M462 (HANDOUT 1)

0.1. **Essential Information.** .
Professor: Donu Arapura
Office: Math 642
Office hrs: TBA
Book: do Carmo, Differential Geometry of Curves and Surfaces
There may be some additional notes on the web at www.math.purdue.edu/∼dvb

Grading Policy: Grade based on 2 midterms worth 30% a piece and weekly homework worth the remaining 40% (generally due on Thursdays). Since it's a small class, I will expect students to participate i.e. attend. I may ask for "volunteers" to present their solutions, and you get bonus points for this.

0.2. **What is Differential Geometry?** In a nutshell, we apply calculus to study curves, surfaces, and eventually higher dimensional versions of these called manifolds. To begin the story at the beginning, a curve is a C^1 function $F : [a, b] \to \mathbb{R}^3$ i.e. a vector valued function

$$F(t) = (f_1(t), f_2(t), f_3(t))$$

on the interval $[a, b] = \{x \in \mathbb{R} \mid a \leq t \leq b\}$. Recall that C^1 means that the functions f_i are continuous and the derivative f_i' exist and are continuous as well. We can think of $F(t)$ as the trajectory of a particle moving in space, and

$$F'(t) = (f_1'(t), f_2'(t), f_3'(t))$$

as its velocity, also called the tangent vector which may be denoted by T later. The second derivative (whose existence requires an additional C^2 assumption, which we now make)

$$F''(t) = (f_1''(t), f_2''(t), f_3''(t))$$

is the acceleration. While we are on the subject of derivatives of vectors, we recall the following fact, if F, G are two parametrized curves, then

$$(F \cdot G)' = (f_1 g_1 + f_2 g_2 + f_3 g_3)' = \ldots = F' \cdot G + F \cdot G'$$

This leads to the first result.

Theorem 0.3. *If F is a C^2 parameterized curve with a unit speed, i.e.*

$$||F'|| = \sqrt{f_1^2 + f_2^2 + f_3^2} = 1$$

then the velocity and acceleration F' and F'' are orthogonal.

(Reminder: two vectors are orthogonal if their dot product is zero. If neither vector is zero, then they must be perpendicular.)

Proof. The assumption is equivalent to

$$(F' \cdot F') = ||F'||^2 = 1$$

Therefore

$$0 = (F' \cdot F')' = 2(F' \cdot F'')$$

□

To understand the meaning of the unit speed assumption, recall that the integral of the speed

$$s(t) = \int_a^t ||F'(t)||dt$$

gives the length of the curve from a to t, also called the arclength. When the particle travels at unit speed, then

$$s(t) = \int_a^t 1dt = t - a$$

After translating t so that $a = 0$, we have $s = t$. We say that curve is parameterized by arclength if $s = t$. Conversely, if the curve is parameterized by arclength, then $||F'(t)|| = \frac{ds}{dt} = 1$.

Now suppose that F is a curve such that $F'(t) \neq 0$ for all $t \in [a, b]$. Such a curve is called regular, and points where the derivative is zero are called singular. Since $\frac{ds}{dt} = ||F'(t)|| \neq 0$. This means by the inverse function theorem, we can rewrite t as a function of s. In other words, for such a curve we can reparameterize by arclength. From now, we usually assume that our curves are parameterized by arclength, because it simplifies things.

Now let us suppose that $F(s)$ is a curve parametrized by arclength. At the points where $F'' \neq 0$, we make can make into a unit vector

$$N(s) = F''((s)/||F''(s)||$$

called the normal. From the last theorem, we deduce

Corollary 0.4. *The tangent and normal vectors $T(s) = F'(s), N(s)$ are orthogonal.*

Note that T, N are also unit vectors. We want to complete this to an orthonormal basis (varying with s). Recall, from linear algebra, a basis of $\mathbb{R}^3$ consists of 3 linearly independent vectors. They form an orthonormal basis if in addition they are all unit vectors and the dot product of any two is zero. So to complete to such a basis, we choose a third unit vector $B(s)$ perpendicular to the plane spanned by T and N. There are actually two possible solutions to this, we use the right hand rule to pick out the one we want: curl the fingers of your right hand to point from T to N, B points in the direction of the thumb. Alternatively, it is given by the vector cross product

$$B(s) = T(s) \times N(s)$$

Note that do Carmo uses a $\wedge$ symbol for this operation. The vector B is called the binormal. The last formula gives a rigorous way to define and calculate it (more about this later on). The ordered basis consisting of T, N, B is called the Frenet frame.

0.5. **Homework 1.** (Due Thurs August 29, in class)

Do do Carmo page 5 problem # 4, page 7 problem # 2.