

# THE JACOBIAN OF A RIEMANN SURFACE

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Fix a compact connected Riemann surface  $X$  of genus  $g$ . The set of divisors  $Div(X)$  forms an abelian group. A divisor is called principal if it equals  $div(f) = \sum_p ord_p(f)p$  for some  $f \in \mathbb{C}(X)^*$ ; recall that  $ord_p$  is the discrete valuation measuring the order of the zero or pole at  $p$ . From the definition the operation  $div$  is a homomorphism. Therefore the set of principal divisors  $Princ(X)$  forms a subgroup of  $Div(X)$ . The quotient

$$Cl(X) = \frac{Div(X)}{Princ(X)}$$

is called the divisor class group of  $X$ . Let  $Div^0(X)$  be the kernel of the degree homomorphism. Recall that we have proved that this contains the principal divisors. Thus we have an exact sequence

$$0 \rightarrow Cl^0(X) \rightarrow Cl(X) \rightarrow \mathbb{Z} \rightarrow 0$$

where  $Cl^0(X) = Div^0(X)/Princ(X)$ . The structure is given as follows.

**Theorem 0.1.**  *$Cl^0(X)$  can be given the structure of a  $g$  dimensional complex torus, i.e. it is the quotient of a  $g$  dimensional complex vector space by a lattice.*

The Jacobian is  $J(X) = Cl^0(X)$  with the above structure of a torus. It is now not just a group, but a geometric object in its own right. Some of the significance of this group, both in geometry and arithmetic, will be discussed below.

## 1. LINE BUNDLE ASSOCIATED TO A DIVISOR

Given a divisor  $D = \sum n_p p$ , recall that we can associate a sheaf  $\mathcal{O}_X(D)$ . By construction, when  $U$  is a coordinate disc,

$$\mathcal{O}(D)(U) = \mathcal{O}_X(U) \cdot \prod_{p \in U} (z - p)^{n_p} \cong \mathcal{O}_X(U)$$

where the isomorphism is as  $\mathcal{O}(U)$ -modules. We give a name to such an object. A line bundle<sup>1</sup> is a sheaf  $\mathcal{M}$  such that each  $\mathcal{M}(U)$  is a module over  $\mathcal{O}(U)$ , in a way compatible with restriction, and such that  $\mathcal{M}$  is locally isomorphic to  $\mathcal{O}_X$ . The last phrase means that we have an open cover  $\{U_i\}$  such that we have a family of isomorphisms

$$\phi_U : \mathcal{M}(U) \cong \mathcal{O}_X(U), \quad U \subset U_i$$

compatible with restriction. The last condition can be expressed more succinctly by saying that  $\mathcal{M}$  is locally free of rank one. A collection of such isomorphisms is called a local trivialization.

**Example 1.1.**  $\mathcal{O}_X$  and more generally  $\mathcal{O}_X(D)$  are line bundles.

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<sup>1</sup>This is not the usual definition, but it is equivalent to it. However, we would rather not get into this part of the story.

**Example 1.2.** A second example is the sheaf  $\Omega_X^1$  of holomorphic 1-forms. For any coordinate disk  $U$ ,

$$\Omega_X^1(U) = \mathcal{O}(U)dz \cong \mathcal{O}(U)$$

An isomorphism of line bundles  $\eta : \mathcal{M} \cong \mathcal{L}$  is a family of module isomorphisms  $\eta_U : \mathcal{M}(U) \cong \mathcal{L}(U)$  compatible with restriction. A line bundle is trivial if it is isomorphic to  $\mathcal{O}_X$ .

**Lemma 1.3.**  $\mathcal{O}(D)$  is trivial if and only if  $D$  is principal.

*Proof.* Let  $D = \sum n_p p$ . Suppose that  $\eta : \mathcal{O} \cong \mathcal{O}(D)$  is an isomorphism. Then we have a function  $f = \eta(1) \in H^0(X, \mathcal{O}_X(D))$  which necessarily satisfies  $\text{div}(f) \geq -D$ . In fact, we claim that equality holds. To see this choose a coordinate disk centered at  $p$  which contains no other points of the support of  $D$ . Then since  $\eta$  is surjective, we must have  $\mathcal{O}(U)f = \mathcal{O}(U)z^{-n_p}$ . This implies that  $\text{ord}_p(f) = -n_p$  for each  $p$ . Therefore  $\text{div}(f) = -D$ . So that  $D = \text{div}(f^{-1})$ .

