V-filtrations and vanishing cycles for $\mathcal{D}_X$-modules, II

Joe

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We now move to the setting of (as always, coherent) $\mathcal{D}_X$-modules equipped with a good (increasing) filtration $F_\bullet$. We call refer to these as "filtered $\mathcal{D}_X$-modules". Here is the key example:

Example

Let $(L, \nabla, F^\bullet)$ be a variation of Hodge structure on X. Recall that ∇ is a flat connection on the vector bundle $E := L \otimes \mathcal{O}_X$, giving it a $\mathcal{D}_X$-module structure. Here F is a decreasing filtration on E, but

$$F_p := F^{-p}$$

is increasing and good. The pair $(E, F_\bullet)$ is a filtered $\mathcal{D}_X$-module.

Every Hodge module will have an underlying filtered $\mathcal{D}_X$-module. But in order for the category of Hodge modules to have the desired properties from the introduction, we have to put conditions on the possible filtrations F.

For example, in the setting of non-filtered $\mathcal{D}_X$-modules we verified that

$$j^* M' \cong j^* M \implies M' \cong M$$

for M, M' strictly supported on Z and a nontrivial open embedding $j : U \to Z$. This fails, however, in the filtered case, as the following example demonstrates.

Example

Let $M = \mathbb{C}[t]$ (as a $\mathbb{C}[t, \partial_t]$-module) with the filtration

$$\cdots = [0]_0 = [t^2\mathbb{C}[t]]_1 \subset [\mathbb{C}[t]]_2 = \dots$$

where $[-]_p$ designates F_p; this is a good filtration. But notice that it induces the same filtration on $M|_{t \neq 0}$ as the good filtration

$$\cdots = [0]_0 = [\mathbb{C}[t]]_1 \subset [\mathbb{C}[t]]_2 = \dots$$

References (for most of this talk)

- Saito, “Modules de Hodge Polarisables”, sections 3.2, 3.4, and 5.1
- Popa, “Lecture notes on the Hodge filtration on D-modules”
- Schnell, ”An overview of Morihiko Saito’s theory of mixed Hodge modules”
- Peters and Steenbrink, “Mixed Hodge Structures”

Setting

- X is an affine (for simplicity) smooth complex variety.
- $t : X \to \mathbb{C}$ is an algebraic function, such that $D \overset{i}{\hookrightarrow} X$, the vanishing locus of t, is smooth.
- $U \overset{j}{\hookrightarrow} X$ is the complement of D in X.
- (M, F) is a coherent $\mathcal{D}_X$-module equipped with a good (increasing) filtration F. We will always assume that M admits a (rational) V-filtration.

Given (M, F), define

- $F_p V^\alpha M := F_p M \cap V^\alpha M$
- $F_p V^{>\alpha} M := F_p M \cap V^{>\alpha} M$
- $F^p \mathrm{gr}_V^\alpha M := (F_p V^\alpha M)/(F_p V^{>\alpha} M)$

Lemma C

The following are equivalent:

1. The inclusion $F_p V^{>0} M \subset V^{>0} M \cap j_* j^* F^p M := \{u \in V^{>0} : j^*(u) \in j^* F_p M\}$ is an equality
2. For all $\alpha > 0$, $t : F^p V^\alpha M \to F^p V^{\alpha+1} M$ is surjective.

Proof (1 implies 2): Using the hypothesis, it is enough to show the surjectivity of

$$t : V^\alpha M \cap j_* j^* F^p M \to V^{\alpha+1} M \cap j_* j^* F^p M$$

This in turn follows from the surjectivity of

$$t : V^\alpha M \to V^{\alpha+1} M$$

(part of the definition of V-filtration) plus the invertibility of t on U.

Lemma C

The following are equivalent:

1. The inclusion $F_p V^{>0} M \subset V^{>0} M \cap j_* j^* F^p M := \{u \in V^{>0} : j^*(u) \in j^* F_p M\}$ is an equality
2. For all $\alpha > 0$, $t : F^p V^\alpha M \to F^p V^{\alpha+1} M$ is surjective.

