# INTRODUCTION TO THE HODGE THEORY OF ALGEBRAIC VARIETIES

DONU ARAPURA

## 1. Projective varieties

Let $\mathbb{P}^n$ be $n$-dimensional complex projective space. This is the set of 1-dimensional subspaces of $\mathbb{C}^{n+1}$. A subset $X \subset \mathbb{P}^n$ is called an algebraic subvariety if it is the zero set of a collection of homogeneous polynomials called defininining polynomials. An algebraic subvariety of some $\mathbb{P}^n$ is called a projective algebraic variety. A subvariety of $\mathbb{P}^n$ is called nonsingular or smooth if the Jacobian of these polynomials has the expected rank, locally. It follows that it is a complex manifold. Conversely:

**Theorem 1.1** (Chow)**.** *Any complex submanifold of $\mathbb{P}^n$ is an algebraic subvariety.*

**Example 1.2.** *Any compact Riemann surface can be realized as an algebraic subvariety of $\mathbb{P}^3$.*

**Example 1.3.** *Grassmians and flag varieties are projective algebraic.*

**Example 1.4.** *The product of two projective algebraic varieties is projective algebraic.*

**Example 1.5.** *The set of all subvarieties of $\mathbb{P}^n$ has the structure of a countable union of projective algebraic varieties.*

A map $f : X \to Y$ between smooth projective varieties is called a morphism if its graph is a subvariety of $X \times Y$ (this definition isn't quite correct for singular varieties).

**Corollary 1.6** (to Chow)**.** *$f$ is a morphism if and only if it is holomorphic.*

## 2. Käher metrics

A Hermitean metric on a complex manifold is a metric

$$\sum g_{ij} dz_i \otimes d\bar{z}_j$$

whose associated Käher form

$$\omega = \frac{\sqrt{-1}}{2} \sum g_{ij} dz_i \wedge d\bar{z}_j$$

is closed. A complex manifold is called Kähler if it possessess such a metric.

The main example for us is:

**Example 2.1.** *$\mathbb{P}^n$ has a unique $U(n+1)$-invariant metric of volume $1$ called the Fubini-Study metric. This metric is Kähler.*

**Proposition 2.2.** *The restriction of a Kähler metric to a submanifold is Kälher.*

**Corollary 2.3.** *Any smooth projective algebraic variety is Kähler.*

Not all examples arise this way:

**Example 2.4.** *The Euclidean metric on a complex torus is Kähler. However, if* $L \subset \mathbb{C}^n$ *is a generic lattice with* $n > 1$, $\mathbb{C}^n/L$ *will not be an algebraic variety.*

The importance of Kähler metrics stems from the following principle:

**Principle 2.5.** *An identity involving natural differential operators of order* $\leq 1$ *holds for a Kähler manifold if it holds for Euclidean space.*

Operators for which this applies are $d, \partial, \overline{\partial} \ldots$ This principle can be used to prove a series of nontrivial identities called the Kähler identities.

## 3. The Hodge theorem

Recall that the de Rham cohomology of a manifold is

$$H^*(X) = \frac{\ker d}{\text{image } d}$$

where $d$ is the exterior derivative on differential forms. When $X$ is a a Riemannian manifold, there is a distinguished representative for each cohomology class. The Laplacian is defined as $\Delta = dd^* + d^*d$, where $d^*$ is the adjoint to $d$. A form is called harmonic if $\Delta\alpha = 0$.

**Theorem 3.1** (Hodge)**.** *Let* $X$ *be a compact Riemannian manifold. Then any de Rham cohomology class has a unique harmonic representative.*

Recall the complex valued form is said to be of $(p, q)$-type if it can be expressed in local analytic coordinates as

$$\sum f_{i_1,\ldots i_p;j_1,\ldots j_q} dz_{i_1} \wedge \ldots dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots d\bar{z}_{j_q}$$

This together with the Kähler identities implies

**Theorem 3.2** (Hodge Decomposition)**.** *Suppose that* $X$ *is a compact Kähler manifold. A form is harmonic if and only if its complex conjugat is harmonic if and only if all of its* $(p, q)$ *components are harmonic. Therefore there is a decomposition*

$$H^i(X) = \bigoplus_{p+q=i} H^{pq}$$

*where* $H^{pq}$ *is the space of* $(p, q)$ *harmonic forms. This satisfies* $\overline{H^{pq}} = H^{qp}$

