PURITY FOR INTERSECTION COHOMOLOGY AFTER DELIGNE-GABBER

D. ARAPURA

These notes are my translation and slight expansion of Deligne's unpublished paper "Pureté de la cohomologie de MacPherson-Goresky" where Gabber's purity theorem was first written up. Although this paper was superseded by [BBD], it has the advantage of brevity. The theorem says in effect that intersection cohomology satisfies the Weil conjectures and therefore hard Lefschetz. In fact, I believe this was the first proof of this fact. This is an interesting inversion of history: for ordinary cohomology the first correct proof of Lefschetz' s theorem was due Hodge; Deligne's arithmetic proof came much later. Here the arithmetic came first, and seems to have influenced the subsequent Hodge theory, e.g. [CKS], [S].

I've included a few pages of introductory material, references, and probably several errors, but otherwise it's pretty much identical to the original. My thanks to Shenghao Sun for suggesting various corrections.

0. Background

This is a quick synopsis of some background material, including some relevant parts of [Weil2]. See also [KW].

0.1. **ℓ-adic sheaves.** Given a scheme X the étale topology X_{et} is a Grothendieck topology where the "open sets" are étale (= flat and unramified) maps $U \to X$, and covers are surjective families. A sheaf F is contravariant functor on X_{et} such that elements of $F(U)$ can be patched uniquely on a cover $\{U_i\}$, i.e. the diagram

$$F(U) \to \prod F(U_i) \rightrightarrows \prod F(U_i \times_X U_j)$$

is an equalizer. For example, given an abelian group A, the presheaf

$$A(U) = A^{\#\text{ of components of }U}$$

with obvious restrictions is a sheaf, called the constant sheaf associated to A. For another important example, if n is invertible in $\mathcal{O}_X$, then the presheaf $\mu_n(U)$ of nth roots of unity in $\mathcal{O}(U)$ is a sheaf.

Standard sheaf theoretic constructions such as an inverse and direct images and extension by zero exist for étale sheaves. A sheaf F is locally constant if there is an étale cover $\pi : Y \to X$ such that π^*F is constant. For example, μ_n is locally constant.

When X is a normal variety, the étale fundamental group $\pi_1^{et}(X)$ can be identified with the Galois group of the union of function fields of étale extensions. In general, there is a more abstract definition, which makes it clear that it depends on a choice of geometric point $\bar{x} \to X$ (although the isomorphism class won't) just as in the topological case. This is a profinite group, which is the profinite completion of the usual fundamental group when X is a complex variety. When X is connected, the category of locally constant sheaves is equivalent to the category of continuous representations of $\pi_1^{et}(X,\bar{x})$ via $F \mapsto F_{\bar{x}}$. This is analogous to the correspondence representation of $\pi_1(X,x)$ and locally constant sheaves in the usual topology.

The category of sheaves of abelian groups on X_{et} is abelian with enough injectives, so $H^i(X_{et},F)$ can be defined as the ith derived functor of $X \mapsto F(X)$. When X is a variety, cohomology with compact support is defined as $H^i_c(X_{et},F) = H^i(\bar{X}_{et}, j_!F)$, where $j : X \to \bar{X}$ is a compactification. It is independent of the choice of $\bar{X}$. When X is a complex variety, and $F = \mathbb{Z}/n$, then these cohomology groups coincide with singular cohomology.

In general, this cohomology will not work as expected unless F is a torsion sheaf of exponent prime to $char X$. In order to get values over a field of characteristic zero, we can use inverse limits. Fix a field k of characteristic p, and let ℓ be a prime number coprime to p. Suppose that X is a k-scheme. Then ℓ-adic cohomology $H^i(X_{et},\mathbb{Z}_\ell) = \varprojlim H^i(X_{et},\mathbb{Z}/\ell^n)$ by definition. More generally a $\mathbb{Z}_\ell$-sheaf is really a special kind of pro-object $\ldots \to K_2 \to K_1$ in the category of sheaves on the étale site X_{et}, where each K_i is a $\mathbb{Z}/\ell^i$-module. Its cohomology is defined as $\varprojlim H^*(X_0,K_i)$. For our purposes, we can pretend as if we are working with $\varprojlim K_i$, although the homological algebra is far from straight forward. We will mostly ignore these subtleties. The systems $K_\bullet$ are subject to a some addition constraints:

(1) Each K_i is constructible i.e. the stalks are finite and there exist a partition of X into locally closed sets $\{S_j\}$ such that $K_i|_{S_j}$ is locally constant.
(2) The structure maps $K_i \otimes \mathbb{Z}/\ell^{i-1} \to K_{i-1}$ are isomorphisms.

For example, we could require that K_i is locally constant, in which case the sheaf is called *lisse*. Lisse sheaves correspond to continuous representations $\pi_1^{et}(X_0) \to GL_N(\mathbb{Z}_\ell)$. The Tate sheaf $\mathbb{Z}_\ell(n) = \ldots \mu_{\ell^2}^{\otimes n} \to \mu_\ell^{\otimes n}$ on Spec(k) is lisse, and corresponds to the nth power of the cyclotomic character.