Proof (2 implies 1): Conversely, if for $\alpha > 0$

$$m \in V^{>0} \cap j_* j^* F^p M$$

then for some $N > 0$

$$t^N m \in F^p V^{>0} M$$

We conclude by recalling that $t : V^{>0} M \to V^{>0} M$ is bijective. □

Lemma D

Assume that $\partial_t : \mathrm{gr}_V^1 M \to \mathrm{gr}_V^0 M$ is surjective. Then the following are equivalent:

1. The inclusion $F_p M \supset \sum_{i \geq 0} \partial_t^i \cdot F_{p-i} V^{>0} M$ is an equality.
2. For all $\alpha \leq 1$,

$$\partial_t : F^p \mathrm{gr}_V^\alpha M \to F^{p+1} \mathrm{gr}_V^{\alpha-1} M$$

is surjective

Before stating the next lemma, recall from last time that if M is supported inside D, then the V-filtration on M has a simple description. Kashiwara's equivalence gives us an isomorphism

$$\phi : M \xrightarrow{\sim} \bigoplus_{n \leq 0} M^n = M^0 \otimes \mathbb{C}[\partial_t]$$

where $M^n = \ker(\partial_t t - n) = \partial_t^{-n} M^0$. It is worth noting that

$$M^0 = \ker(\partial_t t : M \to M) = \ker(t : M \to M)$$

We have that

$$V^\alpha M = \phi^{-1}\Big(\bigoplus_{n \geq \lceil \alpha \rceil} M^n \Big)$$

In particular $V^0 M = M^0$, and $\mathrm{Gr}_V^\alpha M \neq 0$ only if $\alpha \in \mathbb{Z}_{\leq 0}$.

Lemma E

Assume that (M, F) is supported within D. Let $F_p M = F_p M \cap M_0$. The following are equivalent

1. $F_p M = \sum_{i \geq 0} \partial_t \cdot F_{p-i} M_0$
2. $\partial_t : F_p \mathrm{Gr}_V^{\alpha} M \to F_{p+1} \mathrm{Gr}^{\alpha-1} M$ is surjective $(\alpha < 1)$.

Proof: Let $F'_p M := \sum_{i \geq 0} \partial_t \cdot F_{p-i} M_0$. We have that

$$F' = F \iff F'_p \mathrm{Gr}_V^{-i} M = F_p \mathrm{Gr}_V^{-i} M$$

for all i. By design, if $i \in \mathbb{Z}_{\geq 0}$ then $F'_p \mathrm{Gr}_V^{-i} M = \partial^i \cdot F_{p-i} \mathrm{Gr}_V^0 M$. So

$$F' = F \iff F_p \mathrm{Gr}_V^{-i} M = \partial^i \cdot F_{p-i} \mathrm{Gr}_V^0 M$$

By induction on i one sees this is equivalent to condition 2. $\square$

Definition

We say that a filtered (coherent) $\mathcal{D}_X$-module (M, F) has a V-filtration along $D = [f = 0]$ if for each p:

- M_f has a V-filtration along $X_0 = [t = 0]$
- $t : F_p V^\alpha M_f \to F_p V^{\alpha+1} M_f$ is surjective $(\alpha > 0)$
- $\partial_t : F_p \mathrm{Gr}_V^\alpha M_f \to F_{p+1} \mathrm{Gr}^{\alpha-1} M_f$ is surjective $(\alpha > 1)$

Definition

We say that a filtered (coherent) $\mathcal{D}_X$-module (M, F) is regular and quasiunipotent along D if (M, F) has a V-filtration along D and each

$$\mathrm{Gr}_\bullet^F \mathrm{Gr}_i^W \mathrm{Gr}_V^\alpha M$$

is coherent over $\mathrm{Gr}_\bullet^F \mathcal{D}_{X_0}$. Here W is the monodromy filtration induced by the nilpotent map

$$(\partial_t t - \alpha) : \mathrm{Gr}_V^\alpha M \to \mathrm{Gr}_V^\alpha M$$

As a consequence of lemmas C and D, we have the following:

Corollary

Let (M, F) be regular and quasiunipotent along t. Assume in addition that

- $\partial_t : \mathrm{gr}_V^1 M_f \to \mathrm{gr}_V^0 M_f$ is surjective
- $\partial_t : F_p \mathrm{Gr}_V^{\alpha} M_f \to F_{p+1} \mathrm{Gr}^{\alpha-1} M_f$ is surjective (for each p)

Then

$$F_p M = \sum_{i \geq 0} \partial_t^i \cdot (V^{>0} M \cap j_* j^* F_{p-i} M)$$

$\mathrm{can}_f : \psi_{f,1}M \rightleftarrows \phi_{f,1}M : \mathrm{var}_f$

Proposition

Assume that (M, F) is regular and quasi-unipotent with respect to all $f : X \to \mathbb{C}$. Then (M, F) has a strict support decomposition iff for all f,

$$\phi_{f,1}M = \ker(\mathrm{var}_f) \oplus \mathrm{im}(\mathrm{can}_f)$$

Proof:

Suppose first that (M, F) has a strict support decomposition. Given a $D = [f = 0]$, we want to show that

$$\phi_{f,1}M = \mathrm{gr}_V^0 M_f = \ker(\mathrm{var}_f) \oplus \mathrm{im}(\mathrm{can}_f)$$

(as filtered modules).

Proof (cont.): We proceed much as in the nonfiltered case. We reduce to the case where M has strict support Z.

- If D contains Z, then
$$\mathrm{gr}_V^1 M_f = 0$$
implying that $\mathrm{gr}_V^0 M_f = \ker(\mathrm{var}_f)$ and $\mathrm{im}(\mathrm{can}_f) = 0$. The filtrations obviously coincide.
- If D does not contain Z, we already know (ignoring filtrations) that
$$\mathrm{gr}_V^0 M_f = \mathrm{im}(\mathrm{can}_f) \text{ and } \ker(\mathrm{var}_f) = 0$$
So we have a filtered iso if the RHS is given the "induced" filtration. (Question: does this agree with the filtration induced by $\mathrm{gr}_V^0 M_f$? We would need to know that each induced map
$$\partial_t : F_p(\mathrm{gr}_V^1 M) \to F_{p+1}(\mathrm{gr}_V^0 M)$$
is surjective...)

Proof (cont.): For the converse, suppose that for all f,

$$\mathrm{gr}_V^0 M_f = \ker(\mathrm{var}_f) \oplus \mathrm{im}(\mathrm{can}_f)$$

compatibly with the filtrations induced by F. We know from last time that

$$M_f = M' \oplus \mathcal{H}^0_{X_0} M_f$$

where M' has no sub- or quotient objects supported in X_0 (equivalently, in $\iota^f(D)$). We need to see that the filtrations agree.

(The filtrations on the summands are the induced filtrations; it is not automatic that this gives a filtered direct sum.)

As a start, we claim that

$$F_pV^0M_f = F_pV^0M' \oplus F_pV^0\mathcal{H}^0_{X_0}M_f$$

for all p. Given $m \in F_pV^0M_f$, by uniqueness of V we have $m = m_1 + m_2$ for some $m_1 \in V^0M$ and $m_2 \in V^0\mathcal{H}^0_{X_0}M_f$. It is enough to show that $m_2 \in F_pV^0\mathcal{H}^0_{X_0}M_f$. Because

$$F_p\mathrm{gr}^0_V M_f = F_p\ker(\mathrm{var}_f) \oplus F_p\mathrm{im}(\mathrm{can}_f)$$

this follows from the fact that the isomorphism

$$\ker(t : M \to M) \xrightarrow{\sim} \ker(t : \mathrm{gr}^0_V M \to \mathrm{gr}^1_V M)$$

from last time is filtered. (We omit the straightforward proof of this, which uses the assumption on $t : F_pV^\alpha M_f \to F_pV^{\alpha+1}M_f$.)

