

Introduction to differential forms II

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9 Parameterized Surfaces

Recall that a parameterized curve is a C^1 function from an interval $[a, b] \subset \mathbb{R}^1$ to \mathbb{R}^3 . If we replace the interval by a subset of the plane \mathbb{R}^2 , we get a parameterized surface. Let's look at a few of examples

1) The upper half sphere of radius 1 centered at the origin can be parameterized using cartesian coordinates

$$\begin{cases} x = u \\ y = v \\ z = \sqrt{1 - u^2 - v^2} \\ u^2 + v^2 \leq 1 \end{cases}$$

2) The upper half sphere can be parameterized using spherical coordinates

$$\begin{cases} x = \sin(\phi) \cos(\theta) \\ y = \sin(\phi) \sin(\theta) \\ z = \cos(\phi) \\ 0 \leq \phi \leq \pi/2, 0 \leq \theta < 2\pi \end{cases}$$

3) The upper half sphere can be parameterized using cylindrical coordinates

$$\begin{cases} x = r \cos(\theta) \\ y = r \sin(\theta) \\ z = \sqrt{1 - r^2} \\ 0 \leq r \leq 1, 0 \leq \theta < 2\pi \end{cases}$$

An orientation on a curve is a choice of a direction for the curve. For a surface an orientation is a choice of “up” or “down”. The easiest way to make this precise is to view an orientation as a choice of (an upward or outward pointing) unit normal vector field \mathbf{n} on S . A parameterized surface S

$$\begin{cases} x = f(u, v) \\ y = g(u, v) \\ z = h(u, v) \\ (u, v) \in D \end{cases}$$

is called *smooth* provided that f, g, h are C^1 , the function that they define from $D \rightarrow \mathbb{R}^3$ is one to one, and the tangent vector fields

$$\mathbf{T}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right)$$

$$\mathbf{T}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$$

are linearly independent. In this case, once we pick an ordering of the variables (say u first, v second) an orientation is determined by the normal

$$\mathbf{n} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|}$$

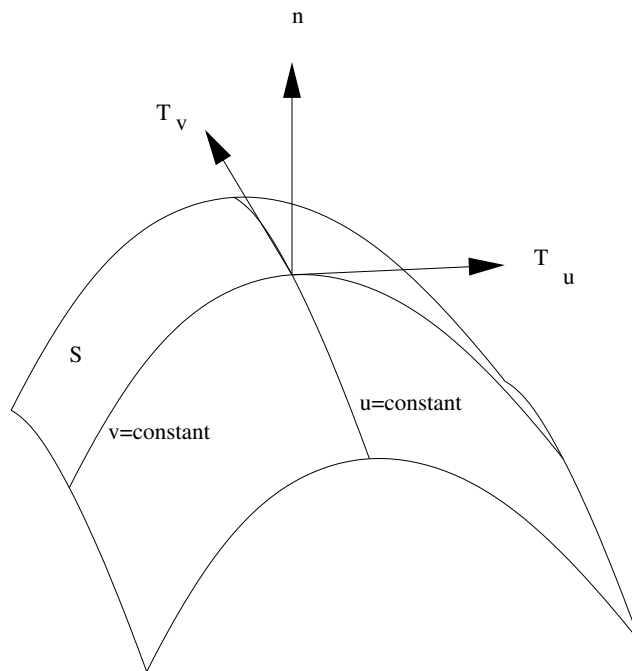


FIGURE 1

If we look at the examples given earlier. (1) is smooth. However there is a slight problem with our examples (2) and (3). Here $\mathbf{T}_\theta = 0$, when $\phi = 0$ in example (2) and when $r = 0$ in example (3). To deal with scenario, we will consider a surface smooth if there is at least one smooth parameterization for it.

10 Surface Integrals

Let S be a smooth parameterized surface

$$\begin{cases} x = f(u, v) \\ y = g(u, v) \\ z = h(u, v) \\ (u, v) \in D \end{cases}$$

with orientation corresponding to the ordering u, v . The symbols dx etc. can be converted to the new coordinates as follows

$$\begin{aligned} dx &= \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \\ dy &= \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \\ dx \wedge dy &= \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right) \wedge \left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \right) \\ &= \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) du \wedge dv = \frac{\partial(x, y)}{\partial(u, v)} du \wedge dv \end{aligned}$$

In this way, it is possible to convert any 2-form ω to uv -coordinates.

DEFINITION 10.1 *The integral of a 2-form on S is given by*

$$\int \int_S F dx \wedge dy + G dy \wedge dz + H dz \wedge dx = \int \int_D \left[F \frac{\partial(x, y)}{\partial(u, v)} + G \frac{\partial(y, z)}{\partial(u, v)} + H \frac{\partial(z, x)}{\partial(u, v)} \right] du dv$$

In practice, the integral of a 2-form can be calculated by first converting it to the form $f(u, v) du \wedge dv$, and then evaluating $\int \int_D f(u, v) du dv$.