We can formally tensor the category of (construcible, lisse) $\mathbb{Z}_\ell$-sheaves by $\mathbb{Q}_\ell$ to get the category of (constructible, lisse) $\mathbb{Q}_\ell$-sheaves. The notions of constructible (or lisse) $\bar{\mathbb{Q}}_\ell$ sheaves are defined in a similar fashion. There is a triangulated category $D(X_0) = D^b_c(X_0,\bar{\mathbb{Q}}_\ell)$ which is morally the bounded derived category of constructible $\bar{\mathbb{Q}}_\ell$ sheaves [E]. This has the standard operations and usual properties. Recently, Bhatt and Scholze [BS] have given a more natural definition of $D(X_0)$ as a derived category of sheaves on their pro-étale topology.

0.2. **Weil II.** Here is a summary of Deligne's second paper on the Weil conjectures [Weil2].

Fix a scheme X_0 of finite type over a finite field $\mathbb{F}_q$ with q elements. Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_q$. We denote objects over $\mathbb{F}_q$ with subscript 0, suppression of the subscript indicates the extension of scalars to $\mathbb{F}$. Thus $X = X_0 \times \text{Spec } \mathbb{F}$.

Next, we need to understand the Frobenius. Since ths is potentially confusing, let us start with an example. Suppose $X = \mathbb{A}^1_{\mathbb{F}} = \text{Spec } \mathbb{F}[x]$. There are at least 3 different things which could rightfully called the Frobenius, the absolute Frobenius Fr_X

$$\sum a_i x^i \mapsto (\sum a_i x^i)^q$$

the relative Frobenius $Fr_{X/\mathbb{F}}$

$$\sum a_i x^i \mapsto \sum a_i (x^i)^q$$

and the arithmetic Frobenius ϕ

$$\sum a_i x^i \mapsto \sum a_i^q x^i$$

Such morphisms exist in general, and $Fr_X = Fr_{X/\mathbb{F}} \circ \phi$. It turns out that Fr_X acts trivially on étale cohomology. The interesting thing to consider is the action of $Fr_{X/\mathbb{F}}$ or equivalently ϕ^{-1}. This is the thing that we will refer to simply as Frobenius and denote by F below. The basic result of Weil II is the following.

Theorem 1. *Suppose that X is a variety. Then the eigenvalues of F on $H^i_c(X_{et}, \mathbb{Q}_\ell)$ are algebraic numbers all of whose eigenvalues have absolute value $q^{w/2}$ with $0 \le w \le i$; all of the eigenvalues satisfy $w = i$ if X is proper and smooth.*

By using Lefschetz pencils, there is no loss in assuming that X admits a surjection $f : X \to \mathbb{P}^1$. Using this and the Leray spectral sequence,

$$E_2 = H^i(\mathbb{P}^1, R^j f_! \mathbb{Q}_\ell) \Rightarrow H^{i+j}_c(X, \mathbb{Q}_\ell)$$

Deligne reduces the theorem to a statement on curves but with general coefficients. Given a $\mathbb{Q}_\ell$ vector space with F-action, call it pure of weight w if the eigenvalues are algebraic numbers all of whose conjugates have absolute value $q^{w/2}$. Given a $\bar{\mathbb{Q}}_\ell$-sheaf K_0 on X_0 and a closed point $x \in X_0$, with $\bar{x} = \mathrm{Spec}(\overline{k(x)}) \to X$, the stalk $K_{\bar{x}}$ carries an action by $Gal(\overline{k(x)}/k(x))$. In particular, the $N(x)$th power Frobenius acts on it, where $N(x) = \#k(x)$. K is *pointwise pure* of weight $w \in \mathbb{Z}$ if for each closed point $x \in X_0$ the eigenvalues of Frobenius at x on the stalk are algebraic numbers having absolute value $N(x)^{w/2}$. For instance, $\bar{\mathbb{Q}}_\ell(n)$ is pointwise pure of weight $-2n$. A sheaf (vector space) is *mixed* (of weight $\le w$ or $\ge w$) if it is an extension of pointwise pure sheaves (spaces) of this type.

The key estimate is:

Theorem 2 ([Weil2, 6.2.5c])**.** *Let X_0 be a smooth proper curve over $\mathbb{F}_q$, $j : U_0 \to X_0$ the inclusion of an open dense set, and K_0 a pointwise pure lisse sheaf of weight n on U_0, then $H^i(X_{et}, j_*K)$ is pure of weight $n + i$.*

Sketch for $i \neq 1$. The case $i = 1$ is really the heart of the proof. A simplified proof due to Laumon can be found in the book by Kiehl-Weissauer [KW], and for another see [K]. The remaining cases of the first purity statement for $i = 0, 2$ are not that difficult. The case of $i = 2$ can be reduced to the case of $i = 0$ by duality. So we turn to this case. The basic idea is to show that the eigenvalues of F on $H^0(X, j_*K)$ are among the eigenvalues of the stalks which have the correct weight by hypothesis. In particular, if the support of K is zero dimensional, there is nothing to prove.