Conversely, if  $D = \text{div}(g)$ . The map  $1 \mapsto g^{-1}$  gives an isomorphism  $\mathcal{O} \cong \mathcal{O}(D)$ .  $\square$

**Corollary 1.4.**  $\mathcal{O}(E) \cong \mathcal{O}(D)$  if and only if  $D - E$  is principal.

For the proof, we need to construction of the tensor product of line bundles  $\mathcal{M}_1 \otimes \mathcal{M}_2$ . This is a new line bundle equipped with a bilinear pairing  $\mathcal{M}_1(U) \times \mathcal{M}_2(U) \rightarrow \mathcal{M}_1 \otimes \mathcal{M}_2(U)$ . When  $\mathcal{M}_i = \mathcal{O}(D_i)$ ,  $\mathcal{M}_1 \otimes \mathcal{M}_2 \cong \mathcal{O}(D_1 + D_2)$  and the pairing simply takes  $f_i \in \mathcal{O}(D_i)(U)$  to the product  $f_1 f_2 \in \mathcal{O}(D_1 + D_2)(U)$ .

*Proof.* If  $D - E$  is principal, then we have an isomorphism  $\mathcal{O} \cong \mathcal{O}(D - E)$ . Multiplying both sides by  $\mathcal{O}(E)$ , yields an isomorphism  $\mathcal{O}(E) \cong \mathcal{O}(D)$ . Conversely given a such an isomorphism, we can multiply by  $\mathcal{O}(-E)$  to conclude that  $\mathcal{O} \cong \mathcal{O}(D - E)$ , and therefore that  $D - E$  is principal.  $\square$

**Proposition 1.5.** Every line bundle is isomorphic to  $\mathcal{O}(D)$  for some divisor  $D$ .

*Sketch.* Let  $\mathcal{M}$  be a line bundle. The key fact, which we won't prove is that  $\dim H^i(X, \mathcal{M}) < \infty$  [GH]. With this in hand, we can imitate the proof of Riemann's inequality to conclude that  $H^0(X, \mathcal{M} \otimes \mathcal{O}(E)) \neq 0$  when  $\text{deg } E \gg 0$ . Choose a nonzero element  $f \in H^0(X, \mathcal{M} \otimes \mathcal{O}(E))$ . We can define a morphism  $\eta : \mathcal{O}_X \rightarrow \mathcal{M} \otimes \mathcal{O}(E)$  which sends 1 to  $f$ . This need not be an isomorphism because  $f$  need not generate  $\mathcal{M} \otimes \mathcal{O}(E)(U)$  as a module. Under a local trivialization  $\phi : \mathcal{M} \otimes \mathcal{O}(E)(U) \cong \mathcal{O}(U)$ ,  $f$  corresponds to a function. This generates precisely if  $f$  has no zeros. In the general the image of  $\phi\eta$  is  $\mathcal{O}(U) \prod (z - p)^{\text{ord}_p(f)}$ . Thus we conclude that the image of  $\eta$  is precisely  $\mathcal{O}(E')$ . Thus we have an isomorphism  $\mathcal{M} \otimes \mathcal{O}(E - E')$ , where  $E' = \text{div}(f)$ . Thus we have an isomorphism  $\mathcal{O} \cong \mathcal{M} \otimes \mathcal{O}(E - E')$ . Tensoring with  $\mathcal{O}(E' - E)$  yields an isomorphism  $\mathcal{M} \cong \mathcal{O}(E - E')$ .  $\square$

