V-filtrations and vanishing cycles for $\mathcal{D}_X$-modules

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Let $\mathcal{A}$ be an abelian category of "sheaf-like" objects on a smooth complex variety X.

For example, $\mathcal{A}$ could be $\mathrm{Sh}(X)$, coherent $\mathcal{D}_X$-modules, the category of perverse sheaves on X, or the category of Hodge modules on X (which we haven't defined yet).

Definition

Let $Z \subset X$ be an irreducible closed subset. An object $M \in \mathcal{A}$ has *strict support* Z if $\mathrm{supp}(M) = Z$ and M has no sub- or quotient objects with support properly contained in Z.

Example

The $\mathcal{D}_X$-module $\mathcal{O}_X$ has strict support X (it is simple and $\operatorname{supp}(\mathcal{O}_X) = X$). Note that $\mathcal{O}_X$ typically does not have strict support X as an $\mathcal{O}_X$-module.

Example

Consider $M = \mathbb{C}[t^{\pm 1}]$ as a $\mathbb{C}[t, \partial_t]$-module. We have $\operatorname{supp}(M) = X$, yet M has $\mathbb{C}[t]$ as a subobject, and the quotient

$$\mathbb{C}[t^{\pm 1}]/\mathbb{C}[t] = \langle t^{-1}, t^{-2}, t^{-3}, \dots \rangle$$

is supported on $[t = 0]$. So M does not have strict support X.

We want to define the category of (pure) Hodge modules on a smooth complex variety X so that, loosely speaking, the following desiderata are true:

1. If M is a Hodge module, then $M \cong \bigoplus_Z M_Z$, where Z runs over irreducible closed subsets of X, and each M_Z has strict support Z. (we call this a *strict support decomposition*).

2. If M is a Hodge module of strict support Z, then there is a nontrivial open embedding $j : U \hookrightarrow Z$ such that $j^* M$ "is a variation of Hodge structure". Moreover, M is uniquely determined by $j^* M$.

3. In addition, given an algebraic function $f : X \to \mathbb{C}$ we want to define the nearby cycles "$\psi_f(M)$" of M. This object will have a monodromy filtration, whose graded pieces should again be (pure) Hodge modules (of the appropriate weights).

We're about to learn about V-filtrations. Why do this?

In the setting of (possibly filtered) coherent $\mathcal{D}_X$-modules M, V-filtrations

1. are used to define nearby and vanishing cycles for (possibly filtered) D_X-modules.
2. detect the existence of a strict support decomposition
3. can be used to give criteria for when objects are determined by their restriction to an open subset.

The plan:

- Now, we will discuss V-filtrations on (non-filtered) $\mathcal{D}_X$-modules.
- Later, we will move to the setting of modules with a good filtration $F^\bullet$.

References:

- Saito, "Modules de Hodge Polarisables", section 3.
- Popa, "Lecture notes on the V-filtration".

Setting

- X is an affine (for simplicity) smooth complex variety.
- $t : X \to \mathbb{C}$ is an algebraic function, such that $D \stackrel{i}{\hookrightarrow} X$, the vanishing locus of t, is smooth.
- $U \stackrel{j}{\hookrightarrow} X$ is the complement of D in X.
- M is a coherent (left) $\mathcal{D}_X$-module. In general, we will say "$\mathcal{D}_X$-module" when we mean "coherent (left) $\mathcal{D}_X$-module".

Definition of V-filtration

Definition

A (rational) V-filtration of M along t is a decreasing, exhaustive filtration $V^\bullet$ of M by coherent $\mathcal{D}_X[t, \partial_t t]$-submodules, indexed by the ordered group $\mathbb{Q}$; it must satisfy the following conditions:

- Discreteness
- $t \cdot V^\alpha M \subset V^{\alpha+1} M$, with equality if $\alpha > 0$.
- $\partial_t \cdot V^\alpha M \subset V^{\alpha-1} M$
- Let $V^{>\alpha} M = \bigcup_{\alpha' > \alpha} V^{\alpha'} M$. The action of $\partial_t t - \alpha$ on
$$\mathrm{gr}^\alpha M = V^\alpha M / V^{>\alpha} M$$
is nilpotent.