In general, note that the statement is unaffected by an extension of the ground field $\mathbb{F}_q$. After such an extension, we can choose a nonempty $j' : U_0 \subset U_0$ so that $K|_{U'}$ is lisse. There is an exact sequence $0 \to L \to j_*K \to j_*j'_*j'^*K$, where L is pure weight n with zero dimensional support. Therefore we can assume that $U = U'$. Choose a $\mathbb{F}$-rational point $\bar{x} \in U$, We have an exact sequence

$$0 \to \pi_1^{et}(U, \bar{x}) \to \pi_1^{et}(U_0, \bar{x}) \to Gal(\mathbb{F}/\mathbb{F}_q) \to 1$$

After identifying

$$H^0(X, j_*K) = L_{\bar{x}}^{\pi_1^{et}(U,\bar{x})}$$

we see that the eigenvalues of $\phi^{-1} \in Gal(\mathbb{F}/\mathbb{F}_q)$ are among the eigenvalues of $L_{\bar{x}}$. □

Corollary 1. *With the same hypothesis as above, $H^i_c(U_{et},K)$ (resp. $H^1(U_{et},K)$) is mixed of weight $\leq n+1$(resp. $\geq n+1$)*

Proof. The exactness

$$H^0(X, j_*K/j_!K) \to H^1_c(U,K) \to H^1(X, j_*K)$$

together implies the bounds on the weight of $H^i_c(U,K)$. The second part follows from this by Poincaré-Verdier duality. ☐

There are some variations on these notions. Instead of assuming the eigenvalues are algebraic, we can measure the norms using a fixed (highly noncanonical!) isomorphism $\iota : \bar{\mathbb{Q}}_\ell \cong \mathbb{C}$, in which case we speak of ι-purity etc. A complex K is mixed of weight $\leq w$ if each sheaf $\mathcal{H}^i(K)$ is mixed of weight $\leq w+i$. It is *pure of weight w* if both K and the Verdier dual DK are mixed of weight $\leq w$ and $\leq -w$ respectively. When X is smooth $DK = \mathbb{R}\mathcal{H}om(K, \bar{\mathbb{Q}}_\ell(\dim X)[2\dim X])$, so a pointwise pure lisse sheaf is pure of the same weight.

The earlier results are refined as follows:

Theorem 3 ([Weil2, 3.2.1])**.** *If $f_0 : X_0 \to S_0$ is a morphism of schemes of finite type over $\mathbb{F}_q$, and K_0 is a mixed sheaf with weights $\leq n$. Then for each i, $R^if_!K$ is mixed with weights $\leq n+i$.*

Corollary 2. *$\mathbb{R}f_!$ takes mixed complexes of weight $\leq w$ to mixed complexes of weight $\leq w$.*

Corollary 3. *If f is proper, $\mathbb{R}f_*$ takes pure complexes to pure complexes of the same weight.*

Proof. Apply the identity $D\mathbb{R}f_* = \mathbb{R}f_*D$ and the previous corollary. ☐

0.3. **Topological consequences.** In addition to arithmetic applications, Deligne deduces a number of topological consequences. Among these are an arithmetic proof of the hard Lefschetz theorem. The corresponding statement for intersection cohomology was originally proved this way by Gabber.

Theorem 4. *Let K be a pure lisse ℓ-adic sheaf on a smooth connected variety X. Then the corresponding representation of $\pi_1^{et}(X)$ is semisimple.*

Sketch. First we can reduce to the case where X is curve by an appropriate form of weak Lefschetz. The group $G = \pi_1^{et}(X_0)$, which is an extension of $Gal(\mathbb{F}_q)$ by $H = \pi_1^{et}(X)$, acts on $V = K_{\bar{x}}$. We proceed by induction by the length of a Jordan-Hölder series as a G-module. If V is an irreducible G-module, then the Zariski closure of $G \subset GL(V)$ is reductive. Since the closure of H is a normal subgroup of the closure of G, it is also reductive. Therefore V is a semisimple H-module.

Now suppose that V is not irreducible, so it fits into an extension

$$0 \to L \to V \to B \to M \to 0$$

of G-modules. By induction, both L and M are semisimple H-modules, so it suffices to prove that this sequence splits as H-modules. But the extension class ξ defines a $Gal(\mathbb{F}_q)$-invariant element of

$$Ext^1_H(M,L) \cong H^1(X_{et}, M^* \otimes L)$$

However, $M^* \otimes L$ is pointwise pure of weight zero. So its cohomology has weights ≥ 1, which forces $\xi = 0$. ☐

Corollary 4. *Given a complex smooth projective family $X \to Y$. The monodromy representation $\pi_1(Y) \to GL(H^i(X_y, \mathbb{Q}))$ is semisimple.*

Proof. This follows by specialization and comparison theorems for classical and étale cohomology. □

Consider an n dimensional smooth complex projective variety $X \subset \mathbb{P}^N$. Blowing up the base locus of a Lefschetz pencil on X yields a map $\tilde{X} \to \mathbb{P}^1$. Remove the discriminant D to get an open $U \subset \mathbb{P}^1$. Fix a base point $s \in U$. Topologically the singular fibres $\tilde{X}_d, d \in D$ are obtained by collapsing an embedded sphere $\delta_d \subset \tilde{X}_s$, called vanishing cycle, to a point. One form of the hard Lefschetz theorem says

Theorem 5.

$$H^{n-1}(\tilde{X}_s, \mathbb{Q}) = H^{n-1}(\tilde{X}_s, \mathbb{Q})^{\pi_1(U,s)} \oplus V$$

where V is the span of the classes of vanishing cycles.