We have shown so far that

$$F_p V^0 M_f = F_p V^0 M' \oplus F_p V^0 \mathcal{H}^0_{X_0} M_f$$

for all p. Using the discreteness of V, and the condition that

$$\partial_t : F_p \mathrm{gr}_V^\alpha M_f \to F_{p+1} \mathrm{gr}_V^{\alpha-1} M_f$$

is surjective for $\alpha < 1$, we can deduce a similar decomposition for all $\alpha < 0$: if $m \in F_p V^\alpha M_f$, then $m = \partial_t m' + m''$ for $m' \in F_{p-1} V^{\alpha+1} M_f$ and $m'' \in F_p V^{>\alpha} M_f$. By induction m' and m'' have the desired decomposition, giving one for m. □

Theorem (Malgrange, Kashiwara)

Let M be a regular holonomic $\mathcal{D}_X$-module such that ${}^p\psi_f(\mathrm{DR}(M))$ has quasi-unipotent monodromy. Then M has a (rational) V-filtration along t.

Moreover, in this case each $\mathrm{gr}_V^\alpha M$ is a regular holonomic $\mathcal{D}_{X_0}$-module.

Now we want to compare the $\mathcal{D}_X$-module version of vanishing cycles with the "previous" notion, on the perverse sheaf side. Actually, it is nontrivial that the "previous" notion makes sense for perverse sheaves:

Theorem (Gabber)

Let $K^\bullet$ be a perverse sheaf. Then for any $f : X \to \mathbb{C}$, the following complexes are perverse:

$$^p\psi_f(K^\bullet) := \psi_f K^\bullet[-1]$$

$$^p\phi_f(K^\bullet) := \phi_f K^\bullet[-1]$$

There are morphisms

$$\mathrm{can}_f : {}^p\psi_f(K^\bullet) \to {}^p\phi_f(K^\bullet)$$

$$\mathrm{var}_f : {}^p\phi_f(K^\bullet) \to {}^p\phi_f(K^\bullet)(-1)$$

and a monodromy action

$$T : {}^p\psi_f(K^\bullet) \to {}^p\psi_f(K^\bullet)$$

Here for a perverse sheaf (with $\mathbb{Q}$-coefficients) P we write

$$P(k) := P \otimes_{\mathbb{Q}} \mathbb{Q}(k)$$

where $\mathbb{Q}(k) := (2\pi\sqrt{-1})^k\mathbb{Q} \subset \mathbb{C}$. This is called the "$k$-th Tate twist of P".

Lemma

Let $\tau : M \to M$ be a morphism in an F-linear abelian category, for a field F. Assume that $g(\tau) = 0$ for some nonzero $g(T) \in F[T]$. If

$$g = g_1 g_2$$

for relatively prime g_1,g_2, then $\ker(g_1(\tau)) \hookrightarrow M$ is a direct summand.

Proof: Application of the Chinese remainder theorem. $\square$

As a consequence, over $\mathbb{C}$ there are decompositions

$$^{p}\psi_f(K^\bullet) \cong \bigoplus_{\lambda \in \mathbb{C}^\times} {}^{p}\psi_{f,\lambda}(K^\bullet)$$

$$^{p}\phi_f(K^\bullet) \cong \bigoplus_{\lambda \in \mathbb{C}^\times} {}^{p}\phi_{f,\lambda}(K^\bullet)$$

where $\psi_{f,\lambda}$ and $\phi_{f,\lambda}$ are the "generalized eigenspaces" of eigenvalue λ.

Remark: If $0 < \lambda < 1$ then

$$\mathrm{can}_f : {}^{p}\psi_{f,\lambda} \to {}^{p}\phi_{f,\lambda}$$

is an isomorphism.

Theorem (Kashiwara, Malgrange, ...)