Remark: Each $\mathrm{gr}^\alpha M$ is naturally a coherent $\mathcal{D}_X$-module, supported on D.

Remark on conventions: Given a V-filtration $V^\bullet$ on M, one obtains an *increasing* filtration $V_\bullet$ by setting

$$V_\alpha = V^{-\alpha}$$

This filtration satisfies e.g. $t \cdot V_\alpha \subset V_{\alpha-1}$, and the action of

$$\partial_t t + \alpha = t\partial_t + \alpha + 1$$

on $V^\alpha / V^{<\alpha}$ is nilpotent. It is common for "V-filtration" to refer to this kind of increasing filtration.

Example

Take $M = \mathcal{O}_X$. Define

$$V^\alpha \mathcal{O}_X = \begin{cases} t^{\lceil \alpha \rceil - 1}\mathcal{O}_X, & \text{if } \alpha > 0 \\ \mathcal{O}_X, & \text{otherwise} \end{cases}$$

So this is the filtration

$$\cdots = [\mathcal{O}_X]^0 = [\mathcal{O}_X]^1 \supset [t\mathcal{O}_X]^2 \supset [t^2\mathcal{O}_X]^3 \supset \ldots$$

where $[-]^\alpha$ designates V^α, and we omit the non-integer steps. One easily checks that this is a V-filtration; the key point is that

$$\partial_t t(t^k) = (k+1)t^k$$

Interesting graded pieces: $\mathrm{gr}_V^i = \langle [t^{i-1}] \rangle$, $i \geq 0$.

Example

Now take $M = j_* j^* \mathcal{O}_X$, i.e. $\mathcal{O}_U$ regarded as a $\mathcal{D}_X$-module. Define $V^\alpha \mathcal{O}_U$ to be the $\mathcal{O}_X$-submodule generated by $t^{\lceil \alpha \rceil - 1}$.

So this is the filtration

$$\cdots \supset [(t^{-2}, t^{-1}, 1, t, \dots)]^{-1} \supset [(t^{-1}, 1, t, \dots)]^0 \supset [(1, t, \dots)]^1 \supset \dots$$

where $[-]^\alpha$ designates V^α. This is a V-filtration.

Interesting graded pieces: $\mathrm{gr}_V^i = \langle [t^{i-1}] \rangle$, $i \in \mathbb{Z}$.

Example

Take $X = \mathrm{Spec}(\mathbb{C}[t])$, and let $M = \mathbb{C}\langle t^{\frac{k}{2}} : k \in \mathbb{Z}\rangle$, with t, ∂_t acting in the way you'd expect. Define $V^\alpha M$ to be the $\mathcal{O}_X$-submodule generated by $t^{\lceil \alpha - 1/2 \rceil - 1/2}$.

So this is the filtration

$$\cdots \supset [(t^{-\frac{3}{2}}, t^{-\frac{1}{2}}, t^{\frac{1}{2}}, \ldots)]^{-\frac{1}{2}} \supset [t^{-\frac{1}{2}}, t^{\frac{1}{2}}, \ldots)]^{\frac{1}{2}} \supset [t^{\frac{1}{2}}, t^{\frac{3}{2}}, \ldots)]^{\frac{3}{2}} \supset \ldots$$

where $[-]^\alpha$ designates V^α. This is a V-filtration.

Interesting graded pieces:
$\mathrm{gr}^{-\frac{1}{2}}(M) = \langle [t^{-\frac{3}{2}}] \rangle$, $\mathrm{gr}^{\frac{1}{2}}(M) = \langle [t^{-\frac{1}{2}}] \rangle$, $\ldots$.