**Corollary 3.3.** *If* $i$ *is odd then the* $i$*th Betti number is even.*

**Theorem 3.4** (Hard Lefschetz)**.** *Suppose that* $X$ *is an* $n$ *dimensional compact Kähler manifold. Let* $L = [\omega]$ *be cup product with the Kähler form. Then*

$$L^i : H^{n-i}(X) \to H^{n+i}(X)$$

*is an isomorphism.*

## 4. Hodge structures

We define a Hodge structure of weight $i$ to consist of a finitely generated Abelian group $H_{\mathbb{Z}}$ together with a decomposition

$$H = H_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=i} H^{pq}$$

such that $\overline{H^{pq}} = H^{qp}$ We define a morphism of Hodge structures $(H_{\mathbb{Z}}, \dots) \to (G_{\mathbb{Z}}, \dots)$ to be homomorphism of Abelian groups $f_{\mathbb{Z}} : H_Z \to G_{\mathbb{Z}}$ such $f \otimes \mathbb{C}$ preserverse the decomposition. Given a Hodge structure, we define the Hodge filtration by

$$F^p H = \bigoplus_{p' \geq p} H^{p'q}$$

Given a Hodge structure $H$ of weight and an integer $m$, the $m$th Tate twist $H(m)$ of $H$ has the same Abelian as before, with decomposition

$$[H(m)]^{p,q} = H^{p-m,q-m}$$

$\mathbb{Q}(m)$ is the unique Hodge structure of weight $2m$.

Given a map of compact oriented manifolds $f : X \to Y$, recall there are maps

$$f^* : H^i(Y) \to H^i(X)$$

and

$$f_* : H^i(X) \to H^{i+c}(Y)$$

where $c = \dim_{\mathbb{R}} Y - \dim_{\mathbb{R}} X$ is the difference of real dimensions. The Gysin maps $f_*$ are defined using Poincaré duality.

**Theorem 4.1** (Hodge Decomposition II)**.** *The Hodge structure on $H^i(X) = H^i(X, \mathbb{Z}) \otimes \mathbb{C}$, where $X$ is a compact Kähler manifold is bifunctorial. More precisley, given a holomorphic map $f : X \to Y$ there are morphisms*

$$f^* : H^i(Y) \to H^i(X)$$

*and*

$$f_* : H^i(X)(-c) \to H^{i+2c}(Y)$$

*of Hodge strucutres, where $c$ is the difference of complex dimensions.*

## 5. The Hodge conjecture

Let $X$ be a smooth projective variety. A Hodge class of type $(p, p)$ is an element of $H^{2p}(Z, \mathbb{Q}) \cap H^{p,p}$. Examples of Hodge classes can be constructed as follows: Suppose that $Z \subset X$ is a subvariety of codimension $p$. If $Z$ is nonsingular, we have a Gysin homomorphism $f_* : H^0(Z)(-p) \to H^{2p}(X)$ associated to the inclusion. We have an isomorphism $H^0(Z)(-p) \cong \mathbb{Q}(-p)$ The fundamental class is the image $[Z] = f_* 1l$. It is a Hodge class of type $(p, p)$. More generally if $Z \subset X$ is a singular subvariety. We can let $p : \tilde{Z} \to Z$ be a resolution of singularities, which exists by a fundamental theorem of Hironaka. The class $[Z] = (f \circ p)_* 1$ can be shown to be independent of $\tilde{Z}$, and is the fundamental class of $Z$. This is again a Hodge class of type $(p, p)$. The famous Hodge conjecture asserts that all Hodge classes arise this way:

**Conjecture 5.1** (The Hodge Conjecture)**.** *Every Hodge class is a linear combination of fundamental classes of subvarieties.*

Remarks:

(1) The Hodge conjecture (HC) holds for $(1,1)$ classes.
(2) The analogue of HC for Kähler manifolds is known to be false (Zucker).
(3) If $\dim X = n$, then by Hard Lefschetz HC holds for $(n-p, n-p)$ classes on $X$ if it holds for $(p,p)$. In particular, it holds for $(n-1, n-1)$.
(4) From previous two items, HC holds when $\dim X \leq 3$. The case of $\dim X = 4$ is open!

Part of the significance of HC stems from the theory of motives. Without defining things precisely, we state the basic result.

**Proposition 5.2.** *If HC holds, then*

(1) *Numerical equivalence coincides with homological equivalence.*
(2) *The category of (neq) motives embeds as a full abelian subcategory of the category of Hodge structures.*

The second statement is equivalent to HC.

The *coniveau* filtration on $H^i(X)$ is given by

$$N^pH^i(X) = \sum_{codim S \geq p} \mathrm{Ker}[H^i(X) \to H^i(X - S)]$$
$$= \sum_{codim S = q \geq p} \mathrm{Im}[H^{i-2q}(\tilde{S}) \to H^i(X)]$$

where $S$ runs over closed subvarieties, and $\tilde{S}$ are fixed desingularizations. $N^pH^i(X)$ is a sub-Hodge strucutre of $H^i(X)$ contained in $F^pH^i(X)$. Conversely,

**Conjecture 5.3** (The generalized Hodge conjecture)**.** $N^pH^i(X)$ *is the maximal sub-Hodge strucutre of* $H^i(X)$ *contained in* $F^pH^i(X)$.

Remarks:

(1) The generalized Hodge conjecture (GHC) was formulated in the above way by Grothendieck; he gave a countrerexample to the original formulation of Hodge.
(2) GHC implies HC. More precisely, $N^pH^{2p}(X)$ is the span of fundamental classes, and the space of Hodge cycles is the maximal Hodge structure of $F^pH^{2p}(X)$.
(3) GHC is open even in dim = 3.

Department of Mathematics, Purdue University, West Lafayette, IN 47907, U.S.A.