Let S be the upper half sphere of radius 1 oriented with the upward normal parameterized using spherical coordinates, we get

$$dx \wedge dy = \frac{\partial(x, y)}{\partial(\phi, \theta)} d\phi \wedge d\theta = \cos(\phi) \sin(\phi) d\phi \wedge d\theta$$

So

$$\int \int_S dx \wedge dy = \int_0^{2\pi} \int_0^{\pi/2} \cos(\phi) \sin(\phi) d\phi d\theta = \pi$$

On the other hand if use the same surface parameterized using cylindrical coordinates

$$dx \wedge dy = \frac{\partial(x, y)}{\partial(r, \theta)} dr \wedge d\theta r dr \wedge d\theta$$

Then

$$\int \int_S dx \wedge dy = \int_0^{2\pi} \int_0^1 r dr d\theta = \pi$$

which leads to the same answer as one would hope. The general result is:

THEOREM 10.2 *Suppose that a oriented surface S has two different smooth C^1 parameterizations, then for any 2-form ω , the expression for the integrals of ω calculated with respect to both parameterizations agree.*

(This theorem needs to be applied to the half sphere with the point $(0, 0, 1)$ removed in the above examples.) Complicated surfaces may be divided up into nonoverlapping patches which can be parameterized separately. Then to integrate a 2-form, one would have to sum up the integrals over each patch.

11 Flux

In many situations arising in physics, one needs to integrate a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ over a surface. The resulting quantity is often called a flux. We will simply define this integral, which is usually written as $\int \int_S \mathbf{F} \cdot d\mathbf{S}$ or $\int \int_S \mathbf{F} \cdot \mathbf{n} dS$, to mean

$$\int \int_S F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$$

It is probably easier to view this as a two step process, first convert \mathbf{F} to a 2-form as follows:

$$F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \leftrightarrow F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy,$$

then integrate. Earlier, we learned how to convert a vector field to a 1-form:

$$F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \leftrightarrow F_1 dx + F_2 dy + F_3 dz$$

To complete the triangle, we can convert a 1-form to a 2-form and back via:

$$F_1 dx + F_2 dy + F_3 dz \leftrightarrow F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$$

This operation is usually denoted by $*$.

As a typical example, consider a fluid such as air or water. Associated to this, there is a scalar field $\rho(x, y, z)$ which measures the density, and a vector field \mathbf{v} which measures the velocity of the flow (e.g. the wind velocity). Then the rate at which the fluid passes through a surface S is given by the flux integral $\int \int_S \rho \mathbf{v} \cdot d\mathbf{S}$

12 Green's and Stokes' Theorems

Let C be a closed C^1 curve in \mathbb{R}^2 and D be the interior of C . D is an example of a surface with a boundary C . In this case the surface lies flat in the plane, but more general examples can be constructed by letting S be a parameterized surface

$$\begin{cases} x = f(u, v) \\ y = g(u, v) \\ z = h(u, v) \\ (u, v) \in D \subset \mathbb{R}^2 \end{cases}$$

then the image of C in \mathbb{R}^3 will be the boundary of S . For example, the boundary of the upper half sphere S

$$\begin{cases} x = \sin(\phi) \cos(\theta) \\ y = \sin(\phi) \sin(\theta) \\ z = \cos(\phi) \\ 0 \leq \phi \leq \pi/2, 0 \leq \theta < 2\pi \end{cases}$$

is the circle C given by

$$x = \cos(\theta), y = \sin(\theta), z = 0, 0 \leq \theta \leq 2\pi$$

In what follows, we will need to match up the orientation of S and its boundary curve. This will be done by the *right hand rule*: if the fingers of the right hand point in the direction of C , then the direction of the thumb should be “up”.

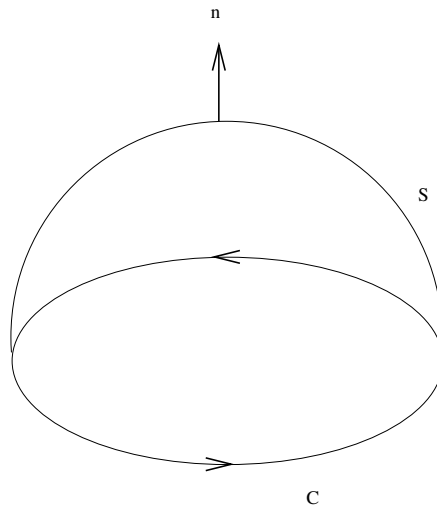


FIGURE 2

Stoke's theorem is really the fundamental theorem of calculus of surface integrals.