Let M be a regular holonomic $\mathcal{D}_X$-module. Denote $e(\alpha) := \exp(-2\pi i\alpha)$. There are canonical isomorphisms

$$\mathrm{DR}(\mathrm{gr}_V^{\alpha} M_f) \xrightarrow{\sim} {}^p\psi_{f,e(\alpha)}(\mathrm{DR}(M)), \text{ for } 0 < \alpha \leq 1$$

$$\mathrm{DR}(\mathrm{gr}_V^{\alpha} M_f) \xrightarrow{\sim} {}^p\phi_{f,e(\alpha)}(\mathrm{DR}(M)), \text{ for } 0 \leq \alpha < 1$$

such that under these isomorphisms

$$\mathrm{DR}(\partial_t : \mathrm{gr}_V^1 M_f \to \mathrm{gr}_V^0 M_f) = \mathrm{can}_f : {}^p\psi_{f,1} \to {}^p\phi_{f,1}$$

and

$$\mathrm{DR}(t : \mathrm{gr}_V^0 M_f \to \mathrm{gr}_V^1 M_f(-1)) = \mathrm{var}_f : {}^p\phi_{f,1} \to {}^p\psi_{f,1}(-1)$$

(We will comment on the "Tate twist" in the last line momentarily.)

In particular,

$$\mathrm{DR}(\partial_t t) = \mathrm{can}_f \circ \mathrm{var}_f = N : {}^p\psi_{f,1} \to {}^p\psi_{f,1}(-1)$$

where

$$N = \frac{\log(T)}{2\pi\sqrt{-1}}$$

(here T is restriction of the monodromy operator to ${}^p\psi_{f,1}$).

Definition

A regular holonomic $\mathcal{D}_X$-module with $\mathbb{Q}$-structure is a tuple (M, F, P, θ) where (M, F) is a filtered regular holonomic $\mathcal{D}_X$-module, P is a perverse sheaf over $\mathbb{Q}$ on X, and

$$\theta : P \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathrm{DR}(M)$$

is an isomorphism.

Tate twists: By definition the k-th Tate twist of $(M, F_\bullet, P, \theta)$ is

$$(M, F_{\bullet - k}, P(k), (2\pi\sqrt{-1})^k \theta)$$

Definition

Let $\mathcal{M} = (M, F, P)$ be a regular holonomic $\mathcal{D}_X$-module with $\mathbb{Q}$-structure.

- $\psi_f \mathcal{M} := \oplus_{0<\lambda\leq 1}(\mathrm{gr}_V^\alpha M_f, F_{\bullet-1}\mathrm{gr}_V^\alpha M_f, {}^p\psi_{f,e(\alpha)}P)$
- $\psi_{f,1}\mathcal{M} := (\mathrm{gr}_V^1 M_f, F_{\bullet-1}\mathrm{gr}_V^1 M_f, {}^p\psi_{f,1}P)$
- $\phi_{f,1}\mathcal{M} := (\mathrm{gr}_V^0 M_f, F_\bullet \mathrm{gr}_V^0 M_f, {}^p\phi_{f,1}P)$

Remark: The shift $F_{\bullet-1}$ in the definition of ψ_f comes from the fact that we "only" have

$$\partial_t : F_p \mathrm{gr}_V^1 M_f \to F_{p+1}\mathrm{gr}_V^0 M_f$$

After making this shift, var_f becomes a morphism

$$t : \mathrm{gr}_V^0 M_f \to \mathrm{gr}_V^1 M_f(-1)$$

Theorem (Kashiwara, Malgrange, ...)

Let M be a regular holonomic $\mathcal{D}_X$-module. Denote $e(\alpha) := \exp(-2\pi i\alpha)$ There are canonical isomorphisms

$$\mathrm{DR}(\mathrm{gr}_V^{\alpha} M_f) \xrightarrow{\sim} {}^p\psi_{f,e(\alpha)}(\mathrm{DR}(M)), \text{ for } 0 < \alpha \leq 1$$

$$\mathrm{DR}(\mathrm{gr}_V^{\alpha} M_f) \xrightarrow{\sim} {}^p\phi_{f,e(\alpha)}(\mathrm{DR}(M)), \text{ for } 0 \leq \alpha < 1$$

References (last part of talk)