Sketch. Classical Picard-Lefschetz theory shows that V is irreducible and that $\dim V$ and $\dim H^{n-1}(\tilde{X}_s)^{\pi_1(U)}$ are complementary. The previous semisimplicity result shows that we have a direct sum. □

The above statement is closer to the way Lefschetz would have formulated it in the 1920's. It implies the more familiar cohomological statement

$$c_1(\mathcal{O}(1))^i : H^{n-i}(X, \mathbb{Q}) \cong H^{n+i}(X, \mathbb{Q})$$

Using these ideas, Deligne's gave the following strong form:

Theorem 6. *Let $X \subset \mathbb{P}^N$ be a projective scheme, $\eta \in H^2(X, \mathbb{Z}_\ell(1))$ be the first Chern class of $\mathcal{O}(1)$ and K a pure complex. Let n be an integer, and suppose that K and DK satisfy the following condition: for each i, the dimension of the support of $\mathcal{H}^i(K)$ is $\leq n - i$.*

Then, for all $i \geq 0$, the iterated cup product

$$\eta^i : H^{n-i}(X, K) \to H^{n+i}(X, K)(i)$$

is an isomorphism.

0.4. **Invariant Cycle Theorem.** The usual form of the local invariant cycle theorem says that given a family of complex projective varieties $f : X \to \Delta$ over a disk, the specialization map

$$H^i(X_0, \mathbb{Q}) \to H^i(X_t, \mathbb{Q})^{\pi_1(\Delta^*)}$$

is surjective. The following gives a strong form of this theorem, in the algebraic setting.

Theorem 7 ([Weil2, 6.2.9])**.** *Suppose that $f : X \to S$ is a morphism to the spectrum of a strict henselian DVR, obtained by base change from a morphism over $\mathbb{F}_q$ to a smooth curve $f : X_0 \to S_0$. Let $s \in S$ denote the closed point and η the generic point. Suppose that K is obtained from a pure complex K_0 on X_0. Set $I = Gal(\bar{\eta}/\eta)$. Then the specialization map*

$$H^i(X_s, K) \to H^i(X_{\bar{\eta}}, K)^I$$

is surjective.

Let us modify the notation of the previous theorem, by supposing that S is the henselization rather strict henselization at s. We now have an exact sequence

$$1 \to I \to Gal(\bar{\eta}/\eta) \to Gal(\mathbb{F}/\mathbb{F}_q) \to 1$$

Thus the Frobenius lifts to the group in the middle. The inertia group $I = Gal(\bar{\eta}/\eta)$ can be identified with the group I occurring in theorem 7. I acts on $V = H^i(X_{\bar{\eta}}, K)$ quasiunipotently. Let $N : V(1) \to V$ be the logarithm of the unipotent part of local monodromy [Weil2, 1.7.2]. Then the associated monodromy filtration M on V is characterized by $NM_i \subseteq M_{i-2}$ and $N^k : Gr^M_k V \xrightarrow{\sim} Gr^M_{-k} V$ [Weil2, 1.6]. These properties show that $\ker N \subset M_0$. The filtration is stable under the lift of Frobenius, so we may talk about the weights of $Gr^M V$ (the norms of eigenvalues are independent of choices).

Specializing [Weil2, 1.8.4] to the situation at hand:

Theorem 8. *Suppose that K_0 is pointwise pure of weight 0, then $Gr^M_k(V)$ is pure of weight $i+k$.*

1. Gabber's purity theorem

We continue the notational conventions of section 0.2. So X_0 is defined over $\mathbb{F}_q$, X denotes the extension to $\mathbb{F} = \bar{\mathbb{F}}_q$ etc.

A *perverse sheaf* on X_0 is an object K_0 of $D(X_0)$ such that for every irreducible subvariety $i : Y \hookrightarrow X$, with generic point η, one has

$$(\mathcal{H}^n \mathbb{R} i^* K)_\eta = 0 \text{ for } n > -\dim Y \tag{A}$$

and

$$(\mathcal{H}^n \mathbb{R} i^! K)_\eta = 0 \text{ for } n < -\dim Y \tag{B}$$

The condition (A) can also be expressed as $\dim Supp\, \mathcal{H}^i K \leq -i$. A systematic account can be found in [BBD]. For example, the category of perverse sheaves is abelian. On a few occasions, it will be convenient to work with perverse sheaves up to translation. If K is perverse, $K[-N]$ will be called N-perverse.

