{H}^n(\{U_i\}, \mathcal{F}) \cong H^n(X, \mathcal{F})$$

for all n.

Proof. This is clearly true for $n = 0$. With the notation of section 3.2, we have an exact sequence

$$0 \to \mathcal{F} \to G(\mathcal{F}) \to C^1(\mathcal{F}) \to 0$$

with

$$H^1(X, \mathcal{F}) = coker[\Gamma(X, G(\mathcal{F})) \to \Gamma(X, C^1(\mathcal{F}))]$$

and

$$H^{n+1}(\mathcal{F}) = H^n(C^1(\mathcal{F})).$$

Lemmas 6.4.3 and 6.4.2 imply that

$$\check{H}^1(\mathcal{F}) \cong coker[\Gamma(X, G(\mathcal{F})) \to \Gamma(X, C^1(\mathcal{F}))]$$

and

$$\check{H}^{n+1}(\mathcal{F}) = \check{H}^n(C^1(\mathcal{F}))$$

for $n > 0$. This proves the result for $n \geq 1$ by induction. □

Corollary 6.4.6. *If every covering admits a Leray refinement, then* $\check{H}^n(X, \mathcal{F}) \cong H^n(X, \mathcal{F})$.

We state a few more general results.

Proposition 6.4.7. *For any sheaf* $\mathcal{F}$

$$\check{H}^1(X, \mathcal{F}) \cong H^1(X, \mathcal{F})$$

Proof. See [G, Cor. 5.9.1] □

Theorem 6.4.8. *If X is a paracompact space (e.g. a metric space), then for any sheaf and all i,*

$$\check{H}^i(X, \mathcal{F}) \cong H^i(X, \mathcal{F})$$

for all i.

Proof. See [G, Cor. 5.10.1] □

6.5 First Chern class

Let $(X, \mathcal{O}_X)$ be a complex manifold or algebraic variety over $\mathbb{C}$. Then we have isomorphisms

$$Pic(X) \cong \check{H}^1(X, \mathcal{O}_X^*) \cong H^1(X, \mathcal{O}_X^*)$$

The exponential sequence is

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \xrightarrow{e^{2\pi i}} \mathcal{O}_X^* \to 1 \tag{6.5}$$

Definition 6.5.1. *Given a line bundle L, its first Chern class $c_1(L) \in H^2(X, \mathbb{Z})$ is the image of L under the connecting map $Pic(X) \to H^2(X, \mathbb{Z})$*

This can be carried out for C^∞ manifolds as well, provided one interprets $\mathcal{O}_X$ as the sheaf of complex valued C^∞ functions, and $Pic(X)$ as group of C^∞ complex line bundles. In this case, c_1 is an isomorphism. It is clear that the construction is functorial:

Lemma 6.5.2. *If $f : X \to Y$ is C^∞ map between manifolds, $c_1(f^*L) = f^*c_1(L)$.*

We want to calculate this explicitly for $\mathbb{P}^1$.

Lemma 6.5.3. *$c_1(\mathcal{O}(1))$ is the fundamental class of $\mathbb{P}^1$.*

Proof. Set $\mathbb{P} = \mathbb{P}^1_{\mathbb{C}}$. We have an isomorphism $H^2(\mathbb{P}, \mathbb{Z}) \cong \mathbb{Z}$ under which the fundamental class maps to 1. We can use the standard covering $U_i = \{x_i \neq 0\}$. We identify U_1 with $\mathbb{C}$ with the coordinate $z = x_0/x_1$. The 1-cocycle (section 6.3) of $\mathcal{O}(1)$ is $g_{01} = z^{-1}$. The logarithmic derivative $d\log g_{01} = -dz/z$ is a 1-cocycle with values in $\mathcal{E}^1$. Since $\mathcal{E}^1$ is soft, this cocycle is coboundary, i.e. there exists forms $\alpha_i \in \mathcal{E}^1(U_i)$ such that $-dz/z = \alpha_1 - \alpha_0$ on the intersection. Therefore $d\alpha_i$ patch to yield a global 2-form $\beta \in \mathcal{E}^2(\mathbb{P})$. $\beta/2\pi i$ gives an explicit representative of the image of $c_1(\mathcal{O}(1))$ in $H^2(X, \mathbb{C})$. We have an isomorphism $H^2(X, \mathbb{C}) \cong \mathbb{C}$ given by integration, under which this class can be identified with the number $\int_{\mathbb{P}} \beta/2\pi i$. In order to evaluate this integral divide the sphere into two hemispheres $H_0 = \{|z| \leq 1\}$ and $H_1 = \{\||z| \geq 1\}$. Let C be the curve $|z| = 1$ with positive orientation. Note the boundary of H_1 is $-C$. Then with the help of Stokes' theorem, we get