Example

Fix a $\beta \in \mathbb{Q}$. Let $M_{\beta,p}$ be the $\mathcal{D}_X$-module generated by expressions of the form

$$e_{j,k} := \begin{cases} \frac{t^{\beta+j}\log^k(t)}{k!}, & \text{if } 0 \le k \le p \\ 0, & \text{otherwise} \end{cases}$$

where $j, k \in \mathbb{Z}$, and t, ∂_t act in the way you'd expect. Then

$$\begin{aligned} \partial_t t \cdot e_{j,k} =& \frac{1}{k!}[(\beta + j + 1)t^{\beta+j}\log^k(t) + kt^{\beta+j}\log^{k-1}(t)] \\ =& (\beta + j + 1)e_{j,k} + e_{j,k-1} \end{aligned}$$

implying that each $e_{j,k}$ is annihilated by a power of $\partial_t t - \beta - j - 1$. There is a V-filtration such that

$$\mathrm{gr}_V^{\beta+j+1} M_{\beta,p} = \langle [e_{j,0}], \ldots, [e_{j,p}] \rangle$$

Example

Take $X = \mathrm{Spec}(\mathbb{C}[t])$, and $M = \mathbb{C}[t, \partial_t]/\mathbb{C}[t, \partial_t]t = \mathbb{C}[\partial_t]$.

Define $V^\alpha M = \{\partial_t^k : 0 \le k \le -\lceil \alpha \rceil\}$.

So this is the filtration

$$\cdots \supset [(\partial_t^2, \partial_t, 1)]^{-2} \supset [(\partial_t, 1)]^{-1} \supset [(1)]^0 \supset [0]^1 = \dots$$

where $[-]^\alpha$ designates V^α. This is a V-filtration.

Interesting graded pieces: $\mathrm{gr}_V^i = \langle [\partial_t^{-i}] \rangle$, $i < 0$. (Notice e.g. that

$$\partial_t t[\partial_t] = [\partial_t t \partial_t] = [\partial_t(\partial_t t - 1)] = [-\partial_t])$$

Example

Generalizing the previous example, suppose that $\mathrm{supp}(M) \subset D = [t = 0]$. Recall that Kashiwara's equivalence gave us an isomorphism

$$\phi : M \xrightarrow{\sim} \bigoplus_{n \leq 0} M^n = M^0 \otimes \mathbb{C}[\partial_t]$$

where $M^n = \ker(\partial_t t - n) = \partial_t^{-n} M^0$. Define the V-filtration:

$$V^\alpha M = \phi^{-1}\Big(\bigoplus_{n \geq \lceil \alpha \rceil} M^n \Big)$$

Key facts: Note that $V^{>0} M = 0$; conversely, one easily checks that $\mathrm{supp}(M) \subset D$ if $V^{>0} M = 0$. Also note that M is determined by $V^0 M = M^0$.

Lemma

There is at most one V-filtration (with respect to t) on M.

Corollary

Let $\phi : M \to N$ be a morphism of $\mathcal{D}_X$-modules equipped with V-filtrations along t. Then ϕ is strictly compatible with these filtrations, namely,

$$\phi(V^\alpha M) = \phi(M) \cap V^\alpha N$$

Proof (Cor.): One immediately checks that both sides of the equation define a V-filtration on $\mathrm{im}(\phi)$. □

Corollary

For each $\alpha \in \mathbb{Q}$, the following functors are exact:

- $M \mapsto V^\alpha M$
- $M \mapsto \mathrm{gr}^\alpha M$

The following result is a prototype of desideratum 2 from the introduction:

Proposition 1

Let M,N be coherent $\mathcal{D}_X$-modules equipped with V-filtrations along t. Assume that M,N have strict support X. Then any isomorphism $\phi_U : j^*M \xrightarrow{\sim} j^*N$ extends to an isomorphism $\phi : M \to N$.

Before proving this, we give two lemmas that are useful more broadly.