For $j : V_0 \hookrightarrow X_0$ an open subset of X_0, one has the *intermediate direct image* functor $j_{!*}$ from perverse sheaves on V_0 to perverse sheaves on X_0. By definition $H^*(X, j_{!*}\mathbb{Q}_\ell)$ is the intersection cohomology of X [BBD, GM] (up to a shift in indices for the second reference). If the complement $i : Y_0 \hookrightarrow X_0$ of V_0 is smooth, pure of dimension d, and the $i^* R^n j_* K$ are lisse, one has

$$j_{!*} K_0 = \tau^{Y_0}_{\leq -d-1} \mathbb{R} j_* K_0 \tag{1}$$

where $\tau^{Y_0}_{\leq k}$ is the functor which to a complex L_0 attaches the subcomplex of L_0 which coincides with L_0 outside of Y_0 and $\tau_{\leq k}$ on Y_0. The general case can be built up from this by working stratum by stratum. An explicit formula can be found in [BBD, 2.1.11].

Let a be the projection from X_0 to $\mathrm{Spec}(\mathbb{F}_q)$. The dualizing complex $D_{X_0} = \mathbb{R} a^! \bar{\mathbb{Q}}_\ell$. For X_0 smooth and pure of dimension N, this is the constant sheaf $\bar{\mathbb{Q}}_\ell$ in degree $-2N$. The duality functor D is the functor $K_0 \mapsto \mathbb{R}\mathcal{H}om(K_0, D_{X_0})$. It interchanges the conditions (A) and (B), so that the category of perverse sheaves is stable under D. The functor $j_{!*}$ is self dual: $Dj_{!*} = j_{!*}D$.

Of course, the preceding discussion is geometric, and it works as well for $\mathbb{F}$ as for $\mathbb{F}_q$.

Theorem 9. *Let $j : U_0 \hookrightarrow X_0$ be an open subset of X_0, and K_0 an N-perverse sheaf on U_0. If K_0 is pure of weight w, $j_{!*}K_0$ is pure of the same weight.*

Corollary 5. *If X is proper, the complex conjugates of eigenvalues of Frobenius on $H^i(X, j_{!*}K)$ have absolute value $q^{(i+w)/2}$.*

Corollary 6. *If X is projective and if η is the first Chern class of an ample invertible sheaf, the iterated cup product*

$$\eta^i : H^{N-i}(X, j_{!*}K) \to H^{N+i}(X, j_{!*}K)$$

is an isomorphism.

Proof. Apply theorem 6. □

Corollary 7. *The hard Lefschetz holds for intersection cohomology of complex projective varieties.*

Proof. This follows from the previous corollary by specialization, cf. [BBD, chap 6]. □

2. Preliminary reductions

For simplicity, we will assume that X is equidimensional. There is no loss in generality in assuming that $N = w = 0$. The proof is by induction on $\dim X_0$. For $\dim X_0 = 0$, the assertion is trivial. For $\dim X_0 = 1$, this is theorem 2.

The problem is local, one can suppose that X_0 is affine, and then projective. Choose a projective embedding $X \subset \mathbb{P}$. We have the following weak Lefschetz property:

Lemma 1. *For H a general hyperplane section of X, the restriction of $K[-1]$ (check) to $U \cap H$ is an perverse sheaf on $U \cap H$, pure of weight 0. The intermediate direct image of this on H coincides with the shifted restriction of $j_{!*}K$ to H.*

Proof. The perversity of the restriction follows from the definition (A), (B) because $\dim \mathit{supp}\mathcal{H}^i(K|_H)$ drops by 1. Also purity is clear, as it is a pointwise condition and the stalks of $K|_H$ are isomorphic to stalks of K along H.

The last statement requires some work. By induction on the number of strata, we can reduce to the case where formula (1) applies. Let $\tilde{X} = \{(x, H) \in X \times \check{\mathbb{P}} \mid x \in H\}$ be the incidence variety, where $\check{\mathbb{P}}$ is the dual space . Consider the cartesian diagram

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\pi} & U \\ \downarrow{\scriptstyle \tilde{j}} & & \downarrow{\scriptstyle j} \\ \tilde{X} & \xrightarrow{\pi} & X \end{array}$$

Since π is smooth, the smooth base change theorem shows that $\pi^*\mathbb{R}j_*K = \mathbb{R}\tilde{j}_*\pi^*K$. As $\tilde{j}$ is a map of $\check{\mathbb{P}}$-schemes, the generic base change theorem ([SGA41/2] Th. finitude) shows that

$$(\pi^*\mathbb{R}j_*K)|_H = \mathbb{R}j_{H*}\pi^*(K|_H)$$

for general H, where $j_H : U \cap H \to H$ is the inclusion. Whence

$$(\tau^Y_{\leq -d-1}\pi^*\mathbb{R}j_*K[-1])|_H = \tau^Y_{\leq -d-2}\mathbb{R}j_{H*}(\pi^*K|_H)$$

□

Returning to the proof of the theorem, from the lemma and induction assumption, one deduces that $j_{!*}K_0$ is pure on the complement of a finite set E_0 of X_0. Replacing U_0 by $X - E_0$, and K_0 by the restriction of $j_{!*}K_0$ to $X_0 - E_0$; we can suppose that U_0 is the complement of a finite set E_0 in X_0. It is also helpful to keep in mind the basic model case is where U_0 is smooth and K_0 is a lisse sheaf concentrated in degree $-\dim X$.