## 2. CLASSIFICATION OF LINE BUNDLES

The key step in the proof of the main theorem is the classification of line bundles. Given a line bundle  $\mathcal{M}$  with a local trivialization  $\phi_i : \mathcal{M}(U_i) \cong \mathcal{O}_X(U_i)$  above. Now consider  $\phi_i \phi_j^{-1}$ . This is a module isomorphism of  $\mathcal{O}(U_{ij})$  with itself, which is necessarily given by a unit  $f_{ij} \in \mathcal{O}(U_{ij})^*$  in the ring; i.e.  $f_{ij}$  is a nowhere zero holomorphic function. We can see from the definition that  $f_{ij} f_{jk} f_{ki} = 1$ . In other

words, this forms a 1-cocycle for the sheaf  $\mathcal{O}_X^*$  of invertible holomorphic functions. If we pick a different trivialization  $\phi'_i : \mathcal{M}(U_i) \cong \mathcal{O}_X(U_i)$ , then the new cocycle  $f'_{ij}$  will differ from  $f_{ij}$  by the coboundary  $\phi'_i \phi_i^{-1}$ . Thus we get a cohomology class  $[f_{ij}] \in H^1(X, \mathcal{O}_X^*)$  which depends only on  $\mathcal{M}$ .

**Theorem 2.1.** *There is a bijection between the set of isomorphism classes of line bundles and  $H^1(X, \mathcal{O}_X^*)$ .*

We can utilize this with the help of the exact sequence

$$0 \rightarrow \mathbb{Z}_X \rightarrow \mathcal{O}_X \xrightarrow{e^{2\pi i}} \mathcal{O}_X^* \rightarrow 1$$

to obtain an exact sequence

$$H^1(X, \mathbb{Z}_X) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X^*)$$

Our first task is to understand  $H^1(X, \mathbb{Z}_X)$ . We have an exact sequence of constant sheaves

$$0 \rightarrow \mathbb{Z}_X \rightarrow \mathbb{R}_X \rightarrow U(1)_X \rightarrow 0$$

which gives rise to an exact sequence

$$H^0(X, \mathbb{R}_X) \rightarrow H^0(X, U(1)_X) \rightarrow H^1(X, \mathbb{Z}_X) \rightarrow H^1(X, \mathbb{R}_X)$$

Since  $X$  is connected, the first map can be identified with the surjection  $\mathbb{R} \rightarrow U(1)$ . Therefore  $H^1(X, \mathbb{Z}_X) \rightarrow H^1(X, \mathbb{R}_X)$  is injective. To analyze the last group, we use the exact sequence

$$0 \rightarrow \mathbb{R}_X \rightarrow C_{\mathbb{R}}^{\infty} \xrightarrow{d} \mathcal{E}_{\mathbb{R},cl}^1 \rightarrow 0$$

where  $C_{\mathbb{R}}^{\infty}$  is the sheaf of real valued  $C^{\infty}$  functions, and  $\mathcal{E}_{\mathbb{R},cl}^1$  is sheaf of real valued closed forms. We obtain an exact sequence

$$C_{\mathbb{R}}^{\infty}(X) \rightarrow \mathcal{E}_{\mathbb{R},cl}^1(X) \rightarrow H^1(X, \mathbb{R}_X) \rightarrow H^1(X, C_{\mathbb{R}}^{\infty})$$

The right hand group is zero by a partition of unity argument similar to what we used earlier [A]. Therefore we obtain a version of de Rham's theorem

**Theorem 2.2.**

$$H^1(X, \mathbb{R}_X) \cong \{ \text{Closed real 1-forms} \} / \{ \text{Exact real 1-forms} \}$$

*In particular, by the usual form of de Rham's theorem, we find that this is a real vector space of dimension  $2g$ .*

Now  $H^1(X, \mathbb{Z}_X) \subset H^1(X, \mathbb{R}_X)$  can be identified a subgroup represented by closed 1-forms  $\alpha$  such that the period  $\int_{\gamma} \alpha \in \mathbb{Z}$  for all closed loops  $\gamma \subset X$ . It suffices to check that  $\int_{\gamma_i} \alpha \in \mathbb{Z}$  where  $\gamma_1, \dots, \gamma_{2g}$  is a homology basis. It follows that  $H^1(X, \mathbb{Z}_X)$  is a lattice inside  $H^1(X, \mathbb{R}_X)$ . Also we have isomorphism

$$H^1(X, \mathbb{Z}_X) \cong H^1(X, \mathbb{Z})$$

$$H^1(X, \mathbb{R}_X) \cong H^1(X, \mathbb{R})$$

with singular cohomology, so we won't distinguish these groups below.