THEOREM 12.1 (Stokes' theorem) *Let S be an oriented smooth surface with smooth boundary curve C . If C is oriented using the right hand rule, then for any C^1 1-form ω on \mathbb{R}^3*

$$\int \int_S d\omega = \int_C \omega$$

If the surface lies in the plane, it is possible make this very explicit:

THEOREM 12.2 (Green's theorem) *Let C be a closed C^1 curve in \mathbb{R}^2 oriented counterclockwise and D be the interior of C . If $P(x, y)$ and $Q(x, y)$ are both C^1 functions then*

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \right) dx dy$$

In vector notation, Stokes' theorem is written as

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} dS = \int_C \mathbf{F} \cdot ds$$

where \mathbf{F} is a C^1 -vector field. In physics, there are two fundamental vector fields, the electric field \mathbf{E} and the magnetic field \mathbf{B} . They're governed by Maxwell's equations, one of which is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

where t is time. If we integrate both sides over S , apply Stokes' theorem and simplify, we obtain Faraday's law of induction:

$$\int_C \mathbf{E} \cdot ds = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \mathbf{n} dS$$

To get a sense of what this says, imagine that C is a wire loop and that we are dragging a magnet through it. This action will induce an electric current; the left hand integral is precisely the induced voltage and the right side is related to the strength of the magnet and the rate at which it is being dragged through.

13 Triple integrals and the divergence theorem

A 3-form is an expression $f(x, y, z)dx \wedge dy \wedge dz$. Given a solid region $V \subset \mathbb{R}^3$, we define

$$\iiint_V f(x, y, z)dx \wedge dy \wedge dz = \iiint_V f(x, y, z)dx dy dz$$

THEOREM 13.1 (Divergence theorem) *Let V be the interior of a smooth closed surface S oriented with the outward pointing normal. If ω is a C^1 2-form on \mathbb{R}^3 then*

$$\iiint_V d\omega = \iint_S \omega$$

In standard vector notation, this reads

$$\iiint_V \nabla \cdot \mathbf{F} dV = \iint_S \mathbf{F} \cdot \mathbf{n} dS$$

where \mathbf{F} is a C^1 vector field.

As an application, consider a fluid with density ρ and velocity \mathbf{v} . If S is the boundary of a solid region V with outward pointing normal \mathbf{n} , then the flux $\iint_S \rho \mathbf{v} \cdot \mathbf{n} dS$ is the rate at which matter flows out of V . In other words, it is minus the rate at which matter flows in, and this equals $-\partial/\partial t \iiint_V \rho dV$. On the other hand, by the divergence theorem, the above double integral equals $\iiint_V \nabla \cdot (\rho \mathbf{v}) dV$. Setting these equal and subtracting yields

$$\iiint_V [\nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t}] dV = 0.$$

The only way this can hold for all possible regions V is that the integrand

$$\nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} = 0$$

This is one of the basic laws of fluid mechanics.

14 Beyond \mathbb{R}^3

It is possible to do calculus in \mathbb{R}^n with $n > 3$. Here the language of differential forms comes into its own. While it would be impossible to talk about the curl of a vector field in, say, \mathbb{R}^4 , the derivative of a 1-form or 2-form presents no problems; we simply apply the rules we've already learned. Integration over higher dimensional "surfaces" or *manifolds* can be defined, and there is an analogue of Stokes' theorem in this setting.

As exotic as all of this sounds, there are applications of these ideas outside of mathematics. For example, in relativity theory one needs to treat the electric $\mathbf{E} = E_1 \mathbf{i} + E_2 \mathbf{j} + E_3 \mathbf{k}$ and magnetic fields $\mathbf{B} = B_1 \mathbf{i} + B_2 \mathbf{j} + B_3 \mathbf{k}$ as part of a single "field" on space-time. In mathematical terms, we can take space-time to be \mathbb{R}^4 - with the fourth coordinate as time t . The electromagnetic field can be represented by a 2-form

$$F = B_3 dx \wedge dy + B_1 dy \wedge dz + B_2 dz \wedge dx + E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt$$

If we compute dF using the analogues of the rules we've learned:

$$\begin{aligned} dF &= \left(\frac{\partial B_3}{\partial x} dx + \frac{\partial B_3}{\partial y} dy + \frac{\partial B_3}{\partial z} dz + \frac{\partial B_3}{\partial t} dt \right) \wedge dx \wedge dy + \dots \\ &= \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} \right) dx \wedge dy \wedge dz + \left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} + \frac{\partial B_3}{\partial t} \right) dx \wedge dy \wedge dt + \dots \end{aligned}$$

Two of Maxwell's equations

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

can be expressed very succinctly in this language as $dF = 0$.

For more informations, see the books "Differential forms and applications to the physical sciences" by H. Flanders and "Calculus on manifolds" by M. Spivak.