Purity is invariant under finite extensions of the base field, and we will tacitly make the required extensions of scalars for the utilized objects to be defined over the base field. For example, we may suppose the points of E_0 are rational over $\mathbb{F}_q$. Let $P_0 \in E_0$. The subscript P will indicate the henselization at P; one may view this as the analogue of passing to the analytic germ. Since the functor $j_{!*}$ is selfdual, it suffices to verify for that $j_{!*}K_0 = \tau^{E_0}_{\leq -1}\mathbb{R}j_*K_0$ is mixed of weight ≤ 0. At P_0, this means that for $n \leq -1$, the cohomology of the "link" $H^n(X_P - P, K)$ has weights $\leq n$. Here's is the dual statement

Lemma 2. *$H^n(X_P - P, K)$ has weights $\leq n$ if and only if the weights of $H^n(X_P - P, DK)$ are $\geq n+1$ for $n \geq 0$.*

Proof. Not yet.

□

Let H be a hyperplane section passing through P. One knows that $H^n(X_P - H_P, DK) = 0$ for $n > 0$ (as $X_P - H_p$ is affine, [SGA4, XIV §3]), so that the map

$$H^n_{H_P - P}(X_P - P, DK) \to H^n(X_P - P, DK)$$

is bijective for $n > 1$, and surjective for $n = 1$. If H is general, one has a Gysin isomorphism

$$H^{n-2}(H_P - P, (DK)(1)) \xrightarrow{\sim} H^n_{H_P - P}(X_P - P, DK),$$

the restriction of DK_0 to $H_0 - P_0$ is an (-1)-perverse sheaf pure of weight 0, and the induction hypothesis yields the desired conclusion for $n \geq 1$.

Let V_0 be an affine neighbourhood of P_0 in $U_0 \cup \{P_0\}$. The obstruction to extending a class $\alpha \in H^n(V_P - P, DK)$ to $V - P$ is in $H^{n+1}(V, \mathbb{R}j_! DK)$. As V is affine, this is zero for $n \geq 0$. The group

$$H^n(V - P, DK) = H^{2N-n}_c(V - P, K)^\vee$$

has weight $\geq n$, one finds that for $n \geq 0$, $H^n(V_P - P, DK)$ has weight $\geq n$.

Dualizing, we see that for $n = -1$, $H^n(V_P - P, K)$ has weight ≤ 0. It remains to eliminate the weight 0 part, and the rest of the argument is devoted to this.

3. An extension lemma

Let H be a general hyperplane section passing through P. The theorem to be proved implies that the restriction map from $H^{N-1}(X_P - P, K)$ (weights $\leq N-1$) to $H^{N-1}(H_P - P, K)$ (weights $\geq N$) is zero.

Lemma 3. *Let $\alpha \in H^{N-1}(X_P - P, K)$. The restriction of α to $H_P - P$ extends to $H - P$.*

We prove the equivalent statement that $\alpha \in \mathcal{H}^{N-1}(j_{!*}K)_P$ extends to an element of $H^{N-1}(H, j_{!*}K)$. Let k be the inclusion of $H - P$ into H and i_P the inclusion of P. For every $L \in D^b_c(H)$ one has a distinguished triangle

$$k_!k^*L \to L \to i_{P*}i_P^*L \to k_!k^*L[1]$$

Taking L to be the restriction of $j_{!*}K$ to H, and passing to cohomology, one finds as the obstruction to extending α the class

$$\partial\alpha \in H_c^N(H-P,K) = H^N(H, k_! K|_{H-P})$$

The image of $\partial\alpha$ in $H^N(H, j_{!*}K)$ is zero by the exact sequence.

Let H' be a general hyperplane section different from H. The variety $H-(H\cap H')$ is affine, so one has $H^N(H-(H\cap H'), k_! K_{H-P}) = 0$. Thus there is no obstruction to extending α to $H-(H\cap H')$, and $\partial\alpha$ is the image under Gysin of a class

$$\xi \in H^{N-2}(H\cap H', K)(-1) \xrightarrow{\sim} H^N_{H\cap H'}(H,K).$$

We propose a canonical choice of ξ. For H general, the restriction of K to $(H-P)$ is $(N-1)$-perverse and pure of weight 0. We have

$$H^N(H, k_! K|_{H-P}) \cong H^N(H, k_{!*}K|_{H-P})$$

and we denote the image of $\partial\alpha$ by $(\partial\alpha)'$. Let $H' = H'_0$ vary in a general pencil $\{H'_t\}_{t\in\mathbb{P}^1}$. The theorem of the fixed part (cf. theorem 7) says that the restriction map identifies $H^{N-2}(H, k_{!*}K_{H-P})$ with the invariant part $H^{N-2}(H\cap H', K)^{\pi_1(U)}$, where $0 \in U \subset \mathbb{P}^1$ is Zariski open. As t varies over U, the groups $H^{N-2}(H\cap H'_t, K)$ form a semisimple local system [Weil2, 6.2.6, 3.4.1(iii)]. Dually, the Gysin morphism sends the invariant part of $H^{N-2}(H\cap H', K)(-1) = H^N_{H\cap H'}(H,K)$ isomorphically to $H^N(H, k_{!*}K_{H-P})$. We normalize ξ by taking it in the invariant part, then it maps bijectively to $(\partial\alpha)'$ in $H^N(H, k_{!*}K_{H-P})$ and therefore to $\partial\alpha$ in $H^N(H, k_! K_{H-P})$. Thus if $\partial\alpha \neq 0$, we have $\xi \neq 0$.