**Theorem 2.3.**  *$H^1(X, \mathbb{Z})$  is a lattice in  $H^1(X, \mathcal{O}_X)$ . Therefore the quotient is a  $g$  dimensional complex torus.*

*Proof.* We have just seen that  $H^1(X, \mathbb{Z})$  is a lattice in  $H^1(X, \mathbb{R})$ . We just have to construct an isomorphism  $\pi : H^1(X, \mathbb{R}) \cong H^1(X, \mathcal{O}_X)$  compatible with the maps from  $H^1(X, \mathbb{Z})$ . The map  $\pi$  is simply the one induced by the map  $\mathbb{R}_X \rightarrow \mathcal{O}_X$  of sheaves. We use the isomorphisms

$$H^1(X, \mathbb{R}) \cong \{\text{harmonic real 1-forms}\}$$

$$H^1(X, \mathcal{O}_X) \cong \{\text{antiholomorphic 1-forms}\}$$

made earlier. With these identifications, given  $\alpha \in H^1(X, \mathbb{R})$ ,  $\pi(\alpha)$  is the  $(0, 1)$  part  $\pi(\alpha) = \alpha^{(0,1)} \in H^1(X, \mathcal{O}_X)$ . The inverse sends  $\beta \in H^1(X, \mathcal{O}_X)$  to  $\beta + \bar{\beta} \in H^1(X, \mathbb{R})$ .  $\square$

### 3. FIRST CHERN CLASS

So far we have shown that the image of  $H^1(X, \mathcal{O}_X)$  in  $H^1(X, \mathcal{O}_X^*)$  is a torus, but we still have to show that this coincides with  $C^0(X)$ . Given class in  $f \in H^1(X, \mathcal{O}_X^*)$ , corresponding to a line bundle  $\mathcal{M}$ , we need to understand when lies in the image of  $H^1(X, \mathcal{O}_X)$ . We may represent it by a cocycle  $f_{ij}$  on an open cover  $\{U_i\}$ , so that any nonempty intersection  $U_{ij}$  is simply connected. Then we choose logarithms  $\tilde{f}_{ij} \in \mathcal{O}(U_{ij})$  so that  $\exp(2\pi\sqrt{-1}\tilde{f}_{ij}) = f_{ij}$ . If this forms a 1-cocycle, then we are done. In fact, we need a weaker statement we get a 1-cocycle after modifying the choice  $\tilde{f}_{ij} \rightarrow \tilde{f}_{ij} + \gamma_{ij}$ , where  $\gamma_{ij} \in \mathbb{Z}$ . We reformulate this as saying that

$$F_{ijk} := \tilde{f}_{ij} + \tilde{f}_{jk} + \tilde{f}_{ki}$$

can be rewritten as

$$-(\gamma_{ij} + \gamma_{jk} + \gamma_{ki})$$

We note that this collection  $F_{ijk}$  is integer valued satisfies the 2-cocycle identity

$$F_{ijk} - F_{ijl} + F_{ikl} - F_{jkl} = 0$$

The last condition to asking that  $F_{ijk}$  be the coboundary of the 1-cochain  $-\gamma_{ij}$ . So now following the previous pattern, we define

$$H^2(X, \mathbb{Z}_X) = \varinjlim \frac{\{\mathbb{Z}\text{-valued 2-cocycles}\}}{\{\mathbb{Z}\text{-valued 2-coboundaries}\}}$$

where the limit is taken over set of all open covers ordered by refinement. The class associated to the cocycle  $F_{ijk}$  is called the first Chern class of  $\mathcal{M}$ . It is denoted by  $c_1(\mathcal{M})$ . Thus we have proved that

**Proposition 3.1.**  $\mathcal{M}$  lies in the image of  $H^1(X, \mathcal{O}_X)$  if and only if  $c_1(\mathcal{M}) = 0$ .

