Review of D-modules

Basic Ref: Hotta, Takeuchi, Tanisaki= HTT

Let X be a smooth complex algebraic variety. Let D_X be the sheaf of (algebraic) differential operators on X. When $X = \mathbb{A}^n$, (the global sections of) D_X is the Weyl algebra

$$\mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$$

$$[x_i, x_j] = [\partial_i, \partial_j] = 0, [\partial_i, x_j] = \delta_{ij}$$

We can filter D_X by $F_k D =$ sheaf of operators of order $\leq k$. When $X = \mathbb{A}^n$, $Gr_F D_X$ is the polynomial ring in the symbols x_i, ∂_i.

D-modules

In general,

Proposition

$Gr_F D_X = \pi_* \mathcal{O}_{T^*_X}$, where $\pi : T^*_X \to X$ is the cotangent bundle.

A D-module is a sheaf of left modules over D_X. Will generally assume it's quasicoherent over $\mathcal{O}_X$.

Example

Tautologically, D_X is a D-module.

Example

$\mathcal{O}_X$ is a D-module with obvious action. More generally, any vector bundle with integrable connection is a D-module. Such examples are coherent as $\mathcal{O}_X$-modules. Conversely, any $\mathcal{O}_X$-coherent D-module is of this form

Characteristic variety

Example

If $D \subset X$ is a smooth divisor, then $\mathcal{O}_X(*D) = \bigcup \mathcal{O}_X(nD)$ is a D-module. This is coherent (=locally finitely presented) over D_X but not over $\mathcal{O}_X$. (Call a D-module coherent if it coherent over D_X.)

Given a coherent D-module M, there exists a (non unique) filtration $F_\bullet M$ by $\mathcal{O}_X$-submodules such that $F_k D_X F_j M \subset F_{k+j} M$, and F_j is $\mathcal{O}_X$-coherent. This is called a good filtration.

Theorem

1. *$Supp\, Gr_F M \subset T^*_X$ is independent of F. It is called the characteristic variety $Char(M)$.*
2. *(Bernstein's inequality) If $M \neq 0$, then $\dim Char(M) \geq \dim X$.*

Definition

A D-module is called holonomic if $\dim Char(M) = \dim X$.

Example

Integrable connections, and $\mathcal{O}_X(*D)$ *are holonomic,* D_X *isn't. The characteristic variety is, respectively, the zero section of* T^*_X*, the zero section union the conormal bundle to* D*, and the whole of* T^*_X.

Theorem

The category of holonomic modules forms an Artinian abelian category. The simple objects are generically integrable connections on their support.

D-modules on curves

From now on, let X be a smooth curve. We want to understand the structure of a holonomic D-module M. We may as well restrict to the case where M is simple. Then the support of M is either all of X or zero dimensional. Let's suppose it's the second. Then simplicity forces the support to be a point p. Again by simplicity, we must have $M = \mathbb{C}_p$.

So now we suppose that M is simple with X as its support. By the previous theorem, we can find a Zariski open $j : U \to X$ such that $V = M|_U$ is an integrable connection. Explicitly, this means that V is a locally free $\mathcal{O}_U$-module with a connection

$$\nabla : V \to \Omega^1_U \otimes V$$

Integrability is automatic in this case because U is a curve.

The classical Riemann-Hilbert correspondence says that (V, ∇) is determined by the locally constant sheaf, or local system, $L = \ker \nabla^{an}$. (Notice that we switched to the analytic category, because differential equations won't have enough solutions otherwise.) This in turn is given by a representation of $\pi_1(U)$ (the monodromy of ∇). Simplicity of M forces irreducibility of this representation. Otherwise, a nontrivial subrepresentation of V would generate a nontrivial submodule of M.

In general, there are several ways to extend a connection on U to a D-module on X.

Example

Let $D = X - U$. If $V = \mathcal{O}_U$, then $\mathcal{O}_X$ and $\mathcal{O}_X(*D)$ are both extensions of V. What distinguishes them is that $\mathcal{O}_X$ is simple, but $\mathcal{O}_X(*D)$ isn't because it contains $\mathcal{O}_X$.

Proposition

Given an irreducible local system V on U, there is a unique extension to X, which is a simple as D-module. This is called the minimal, or intermediate, extension.

This proves:

Theorem

A simple holonomic D_X-module is either a skyscraper sheaf $\mathbb{C}_p$, or a minimal extension of an irreducible connection from a Zariski open.

One says that M is regular if the connection ∇ is regular in Deligne's sense $\Leftrightarrow$ the system of ODE is regular in the classical sense (solutions don't blow up worse than $O(|z|^{-n})$, for some n, on angular sectors).

It is also useful to understand what happens under de Rham. Given a D-module M, let

$$DR(M) = M^{an} \to \Omega^1_{X^{an}} \otimes M^{an}$$

shifted so that it starts in degree -1. Notice that $DR(M)$ is an object in the derived category $D^b(X^{an}, \mathbb{C})$. It is possible to characterize such objects.

Definition

An object F in the constructible derived category $D^b_c(X^{an}, \mathbb{C})$ is a semiperverse sheaf if

$$\dim supp\, \mathcal{H}^i(F) \leq -i,$$

and perverse if additionally the Verdier dual $DF = \mathbb{R}\mathcal{H}om(F, \mathbb{C}[2])$ is semiperverse.

Lemma

If F is perverse then $\mathcal{H}^i(F) = 0$ unless $i = -1, 0$ and that $\mathcal{H}^0(F)$ has zero dimensional support.

Proof.

Semiperversity of F implies that $\mathcal{H}^i(F) = 0$ for $i > 0$ and that $\mathcal{H}^0(F)$ has zero dimensional support. Semiperversity of DF implies $\mathcal{H}^i(F) = 0$ for $i \leq -2$. □

Theorem (Kashiwara)

If M is a regular holonomic holonomic D-module, then $DR(M)$ is perverse. This gives gives an equivalence between the categories of these D-modules and perverse sheaves.

Sketch of first part.

Set $F = DR(M)$. M is given by connection (V, ∇) on a Zariski open U. Then

$$DR(M)|_U = (\ker \nabla)[1]$$

This implies semiperversity of F. The module $M^* = Ext^1(M, D_X) \otimes \omega_X^{-1}$ is also regular holonomic, and $DF = DR(M^*)$. Therefore DF is also semiperverse. □

Since we have an equivalence of categories, the category of perverse sheaves is also Abelian and Artinian. (This can be proved directly – perhaps, we'll do this later).

Proposition

The simple perverse sheaves are either skyscraper sheaves $\mathbb{C}_p$ or of the form $j_*L[1]$, where L is an irreducible local system on a Zariski open $j : U \to X$.

All of these statements generalize to higher dimensions.