Now fix a general H', and vary H in a general pencil of hyperplanes containing P. Over a proper open of the projective line parameterizing the pencil, ξ provides a global section of the local system whose fibres are $H^{N-2}(H\cap H', K)(-1)$, and an application of the theorem of the fixed part shows that if $\xi \neq 0$, its image $\xi' \in H^N(H', K)$ under Gysin is nonzero. Finally, $H^{N+1}(X-H', j_{!*}K) = 0$ and if $\xi' \neq 0$, its image $\xi'' \in H^{N+2}(X, j_{!*}K)(1)$ under Gysin is not zero.

The cohomology classes (with support) in $H\cap H'$ in H and H', and of H, H' in X give rise to a diagram

$$\begin{array}{ccc} \bar{\mathbb{Q}}_\ell(-1)[-2]_{H\cap H'} & \longrightarrow & \bar{\mathbb{Q}}_{\ell H'} \\ \downarrow & & \downarrow \\ \bar{\mathbb{Q}}_{\ell H} & \longrightarrow & \bar{\mathbb{Q}}_\ell(1)[2] \end{array}$$

Tensoring with $j_{!*}K$ and applying H^N yields a commutative diagram

$$\begin{array}{ccc} H^{N-2}(H\cap H', K)(-1) & \longrightarrow & H^N(H', K) \\ \downarrow & & \downarrow \\ H^N(H, j_{!*}K) & \longrightarrow & H^{N+2}(X, j_{!*}K)(1) \end{array}$$

We see that the image of ξ in $H^N(H, j_{!*}K)$ is zero, and one can conclude that $\xi = 0$, whence the lemma.

4. End of proof

Let S_0 be a smooth curve over $\mathbb{F}_q$, $s_0 \in S_0(\mathbb{F}_q)$, (T_0, s_0, η) the henselization of S at s_0, (T, s, η) the strict henselization, $\bar{\eta}$ a geometric generic point of T and $I = Gal(\bar{\eta}/\eta)$ the local monodromy group or inertia group. We denote by subscripts $T, \bar{\eta}, \ldots$ base change with respect to $T, \bar{\eta}, \ldots$

Lemma 4. *Given $f : Y_0 \to S_0$ and K_0 a pure complex on Y_0. If the induced map $f : f^{-1}(S_0 - s_0) \to S_0 - s_0$ is proper, the restriction morphism $H^n(Y_T, K) \to H^n(Y_{\bar{\eta}}, K)$ has as its image the subgroup $H^n(Y_{\bar{\eta}}, K)^I$ of invariants of local monodromy.*

The proof is the same as that in [Weil2, 6.2.9] (theorem 7). When f is proper, we can replace $H^n(Y_T, K)$ by $H^n(Y_s, K)$.

Let $(H_t)_{t \in \mathbb{P}^1}$ be a general pencil of hyperplane sections of X, $H_0 = H_{s_0}$ the section which contains P_0 (we assume it is defined over $\mathbb{F}_q$), and take S_0 to be a small neighbourhood of s_0 in $\mathbb{P}^1$. Let $\tilde{X}$ be the incidence variety $\{(x,t) \mid x \in X, t \in S, x \in H_t\}$. It is defined over $\mathbb{F}_q$, and the H_t are the fibres of the proper morphism $f : \tilde{X}_0 \to S_0$. The projection of $\tilde{X}_0$ to X_0 identifies the fibre $\tilde{X}_{0s_0}$ with H_0, (P_0, s_0) (denoted simply by s_0) with P_0, the henselization $\tilde{X}_{0P_0}$ with X_{0P_0}. The inverse image $\tilde{K}_0$ of K_0 over $\tilde{X}_0 - P_0$ is also an N-perverse sheaf of weight 0.

Fix the notation

$$\begin{array}{ccccc} H_0 - P_0 & \xrightarrow{u} & \tilde{X}_{T_0} - P_0 & \xleftarrow{v} & \tilde{X}_0 - H_0 \\ \downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow \\ s_0 & \xrightarrow{u'} & T_0 & \xleftarrow{v'} & \eta_0 \end{array}$$

We have $u'^* \mathbb{R}v'_* \mathbb{Q}_\ell = \mathbb{Q}_\ell[0] \oplus \mathbb{Q}_\ell[-1](-1)$; the canonical morphism of $\tilde{K}_0 \otimes f_0^* \mathbb{R}v'_* \mathbb{Q}_\ell$ to $\mathbb{R}v_*(v^* \tilde{K}_0)$ induces a morphism

$$u^* \tilde{K}_0 \oplus \mathbb{R}u^* \tilde{K}_0[-1](-1) \to u^* \mathbb{R}v_* v^* \tilde{K}_0 \tag{2}$$

For a general pencil, this is an isomorphism.

