M462 (HANDOUT 5)

0.1. **Gauss Map.**

Definition 0.2. *A regular surface S is called orientable if it possesses a unit normal vector field $N(p)$. Such a surface possesses precisely two unit normal fields given by $\pm N(p)$. A choice of one of then is called an orientation.*

Most of the standard examples are orientable. The Möbius band or strip, discussed on pages 106-108 of do Carmo, is not.

Definition 0.3. *Given an oriented surface S with unit normal field $N(p)$, we can regard this as a map from S to the unit sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. This is called the Gauss map.*

If S is given parametrically by $x = f_1(u, v), \ldots$, we have seen that

$$N(u, v) = \frac{\alpha \times \beta}{||\alpha \times \beta||}$$

where

$$\alpha = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u})$$
$$\beta = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v})$$

Example 0.4. *If $S = S^2$ with outward pointing normal. By straight forward calculation, we can see that $N : S^2 \to S^2$ is the identity.*

We want to study the derivative of this map dN_p. We can think of this as taking a tangent vector in T_pS to $\mathbb{R}^3$. If we regard N as a column vector, then dN is given by 3×2 matrix with columns $N_u = \partial N/\partial u$ and $N_v = \partial N/\partial v$.

Proposition 0.5. *The image of dN lies in the plane T_pS.*

Proof. We have $\langle N, N\rangle = 1$, leading to $2\langle N_u, N\rangle = 0$ and $2\langle N_v, N\rangle = 0$. □

Thus we can think of dN_p as linear transformation of T_pS to itself. Here is what it means:

Proposition 0.6. *Suppose that $C(t)$ is a curve lying on S with $C(0) = p$, we can restrict N to C, to get a vector field $N(t)$ along C. If $C'(0) \in T_pS$ the tangent vector to C, we have $dN_p(C'(0)) = N'(0)$*

Proof. This follows from the chain rule. □

Given a regular curve C parameterized by arclength the second derivative $C''(s)$ with kn, where n is the unit normal vector to C and k is the curvature of C. If C lies on a regular surface S, the inner product of this with the normal to the surface is called the normal curvature.

Definition 0.7. *The normal curvature to a regular curve C on S is $k_n = k\langle n, N\rangle$,*

To connect this with the Gauss map, we need another definition.

Definition 0.8. *The second fundamental form of the surface S at p is the bilinear form $II_p(\alpha, \beta) = -\langle dN_p\alpha, \beta\rangle$ on T_pS. We also set $II_p(\alpha) = II_p(\alpha, \alpha)$.*

Proposition 0.9. *If $C \subset$ is a regular curve parametrized by arclength, and $C(0) = p$. Then $II_p(C'(0)) = k_n(p)$.*

Proof. Let $N(s)$ denote the restriction of N to C. Since C lies on S, we have $\langle N(s), C'(s)\rangle = 0$. Differentiating yields $\langle N'(s), C'(s)\rangle + \langle N(s), C''(s)\rangle$. Therefore

$$II_p(C'(0)) = -\langle dN_pC'(0), C'(0)\rangle = -\langle N'(0), C'(0)\rangle = \langle N(s), C''(s)\rangle = k_n(p)$$

□

Definition 0.10. *The maximum k_1 and minimum k_2 values of $k_n(p)$ as C varies over all curves through p are called the principal curvatures at p. The average $H(p) = \frac{1}{2}(k_1 + k_2)$ and product $K(p) = k_1k_2$ are called respectively the mean and Gaussian curvatures.*

These can computed using linear algebra. Given a basis of T_pS, we can represent dN_p by a 2×2 matrix.

Theorem 0.11. *The principal curvatures are negatives of the eigenvalues of dN_p. The Gaussian curvature is $\det(dN_p)$ and mean curvature is $-\frac{1}{2}trace(dN_p)$.*

Proof. do Carmo Prop 1 on p 140 shows that II is symmetric: $II(\alpha, \beta) = II(\beta, \alpha)$. Linear algebra then shows that we can choose a basis so that $II(x, y) = \lambda_1 x^2 + \lambda_2 y$ where λ_i are the eigenvalues. The first statement easily follows. Also by linear algebra, the determinant and trace are the product and sum of eigenvalues. □

0.12. **Homework 6 (due Thurs Oct 3).** do Carmo, p 151: 2, 3; p 168: 1