Lemma A

Let $M' \subset M$ be an inclusion of $\mathcal{D}_X$-modules equipped with V-filtrations along t. Assume that $j^*M' \to j^*M$ is an isomorphism. Then for all $\alpha > 0$,

$$V^\alpha M' = V^\alpha M$$

Proof: The previous corollary gives an an exact sequence

$$0 \to V^\alpha M' \to V^\alpha M \to V^\alpha(M/M') \to 0$$

But M/M' is supported on D, and we have seen that this implies that

$$V^\alpha(M/M') = 0$$

when $\alpha > 0$. □

Lemma B

Let M be a $\mathcal{D}_X$-module equipped with V-filtration along t. Assume that M has strict support X. Then

$$M = \mathcal{D}_X \cdot V^{>0} M$$

Proof: The quotient satisfies

$$V^{>0}\Big(M/(\mathcal{D}_X \cdot V^{>0} M)\Big) = 0$$

implying that it is supported within D. □

Now we return to prove the proposition:

Proposition 1

Let M,N be coherent $\mathcal{D}_X$-modules equipped with V-filtrations along t. Assume that M,N have strict support X. Then any isomorphism $\phi_U : j^*M \xrightarrow{\sim} j^*N$ extends to an isomorphism $\phi : M \to N$.

Proof: Consider the composite morphism

$$\phi : M \hookrightarrow j_*j^*M \xrightarrow{\sim} j_*j^*N$$

where the second arrow is induced by ϕ_U. The first arrow is injective because its kernel is supported on D, and yet M has strict support X. We claim that $\mathrm{im}(\phi) = N$. Indeed, by our lemmas,

$$\mathrm{im}(\phi) = \mathcal{D}_X \cdot V^{>0}\mathrm{im}(\phi) = \mathcal{D}_X \cdot V^{>0}N = N$$

(the middle equality uses that $j^*\mathrm{im}(\phi) = j^*N$). □

Now we'd like to pause and discuss how nearby and vanishing cycles along the hypersurface $[t = 0]$ are defined using V.

Definition

- $\psi_{t,1} M = \mathrm{gr}^1_V M$ (unipotent nearby cycles)
- $\phi_{t,1} M = \mathrm{gr}^0_V M$ (unipotent vanishing cycles)

Caveats: these are only the "unipotent parts" of nearby/vanishing cycles; also, we want to define these objects even when $[t = 0]$ is not smooth; finally, we'd like to know what relation these objects have with the previous notions of nearby/vanishing cycles. We will address all of these issues later. For now, note that we have morphisms

$$\text{can} := \partial_t : \psi_{t,1} M \rightleftarrows \phi_{t,1} M : t =: \text{var}$$

such that can $\circ$ var and var $\circ$ can are nilpotent (using $[\partial_t, t] = 1$).

Remark: It can be shown that the following maps are isomorphisms:

- $t : \mathrm{gr}^{\alpha} M \xrightarrow{\sim} \mathrm{gr}^{\alpha+1} M$, if $\alpha \neq 0$;
- $\partial_t : \mathrm{gr}^{\alpha} M \xrightarrow{\sim} \mathrm{gr}^{\alpha-1} M$, if $\alpha \neq 1$.

can $:= \partial_t : \psi_{t,1} M \rightleftarrows \phi_{t,1} M : t =:$ var

The following result is an important step towards Desideratum 1 from the introduction.

Proposition 2

Let $M' = \mathcal{D}_X \cdot V^{>0} M \subset M$. Let $\mathcal{H}^0_D M \subset M$ be the subobject generated by sections supported within D. Then:

1. M' is the smallest subobject of M satisfying $j^* M' \cong j^* M$.
2. $M/M' \cong \int_i^0 \mathrm{coker}(\mathrm{can}) = i_+ \mathrm{coker}(\mathrm{can})$, and
3. $\mathcal{H}^0_D M \cong \int_i^0 \ker(\mathrm{var}) = i_+ \ker(\mathrm{var})$.