Let $\alpha \in H^{N-1}(X_P - P, K)$ be a pure class of weight N. To show that $\alpha = 0$, it suffices to check that the restriction α_η to $X_{P_\eta} = X_P - H_P$ is zero: the induction hypothesis ensures that the image of $H^{N-3}(H_P - P, K)(-1)$ under Gysin has weight $\leq N - 1$.

From Hochshild-Serre, we have a short exact sequence

$$0 \to H^1(I, H^{N-2}(X_{\bar{\eta}}, K)) \to H^{N-1}(X_\eta, K) \to H^{N-1}(X_{\bar{\eta}}, K)^I \to 0$$

For every representation L of I, we have $H^1(I, L) = L_I(-1)$, whence a morphism $L^I(-1) \to H^1(I, L)$ that we can interpret as cup product with a fundamental class $\epsilon \in H^1(I, \mathbb{Q}_\ell(1))$. Lemma 1 yields an extension of α to

$$\tilde{\alpha} \in H^{N-1}(H, j_{!*}K) \xleftarrow{\sim} H^{N-1}(\tilde{X}_T, j_{!*}K)$$

(the last identification by the proper base change theorem). We can assume that $\tilde{\alpha}$ has weight N. From the discussion in section 0, we see that $H^{N-1}(\tilde{X}_{\bar{\eta}}, \tilde{K})^I$ lies in M_0, where M is the monodromy filtration. From theorem 8, we can conclude that $H^{N-1}(\tilde{X}_{\bar{\eta}}, \tilde{K})^I$ has weight $\leq N - 1$, so the restriction $\tilde{\alpha}_\eta$ of $\tilde{\alpha}$ to $H^{N-1}(\tilde{X}_{\bar{\eta}}, \tilde{K})$ dies in $H^{N-1}(\tilde{X}_{\bar{\eta}}, \tilde{K})^I$. Therefore $\tilde{\alpha}_\eta$ is the image $\beta_{\bar{\eta}} \cup \epsilon$ of a unique element $\beta_{\bar{\eta}} \in H^{N-2}(\tilde{X}_{\bar{\eta}}, \tilde{K})^I(-1)$ of weight N. Lemma 2 shows that $\beta_{\bar{\eta}}$ can be lifted to $\beta \in H^{N-2}(\tilde{X}_{\bar{T}} - P, \tilde{K})(-1)$. Let β_η be the restriction of β to $H^{N-2}(X_\eta, \tilde{K})(-1)$.

We have a commutative diagram

$$\begin{array}{ccccccc}
H^{N-2}(X_P - P, K)(-1) & \longrightarrow & H^{N-2}(X_{P_\eta}, K)(-1) & \longrightarrow & H^{N-1}(X_{P_\eta}, K) & \longleftarrow & H^{N-1}(X_P - P, K) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^{N-2}(X_T - P, \tilde{K})(-1) & \longrightarrow & H^{N-2}(\tilde{X}_\eta, \tilde{K})(-1) & \xrightarrow{\epsilon\cup} & H^{N-2}(\tilde{X}_\eta, \tilde{K}) & \longleftarrow & H^{N-1}(\tilde{X}_T, j_{!*}\tilde{K}) \\
\downarrow & & & & \downarrow & & \downarrow \\
H^{N-2}(H - P, \tilde{K})(-1) & & \longrightarrow & & H^{N-2}(H - P, \mathbb{R}v_*\tilde{K}) & \longleftarrow & H^{N-1}(H - P, \tilde{K})
\end{array}$$

in which the last line corresponds to the decomposition (2) of $u^*\mathbb{R}v_*\mathbb{R}v^*\tilde{K}$. The elements constructed so far, which lie in the above groups, can be represented schematically by

$$\begin{array}{ccccccc}
\beta'' & & \longrightarrow & & \alpha_\eta & \longleftarrow & \alpha \\
\uparrow & & & & \uparrow & & \uparrow \\
\beta & \longrightarrow & \beta_\eta & \longrightarrow & \tilde{\alpha}_\eta & \longleftarrow & \tilde{\alpha} \\
\downarrow & & & & \downarrow & & \downarrow \\
\beta' & & \longrightarrow & & \tilde{\alpha}'_\eta & \longleftarrow & \tilde{\alpha}'
\end{array}$$

The primed elements $\beta', \dots, \beta''$ denote the images of the elements already defined in the middle row. The classes β' and $\tilde{\alpha}'$ cannot have the same image $\tilde{\alpha}'_\eta$ unless they are zero because they lie in complementary subspaces induced by (2). Therefore we have $\beta' = 0$. The restriction map from $H^{N-2}(X_P - P, K)(-1)$ to $H^{N-2}(H_P - P, K)(-1)$ is injective. The vanishing of β' implies the vanishing for the image of $\beta'' \in H^{N-2}(X_P - P, K)(-1)$, and therefore that $\alpha_\eta = 0$.

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