We give a more concrete interpretation of  $c_1$ . We can map it to  $H^2(X, \mathbb{C})$  under the embedding  $\mathbb{Z}_X \rightarrow \mathbb{C}_X$ , and then we describe this as a differential form. We have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}_X & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X^* \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow (1/2\pi i)d\log \\ 0 & \longrightarrow & \mathbb{C}_X & \longrightarrow & \mathcal{O}_X & \xrightarrow{d} & \Omega_X^1 \longrightarrow 0 \end{array}$$

with exact rows. The symbol  $d\log f = df/f$ . Using this diagram, we see that the image of  $c_1(\{f_{ij}\})$  in  $H^2(X, \mathbb{C})$  is the coboundary of  $\frac{1}{2\pi\sqrt{-1}}d\log(f_{ij})$ . To compute

this, note that using a trick used earlier, if we view this as a  $C^\infty$ -form then it is a coboundary, That is

$$\frac{1}{2\pi\sqrt{-1}}d\log f_{ij} = \alpha_i - \alpha_j$$

for  $C^\infty$  1-forms  $\alpha_i$ . Differentiating gives a globally defined 2-form  $\beta = d\alpha_i$ . This represents  $c_1(\{f_{ij}\})$ . We can integrate this to get a number  $\int_X \beta$ . We have an isomorphism  $H^2(X, \mathbb{C}) \cong \mathbb{C}$ , so we loose nothing by treating  $c_1$  as a number.

**Proposition 3.2.**  $c_1(\mathcal{O}(D)) = \pm \deg D$ .

With the correct normalization, it should be  $+$ . However, the sign will be somewhat irrelevant for us.

*Proof.* Let  $D = n_1p_1 + \dots + n_kp_k$ . Let  $U_0 = X - \{p_1, \dots, p_k\}$ , and let  $U_i$  be a coordinate disk around  $p_i$  with coordinate  $z_i$ . We can choose  $U_i, i > 0$  small enough so that they are disjoint. Then we can represent  $\mathcal{O}(D)$  by the cocycle  $f_{0i} = z_i^{\pm n_i}$ . Let  $\beta$  be as above. Then

$$c_1(\mathcal{O}(D)) = \int_X \beta = \int_{X-U_1-\dots-U_k} \beta + \sum_1^k \int_{U_i} \beta$$

By Stokes' theorem this equals

$$\sum_1^k \int_{\partial U_i} (\alpha_i - \alpha_0) = \sum \frac{1}{2\pi\sqrt{-1}} \int_{\partial U_i} d\log f_{0i} = \pm \sum n_i$$

□

These two propositions together with the earlier results complete the proof of the main theorem.

#### 4. ABEL-JACOBI MAP

Fix a base point  $p_0 \in X$ . We define a map, called the Abel-Jacobi map, from  $\alpha : X \rightarrow J(X)$  given by  $x \mapsto x - p_0$ .

**Proposition 4.1.** *The map  $\alpha$  is holomorphic.*

Before proving this, it will be convenient to give an alternative description of  $J(X)$ . We can identify

$$H^1(X, \mathcal{O}_X) \cong H^1(X, \Omega_X^1)^*$$

by sending  $\alpha$  (viewed as an antiholomorphic form) to the functional  $\beta \mapsto \int_X \beta \wedge \alpha$ . To see that this is an isomorphism, observe that both spaces are  $g$ -dimensional, and the kernel of the map is trivial because  $\int_X \beta \wedge \bar{\beta} \neq 0$  if  $\beta \neq 0$ . Under this map the image of  $H^1(X, \mathbb{Z}) \subset H^1(X, \mathcal{O}_X)$  is sent to the image of  $H_1(X, \mathbb{Z}) \subset H^1(X, \Omega_X^1)^*$  via  $\gamma \mapsto \int_\gamma -$ . Thus

$$J(X) = \frac{H^1(X, \Omega_X^1)^*}{H_1(X, \mathbb{Z})}$$

We can make it even more explicit by choosing a basis  $\omega_1, \dots, \omega_g \in H^0(X, \Omega_X^1)$ . The dual of the first basis identifies  $H^0(X, \Omega_X^1)^* \cong \mathbb{C}^g$ . Under this the  $H_1(X, \mathbb{Z})$  corresponds to the lattice generated by periods

$$P = \left\{ \left( \int_\gamma \omega_1, \dots, \int_\gamma \omega_g \right) \mid \gamma \in H_1(X, \mathbb{Z}) \right\}$$

Then

$$J(X) \cong \mathbb{C}^g / P$$

The map  $\alpha$  then takes the explicit form

$$x \mapsto \left( \int_{p_0}^x \omega_1, \dots, \int_{p_0}^x \omega_g \right) \pmod{P}$$

Locally around a fixed  $x$ , we can write  $\omega_i = f_i(z)dz$ . Then  $\alpha$  takes the form

$$x \mapsto \left( \int_{p_0}^x f_1(z)dz + C, \dots, \int_{p_0}^x f_g(z)dz + C \right) \pmod{P}$$

and holomorphicity is obvious.