Proof (1): If $M'' \subset M$ satisfies $j^* M'' = j^* M$, then by Lemma A, $V^{>0} M = V^{>0} M''$, implying

$$\mathcal{D}_X \cdot V^{>0} M \subset \mathcal{D}_X \cdot V^{>0} M'' \subset M''$$

can $:= \partial_t : \psi_{t,1} M \rightleftarrows \phi_{t,1} M : t =:$ var

We will indicate a proof of the third statement:

Goal
$\mathcal{H}^0_D M \cong \int_i^0 \mathrm{ker(var)} = i_+ \mathrm{ker(var)}$

Proof (3): We claim that the obvious map $V^0 M \to \mathrm{gr}^0_V M$ induces an isomorphism

$$\ker(t : M \to M) \xrightarrow{\sim} \ker(t : \mathrm{gr}^0_V M \to \mathrm{gr}^1_V M)$$

Claim

$\ker(t : M \to M) \xrightarrow{\sim} \ker(t : \mathrm{gr}_V^0 M \to \mathrm{gr}_V^1 M)$

Proof (3,cont.): First we need to see that

$$\ker(t : M \to M) \subset V^0 M$$

If $tm = 0$ and $m \in V^\alpha M$ for $\alpha < 0$, we have that

$$(-\alpha)^p m = (\partial_t t - \alpha)^p m \in V^{>\alpha} M$$

for some $p > 0$. Repeating this process, and using the discreteness of V, we obtain $m \in V^0 M$.

Claim

$\ker(t : M \to M) \xrightarrow{\sim} \ker(t : \mathrm{gr}_V^0 M \to \mathrm{gr}_V^1 M)$

Proof (3,cont.): Next we need to see that our map is injective. This follows from the equivalence

$$\mathrm{supp}(N) \subset D \iff V^\alpha N = 0 \text{ for all } \alpha > 0$$

applied to $N = \mathcal{D}_X \cdot \ker(t : M \to M)$.

Claim

$\ker(t : M \to M) \xrightarrow{\sim} \ker(t : \mathrm{gr}_V^0 M \to \mathrm{gr}_V^1 M)$

Proof (3,cont.): Finally, we need to see that our map is surjective. There is a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V^{>0}M & \longrightarrow & V^0 M & \longrightarrow & \mathrm{gr}_V^0 M & \longrightarrow & 0 \\ & & \downarrow t & & \downarrow t & & \downarrow t & & \\ 0 & \longrightarrow & V^{>1}M & \longrightarrow & V^1 M & \longrightarrow & \mathrm{gr}_V^1 M & \longrightarrow & 0 \end{array}$$

Part of the definition of V-filtration was that $t : V^{>0}M \to V^{>1}M$ is surjective. Using the snake lemma, the claim is proved.

Recall that we are trying to prove that

$$\mathcal{H}^0_D M \cong \int_i^0 \mathrm{ker}(\mathrm{var}) = i_+ \mathrm{ker}(\mathrm{var})$$

What we know is that

$$\mathrm{ker}(t : M \to M) \xrightarrow{\sim} \mathrm{ker}(t : \mathrm{gr}^0_V M \to \mathrm{gr}^1_V M)$$

(this is a morphism of $\mathcal{D}_D$-modules). Under $\int_i^0$, the left side becomes $\mathcal{H}^0_D M$ by Kashiwara's theorem; the right side is $\int_i^0 \mathrm{ker}(\mathrm{var})$. Part two of the proposition is proved. □

In the discussion so far, the hypersurface $D = [t = 0]$ has been smooth. We would like to start making claims about arbitrary hypersurfaces, using the tools developed so far.

To this end, suppose we have a function $f : X \to \mathbb{C}$ (where $D = [f = 0]$ need not be smooth). Let

$$\iota^f = (\mathrm{id}_X, f) : X \hookrightarrow X \times \mathbb{C}$$

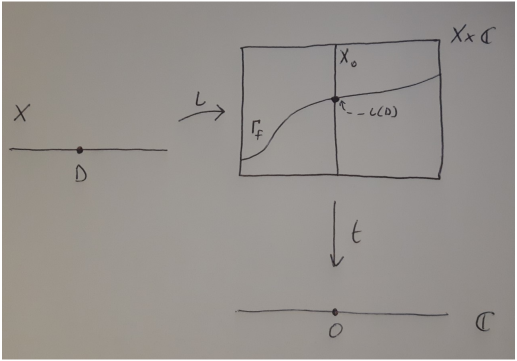
be the graph morphism, and let $t : X \times \mathbb{C} \to \mathbb{C}$ be the projecion. Since $X \cong X_0 := [t = 0]$ is smooth, given a $\mathcal{D}_X$ module M, we can consider V-filtrations along $[t = 0]$ for the $\mathbb{D}_{X\times\mathbb{C}}$-module

$$\int_{\iota^f}^0 M = \iota_+^f M$$

If $\Gamma_f = \iota^f(X) \subset X \times \mathbb{C}$, then $\Gamma_f \cap X_0 = \iota^f(D) \subset X_0$. So using Kashiwara's theorem we have the following:

Key Observation

The functor ι^f_+ induces an equivalence between $\mathcal{D}_D$-modules and $\mathcal{D}_{X\times\mathbb{C}}$-modules supported on $\iota^f(D) = \Gamma_f \cap X_0$.



From now on, for an algebraic function $f : X \to \mathbb{C}$ and a $\mathcal{D}_X$-module M, by

"V-filtration along f for M"

we will mean

"V-filtration along X_0 for $\iota^f_+ M$"

One checks that when $[f = 0]$ is smooth this is compatible with everything we've done so far. Also, we denote

$$M_f := \iota^f_+ M$$

To illustrate the use of the Key Observation, recall a proposition from earlier about modules with a V-filtration along smooth D:

Proposition 2

Let $M' \subset M$ be the smallest subobject satisfying $j^*M' \cong j^*M$. Let $\mathcal{H}^0_D M \subset M$ be the subobject generated by sections supported within D. We have maps

$$\text{can} \; := \partial_t : \text{gr}^1_V M \rightleftarrows \text{gr}^0_V M : t =: \; \text{var}$$

1. $M/M' \cong \int_i^0 \text{coker(can)} = i_+\text{coker(can)}$, and
2. $\mathcal{H}^0_D M \cong \int_i^0 \ker(\text{var}) = i_+\ker(\text{var})$.

We can improve this to the following statement:

Proposition 2 (improved)

Let M be a $\mathcal{D}_X$-module admitting a V-filtration along $D = [f = 0]$ (which may not be smooth). We then have maps

$$\mathrm{can} := \partial_t : \mathrm{gr}_V^1 M_f \rightleftarrows \mathrm{gr}_V^0 M_f : t =: \ \mathrm{var}$$

1. M has no nonzero subobject supported on D iff $\ker(\mathrm{var}) = 0$.
2. M has no nonzero quotient supported on D iff $\mathrm{coker}(\mathrm{can}) = 0$.

(If D is smooth, these are immediate from the old statement.) If D is not smooth and (say) $\ker(\mathrm{var}) = 0$, the old statement says that $\mathcal{H}^0_{X_0} \iota^f_+ M = 0$; Kashiwara implies that $\mathcal{H}^0_D M = 0$. □

We can improve this to the following statement:

Proposition 2 (improved)

Let M be a $\mathcal{D}_X$-module admitting a V-filtration along $D = [f = 0]$ (which may not be smooth). We then have maps

$$\text{can} := \partial_t : \text{gr}_V^1 M_f \rightleftarrows \text{gr}_V^0 M_f : t =: \text{ var}$$

1. M has no nonzero subobject supported on D iff $\ker(\text{var}) = 0$.
2. M has no nonzero quotient supported on D iff $\text{coker}(\text{can}) = 0$.

Remark: We have generally that

$$\text{gr}_V^0(M') = \text{im}(\text{can}_f)$$

$$\text{gr}_V^0(\mathcal{H}^0_{X_0} \iota^f_+ M) = \ker(\text{var}_f)$$

$\mathrm{can}_f := \partial_t : \mathrm{gr}_V^1 M_f \rightleftarrows \mathrm{gr}_V^0 M_f : t =: \mathrm{var}_f$

Now we can characterize modules with strict support (decompositions) using V-filtrations.

Theorem

Let M be a $\mathcal{D}_X$-module admitting a V-filtration along every hypersurface.

