INTERSECTION HOMOLOGY AND PERVERSE SHEAVES

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1. Intersection homology

Goresky and MacPherson modified the definition of homology, to obtain a theory with good properties on singular spaces. The idea was place restrictions on the chains that met the singular set using a function called perversity. Note that their original approach was entirely geometric in the spirit of Lefschetz. One thing that was not obvious from this point of view was the topological invariance of the theory. This was solved later using sheaf theoretic methods.

Here is the set up. Fix an n dimensional pseudomanifold $X \supset X_{n-2} \supset X_{n-3} \ldots$. A perversity is a function from strata $\{X_i - X_{i-1}\}$ (or equivalently the labels $\{n-2, n-3 \ldots\}$) to $\mathbb{Z}$. Originally, Goresky-MacPherson imposed stronger conditions that $p(2) = 0$, and $p(c) \leq p(c+1) \leq p(c)+1$. The topological invariance statement mentioned above is valid only for such perversities. Given a perversity p, the complementary perversity is $q(c) = c-2-p(c)$ figures in the statement of Poincaré duality given below. When X has only even strata (which includes the case of a complex algebraic variety), the so called middle perversity $p(c) = (c-2)/2$ is the most interesting example. It is self dual in the sense that it equals its complement.

Fix a field k. Suppose that X is triangulated in such a way that $X_\bullet$ are subcomplexes. Let C^{-i} be the sheaf associated to the presheaf

$$U \mapsto \{k\text{-valued } i\text{-chains on } U\}$$

where we use possibly infinite, but locally finite, simplicial chains on a triangulation refining the initial one. This is becomes a complex of fine sheaves which realizes the dualizing complex D. Given a chain ξ, let $|\xi|$ denotes its support. Given a perversity p, let $IC_p^{-i}(U) \subset IC^{-i}(U)$ denote the chains ξ satisfying

$$\dim(|\xi| \cap X_{n-c}) \leq i - c + p(c)$$

$$\dim(|\partial\xi| \cap X_{n-c}) \leq i - 1 - c + p(c)$$

To help parse this, note that this looks like a transversality statement when $p(c)$ is omitted, so $p(c)$ measures the deviation from that. Intersection homology $IH_*^p(X,k) = H^{-*}(\Gamma(X, IC_p^*))$ by definition. By taking $p(c) \gg 0$, we see that this includes usual homology. Among other things, Goresky-MacPherson showed that when X is compact q is complementary to p, Poincaré duality holds

$$IH_i^p(X,k)^* \cong IH_{n-i}^q(X,k)$$

This follows from Verdier duality once one observes that IC_p and IC_q are Verdier dual. This was not their original argument, which was more geometric.

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In the next section, X will be an algebraic variety of real dimension n. We will switch to BBD notation

$$ {}^{p}IH^{i}(X) = IH^{p}_{n/2-i}(X) $$