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\mathbb{P}} \beta &= \frac{1}{2\pi i}\left(\int_{H_0} d\alpha_0 + \int_{H_1} d\alpha_1\right) \\
&= \frac{1}{2\pi i}\left(\int_C \alpha_0 - \int_C \alpha_1\right) \\
&= \frac{1}{2\pi i}\int_C \frac{dz}{z} = 1
\end{aligned}$$

Thus $c_1(\mathcal{O}(1))$ is the fundamental class of $H^2(\mathbb{P}^1, \mathbb{Z})$. □

By the same kind of argument, we obtain:

Lemma 6.5.4. *If D is a divisor on a compact Riemann surface X, $c_1(\mathcal{O}(D)) = \deg(D)[X]$*

We are going to generalize this to higher dimensions. A complex submanifold $D \subset X$ of a complex manifold is called a smooth effective divisor if D is locally definable by a single equation. In other words, we have an open covering $\{U_i\}$ of X, and functions $f_i \in \mathcal{O}(U_i)$, such that $D \cap U_i$ is given by $f_i = 0$. We define the $\mathcal{O}_X$-module $\mathcal{O}_X(-D)$ to be the ideal sheaf of D, and $\mathcal{O}_X(D)$ to be the dual. By assumption, $\mathcal{O}_X(-D)$ is locally a principle ideal, and hence a line bundle. The line bundle $\mathcal{O}_X(D)$ is determined by the 1-cocycle $f_i/f_j \in \mathcal{O}(U_i \cap U_j)^*$.

Lemma 6.5.5. *If $H \subset \mathbb{P}^n$ is a hyperplane, then $\mathcal{O}_{\mathbb{P}^n}(H) \cong \mathcal{O}_{\mathbb{P}^n}(1)$.*

Proof. Let H be given by the homogeneous linear form $\ell = \sum_k a_k x_k = 0$. Then for the standard covering $U_i = \{x_i \neq 0\}$, H is defined by

$$\ell_i = \sum a_k \frac{x_k}{x_i} = 0.$$

Thus $\mathcal{O}(H)$ is determined by the 1-cocycle $\ell_i/\ell_j = x_j/x_i$, which is the cocycle for $\mathcal{O}(1)$. $\square$

Theorem 6.5.6. *If D is a smooth effective divisor, then $c_1(\mathcal{O}_X(D)) = [D]$.*

We start with the special case.

Lemma 6.5.7. *$c_1(\mathcal{O}_{\mathbb{P}^n}(1)) = [H]$ where $H \subset \mathbb{P}^n$ is a hyperplane.*

Proof. We have already checked this for $\mathbb{P}^1$ in lemma 6.5.3. Embed $\mathbb{P}^1 \subset \mathbb{P}^n$ as a line. The restriction map induces an isomorphism on second cohomology with $\mathbb{Z}$ coefficients such that $[H]$ maps to the fundamental class of $\mathbb{P}^1$ by (6.3). Since c_1 is compatible with restriction, lemma 6.5.3 implies that $c_1(\mathcal{O}_{\mathbb{P}^n}(1)) = [H]$. $\square$

Proof of theorem 6.5.6. There are several ways to prove this. We are going to outline the proof given in [Hrz] which is reduces it to the previous lemma. Choose an open tubular neighbourhood T of D. This is complex C^∞ line bundle such that D corresponds to the zero section. So there is a classifying map 1.5.3 $X \to \mathbb{P}^n$ such that the line bundle L associated to $\mathcal{O}_{\mathbb{P}^n}(1)$ pulls back to T. We can collapse the complement of T in X to a point resulting in what is called *Thom space* of T. This Thom space will map to the Thom space of L. The Thom space of L can be identified with $\mathbb{P}^{n+1}$ with the zero section corresponding to $\mathbb{P}^n$ embedded as a hyperplane. Thus we obtain a commutative diagram.

$$\begin{tikzcd}
D \arrow[r] \arrow[dr] \arrow[dd] & X \arrow[d] \\
 & X/(X-T) \arrow[d] \\
\mathbb{P}^n \arrow[r] & \mathbb{P}^{n+1}
\end{tikzcd}$$

These maps are very far from holomorphic, but it won't matter for the purposes of the proof. The fundamental class of $\mathbb{P}^n$ pullback to $[D]$ and the C^∞ line bundle associated $\mathcal{O}_X(D)$ is isomorphic to the pullback of the C^∞ line bundle associated $\mathcal{O}_X(1)$. $\square$

Exercise 6.5.8.

1. *Prove lemma 6.5.4.*

2. *Given a vector bundle V of rank r, define* $\det(V) = \wedge^r V$ *and* $c_1(V) = c_1(\det V)$. *Prove that* $\det(V_1 \oplus V_2) \cong \det(V_1) \otimes \det(V_2)$. *Use this to calculate* $c_1(V_1 \oplus V_2)$.