1. M has strict support X iff for all f:

$$\ker(\mathrm{var}_f) = \mathrm{coker}(\mathrm{can}_f) = 0$$

2. M has a strict support decomposition iff for all f:

$$\mathrm{gr}_V^0 M_f = \ker(\mathrm{var}_f) \oplus \mathrm{im}(\mathrm{can}_f)$$

$\mathrm{can}_f := \partial_t : \mathrm{gr}_V^1 M_f \rightleftarrows \mathrm{gr}_V^0 M_f : t =: \mathrm{var}_f$

Proof (part 1): Immediate from (improved) Proposition 2.

Proof (part 2): Suppose first that M has a strict support decomposition. Given a $D = [f = 0]$, we want to show that

$$\mathrm{gr}_V^0 M_f \cong \ker(\mathrm{var}_f) \oplus \mathrm{im}(\mathrm{can}_f)$$

We can reduce to the case where M has strict support Z.

- If D does not contain Z, improved Proposition 2 implies that

$$\mathrm{gr}_V^0 M_f = \mathrm{im}(\mathrm{can}_f) \text{ and } \ker(\mathrm{var}_f) = 0$$

- If D contains Z, then

$$\mathrm{gr}_V^1 M_f = 0$$

  implying that $\mathrm{gr}_V^0 M_f = \ker(\mathrm{var}_f)$ and $\mathrm{im}(\mathrm{can}_f) = 0$.

$\mathrm{can}_f := \partial_t : \mathrm{gr}_V^1 M_f \rightleftarrows \mathrm{gr}_V^0 M_f : t =: \mathrm{var}_f$

Proof (part 2, cont.): For the converse, suppose that for all f,

$$\mathrm{gr}_V^0 M_f = \ker(\mathrm{var}_f) \oplus \mathrm{im}(\mathrm{can}_f)$$

Let M' be the minimal subobject of M_f satisfying

$$M'|_{t \neq 0} \cong (M_f)|_{t \neq 0}$$

We claim that $M'' := M' \cap \mathcal{H}^0_{X_0} M_f = 0$. Our assumption, together with Proposition 2, implies that

$$\mathrm{gr}_V^0 M'' \subset \ker(\mathrm{var}_f) \cap \mathrm{im}(\mathrm{can}_f) = 0$$

implying that M'' itself is zero (since $V^{>0} M'' = 0$), proving the claim. Additionally, it is immediate that M' has no quotients supported in X_0.

Proof (part 2, cont.): Now consider the short exact sequence

$$0 \to M' \oplus \mathcal{H}^0_{X_0} M_f \to M_f \to Q \to 0$$

defining Q. We see immediately that $Q|_{t \neq 0} = 0$. Applying gr_V^0, and using Prop. 2, we get

$$0 \to \mathrm{im}(\mathrm{can}_f) \oplus \ker(\mathrm{var}_f) \to \mathrm{gr}_V^0 M_f \to \mathrm{gr}_V^0 Q \to 0$$

implying that also $\mathrm{gr}_V^0 Q = 0$; this implies $Q = 0$.

We have shown that, for any f, we have

$$M_f = M' \oplus \mathcal{H}^0_{X_0} M_f$$

where M' has no sub- or quotient objects supported in X_0 (equivalently, in $\iota^f(D)$).

$\mathrm{can}_f := \partial_t : \mathrm{gr}_V^1 M_f \rightleftarrows \mathrm{gr}_V^0 M_f : t =: \ \mathrm{var}_f$

Proof (part 2, cont.): Now because M is noetherian, there is a divisor $D = [f = 0]$ such that any subobject of M supported on a proper subset of X is supported within D. As above, write

$$M_f = M' \oplus \mathcal{H}^0_{X_0} M_f$$

for this f. Assume for simplicity that $Z := \mathrm{supp}(M)$ is irreducible.

We claim that M' has strict support Z. If M' has a quotient Q supported within D' but not within D, we have a decomposition as above:

$$M' = M'' \oplus \mathcal{H}^0_{X_0} \iota^{f'}_+ M'$$

where M'' has no quotients supported on D'; but $\mathcal{H}^0_{D'} \iota^{f'}_+ M'$ must be zero as it gives a submodule of M supported within D' but not within D. By induction, the proposition is proved. □