The  $n$ th symmetric product is the quotient

$$S^n X = \underbrace{X \times X \dots X}_n / S_n$$

If  $z_1, \dots, z_n$  are local coordinates of  $X^n$ , then the elementary symmetric functions  $\sigma_i(z_1, \dots, z_n)$  give coordinates on  $S^n X$ . The upshot is that  $S^n X$  is in fact a complex manifold. We can think of elements  $S^n X$  as effective divisors of degree  $n$ . We extend the map  $\alpha$  by

$$\alpha_n : S^n X \rightarrow J(X); \quad x_1 + \dots + x_n \mapsto (x_1 - p_0) + \dots + (x_n - p_0)$$

**Theorem 4.2.**

- (1) (*Abel*) If  $D \in S^n X$ , then the fibre  $\alpha_n^{-1}\alpha_n(D)$  containing  $D$  consists of the set of all effective divisors linearly equivalent to  $D$ . (We say that  $D$  is linearly equivalent to  $D'$  if  $D - D'$  is principal.)
- (2) (*Jacobi*)  $\alpha_g$  is surjective.

*Proof.* In the current set up, Abel's theorem is a tautology (although it wasn't always so!). For Jacobi, we use the following nontrivial fact from several complex variables: for a holomorphic map between compact complex manifolds to be surjective, it suffices that the derivative of the map is surjective at one point. To apply this, it is enough to calculate the derivative of the map  $X^g \rightarrow J(X)$  given by composing the projection  $X^g \rightarrow S^g X$  with  $\alpha_g$ . Let  $\vec{p} = (p_1, \dots, p_g) \in X^g$ . We choose coordinate  $z_i$  about each  $p_i$ , and represent  $\omega_i = f_{ij}(z_j)dz_j$  locally. Since the map is given by  $\vec{p} \mapsto (\int_{p_0}^{p_i} \omega_j)$ , the derivative at this point is represented by the matrix

$$M = \begin{pmatrix} f_{11}(0) & f_{21}(0) & \dots \\ f_{21}(0) & f_{22}(0) & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

Now choose  $p_1$  so that  $\omega_1(p_1) \neq 0$ . After replacing  $\omega_2$  etc. by  $\omega_2 \dots$  minus a multiple of  $\omega_1$ , we can assume that  $\omega_2(p_1) = \dots = 0$ . Continuing in this way, we find  $M$  is triangular with nonzero diagonal entries. Therefore the result is proved.  $\square$

**Corollary 4.3.** *If  $X$  is an elliptic curve i.e.  $g = 1$ , then  $X = J(X)$ . In particular,  $X$  has a natural group structure after choosing a base point  $p_0$*

*Proof.* The map  $\alpha : X \rightarrow J(X)$  is holomorphic and surjective. Given  $p \in X$ , we know that  $L(p) = \mathbb{C}$  which says that  $p$  is the only effective divisor linearly equivalent to  $p$ . Therefore  $\alpha$  has degree 1, and consequently it is an isomorphism.  $\square$

**Corollary 4.4.**  $J(X)$  and  $S^g X$  are bimeromorphically equivalent i.e. there exists a meromorphic map from one to the other which admits a meromorphic inverse.

This should be understood as saying that  $J(X)$  and  $S^g X$  are almost isomorphic. Weil [W] took this as the starting point for his algebraic construction of  $J(X)$ . (Also see [M] for the details in modern language.) The key point is  $S^g X$  is an algebraic variety, i.e. it is defined by polynomials, when  $X$  is a nonsingular projective algebraic curve. We also need to interpret the group law of  $J(X)$  on the other side. Given divisors  $D, D' \in S^g X$ , Riemann-Roch shows  $D + D' - gp_0$  is linearly equivalent to an effective divisor  $D''$ . A more delicate argument shows this  $D''$  is unique for generic choices of  $D, D'$ . Thus we have a partial group law  $\mu : S^g X \times S^g X \dashrightarrow S^g X$  given by  $\mu(D, D') = D''$ . Weil showed that any such partial group can be completed to an algebraic variety where the law is defined everywhere. When applied to  $(S^g X, \mu)$ , we obtain  $J(X)$ .

## 5. JACOBIANS OF FERMAT CURVES

Fix an integer  $n > 2$  and let  $X \subset \mathbb{C}P^2$  be the degree  $n$  Fermat curve  $x^n + y^n = z^n$ .

**Lemma 5.1.** *The genus of  $X$  is  $g = \frac{(n-1)(n-2)}{2}$ .*

*Proof.* This follows directly from a standard formula for genus of a plane curve, which we haven't discussed. So we use the Riemann-Hurwitz formula instead. The affine curve  $x^n + y^n = 1$  projects to the  $x$ -axis. This extends to a degree  $n$  map  $\pi : X \rightarrow \mathbb{C}P^1$ , where the points satisfying  $x^n + y^n = z^n; z = 0$  map to  $\infty$ . The branch points are the  $n$ th roots of unity and all  $n$  sheets come together over these points, i.e.  $e_p = n - 1$ . Therefore

$$2g - 2 = -2n + \sum (e_p - 1) = -2n + n(n - 1)$$

and the lemma follows.  $\square$

The rest of the story is due to Gross and Rohrlich [G]. For integers  $r, s$ , let

$$\eta_{r,s} = \frac{x^{r-1}y^{s-1}}{y^{n-1}} dx$$

where the expression is with respect the affine coordinates  $x, y$ .

**Lemma 5.2.** *The collection  $\eta_{r,s}$  with  $r, s \geq 1$ ,  $r + s \leq n - 1$  gives a basis for  $H^0(X, \Omega_X^1)$ .*

*Proof.* On the affine part of  $X$ ,  $\eta_{r,s}$  is certainly holomorphic away from  $y = 0$ . Differentiating the equation  $x^n + y^n = 1$ , gives the relation  $x^{n-1}dx + y^{n-1}dy = 0$ . Therefore

$$\eta_{r,s} = -\frac{x^{r-1}y^{s-1}}{x^{n-1}} dy$$

which shows that it is holomorphic at  $y = 0$  as well. Let  $\xi = 1/x$ . This is the coordinate around  $\infty$  on  $\mathbb{P}^1$ .  $\pi : X \rightarrow \mathbb{C}P^1$  is unramified over  $\infty$ . Therefore  $\xi$  can be lifted to a coordinate around any point  $p \in X$  lying over  $\infty$ . We have

$$\eta_{r,s} = x^{r-1}y^{s-n} dx = -x^{r-1}y^{s-n}\xi^{-2} d\xi$$

The valuation  $ord_p(x) = -1$ . We have  $y^n = 1 - x^n = \xi^{-n}u$  where  $u = -(1 - \xi^n)$  is a unit. Therefore  $ord_p(y) = -1$ . So  $ord_p(\eta_{r,s}) = -(r - 1) - (s - n) - 2 = n - 1 - (r + s) \geq 0$ . Thus  $\eta_{r,s}$  is holomorphic everywhere.

The forms  $\eta_{rs}$  are easily seen to be linearly independent, and there are exactly  $1 + 2 + \dots + (n - 2) = g$  of them.  $\square$

The Jacobian is a quotient of  $H^0(X, \Omega_X^1)^*$  by the period lattice. We can use the dual basis  $\eta_{rs}^*$  for the numerator. We need generators for homology. Let  $\zeta$  be a primitive  $n$ th root of unity. Consider the map  $X \rightarrow \mathbb{C}P^1$  (different from  $\pi$ ) given by sending  $(x, y)$  in the affine part to  $x^d$ . The branch points are 0 and 1. Let  $\gamma_0 \in \pi_1(\mathbb{C} - \{0, 1\})$  be a loop around 0, and  $\gamma_1$  be a loop around 1. Then the commutator  $\gamma = \gamma_0 \gamma_1 \gamma_0^{-1} \gamma_1^{-1}$  lifts to a loop in  $X$ . The orbit of  $\gamma$  under the action  $(x, y) \mapsto (\zeta^j x, \zeta^j y)$  gives a generating set for  $H_1(X, \mathbb{Z})$  (it's not a basis because there more than  $2g$  elements, but it won't matter). It remains to work out the integrals of  $\eta_{rs}$  along these loops. This can be done on the complex plane, and it reduces to an expression involving the Euler beta function

$$B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt$$

In summary

**Proposition 5.3.** *The period lattice is spanned by the vectors*

$$(\dots, \zeta^{rj+sk} (1 - \zeta^r)(1 - \zeta^s) \frac{1}{n} B\left(\frac{r}{n}, \frac{s}{n}\right), \dots) \in \mathbb{C}^g$$

for  $j, k \in \{0, \dots, n - 1\}$ .

## 6. GALOIS REPRESENTATIONS ASSOCIATED TO CURVES

A central problem in number theory is to understand the Galois group of  $\bar{\mathbb{Q}}$  over  $\mathbb{Q}$ . Generally it is studied through its representations, but this begs the question *where do natural representations of  $G = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  come from?* A natural source comes from algebraic geometry and in particular algebraic curves. Let  $f(x, y) \in \mathbb{Q}[x, y]$  be a rational polynomial such that the associated projective algebraic plane curve  $X$  is nonsingular. For example,  $X$  could be a Fermat curve. Given a subfield  $K \subset \mathbb{C}$ , let  $X(K)$  the set of solutions to the homogenized equation  $F(x, y, z) = 0$  in  $\mathbb{P}^2$  with coordinates in  $K$ . Then  $G$  will act on the set  $X(\bar{\mathbb{Q}})$ . In order to “linearize” this, we pass to divisors. Let  $\text{Div}(X)(\bar{\mathbb{Q}})$  be the group divisors  $\sum n_p p$  with  $p \in X(\bar{\mathbb{Q}})$ . We define  $D \in \text{Div}(X)(\bar{\mathbb{Q}})$  to be principal if  $D = \text{div}(g)$  where  $g$  lies in the field of fractions of  $\bar{\mathbb{Q}}[x, y]/(f)$ . A fact which is not that obvious is that principal divisors in this sense are exactly the elements of the intersection  $\text{Princ}(X) \cap \text{Div}(X)(\bar{\mathbb{Q}})$ . Therefore the quotient  $\text{Div}(X)(\bar{\mathbb{Q}})$  by principal divisors is a subgroup of  $\text{Cl}(X)$ , denoted by  $\text{Cl}(X)(\bar{\mathbb{Q}})$ . Let  $J(X)(\bar{\mathbb{Q}}) \subset J(X)$  denote the degree 0 part. The Galois group  $G$  acts on this, but we want something a bit more “linear”. For any  $n$ , the subgroup of  $n$ -torsion points  $J(X)_n$  is isomorphic to  $(\mathbb{Z}/n\mathbb{Z})^{2g}$ . To see this write  $J(X) = \mathbb{C}^g/P$ , where  $P$  is the period lattice, then

$$J(X)_n = \frac{1}{n}P/P \cong \frac{1}{n}P \otimes \mathbb{Z}/n\mathbb{Z} \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$$

Although, as we saw for the Fermat curve,  $P$  can be very transcendental, this subgroup turns out to be algebraic in nature.

**Theorem 6.1.**  $J(X)_n \subset J(X)(\mathbb{Q})$ .

This is really a corollary of

**Theorem 6.2** (Weil).  $J(X)$  and the group law  $J(X) \times J(X) \rightarrow J(X)$  can be defined by polynomials over  $\bar{\mathbb{Q}}$ . In other words  $J(X)$  is an algebraic group defined over  $\bar{\mathbb{Q}}$ .

*Sketch.* As we saw earlier  $J(X)$  can be reconstructed from  $S^g X$  and the partial group law on it. All of this is defined over  $\bar{\mathbb{Q}}$  (and even over  $\mathbb{Q}$  if  $X(\mathbb{Q}) \neq \emptyset$ ).  $\square$

With the first theorem in hand, we get a representation of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_{2g}(\mathbb{Z}/n\mathbb{Z})$  for each  $n$ . These representations are extremely important, both as a source of Galois representations and for the arithmetic study of  $X$ .

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