- Deligne, SGA 7, II, Exp XIV, section 4
- Mebkhout and Sabbah, "D-modules and cycles évanescents"
- Mutsumi Saito, "A short course on b-functions and vanishing cycles"

Main steps of the proof of the Theorem:

1. Construct certain $\mathcal{D}_X$-modules $M_{\alpha,p}$ by "adjoining" elements of the form $t^{\beta+j}\log^k(t)/k!$ for $j \in \mathbb{Z}$ and $0 \le k \le p$.
2. Construct a canonical isomorphism of $\mathcal{D}_X$-modules

$$\mathrm{gr}_V^\alpha M \xrightarrow{\sim} \varinjlim_p i^* M_{\alpha,p}[-1] =: \psi^{\mathrm{mod}}_{t,e(\alpha)} M$$

$$(i^* := \mathbb{D} \circ i^\dagger \circ \mathbb{D})$$

3. Construct a canonical isomorphism of perverse sheaves

$$i^{-1}\mathrm{DR}(M^{mod}_\alpha) \xrightarrow{\sim} {}^p\psi_{f,e(\alpha)}(\mathrm{DR}(M))$$

$$(M^{mod}_{t,\alpha} = \varinjlim_p M_{\alpha,p})$$

On Step 1:

Fix a $\alpha \in \mathbb{Q}$. Let $N_{\alpha,p}$ be the $\mathcal{D}_{\mathbb{A}^1}$-module generated by expressions of the form

$$e_{\alpha,k} := \begin{cases} \frac{t^{-\alpha}\log^k(t)}{k!}, & \text{if } 0 \leq k \leq p \\ 0, & \text{otherwise} \end{cases}$$

where t, ∂_t act in the way you'd expect. Then

$$\begin{aligned} \partial_t t \cdot e_{j,k} =& \frac{1}{k!}[(-\alpha+1)t^{-\alpha}\log^k(t) + kt^{-\alpha}\log^{k-1}(t)] \\ =&(-\alpha+1)e_{\alpha,k} + e_{\alpha,k-1} \end{aligned}$$

implying that each $e_{\alpha,k}$ is annihilated by a power of $\partial_t t + \alpha - 1$.

Let

$$N_\alpha = \varinjlim_{p} N_{\alpha,p}$$

On Step 1:

We view $N_{\alpha,p}$ as a $\mathcal{D}_{\mathbb{A}^1}$-module (where $\mathbb{A}^1$ has coordinate t). For a $\mathcal{D}_X$-module M, and a function $t : X \to \mathbb{A}^1$ let

$$M_{\alpha,p} = M[t^{-1}] \otimes_{t^{-1}(\mathcal{O}_{\mathbb{A}^1})} t^{-1}(N_{\alpha,p})$$

Here, we abuse notation by identifying the function $t : X \to \mathbb{A}^1$ with the coordinate t on $\mathbb{A}^1$. The $\mathcal{D}_X$-module structure on $M_{\alpha,p}$ is specified as follows: given a derivation Θ,

$$\Theta(m \otimes s) = \Theta(m) \otimes s + m \otimes \tilde{\Theta}(s)$$

where $\tilde{\Theta}$ is the image of Θ under the canonical map

$$\mathcal{T}_X \to t^*\mathcal{T}_{\mathbb{A}^1}$$

On Step 1:

Notice that we then have the following key formula:

$$\begin{aligned}\partial_t t(m \otimes e_{\alpha,k}) &= (\partial_t tm) \otimes e_{\alpha,k} + m \otimes (\partial_t t \cdot e_{\alpha,k}) \\ &= (\partial_t t - \alpha + 1)m \otimes e_{\alpha,k} + m \otimes e_{\alpha,k-1}\end{aligned}$$

$M_{\alpha,p}$ has the following V-filtration:

$$V^\beta M_{\alpha,p} = \bigoplus_{0 \le k \le p} V^{\beta+\alpha-1}(M[t^{-1}]) \otimes e_{\alpha,k}$$

Let

$$M^{mod}_{t,\alpha} = \varinjlim_p M_{\alpha,p}$$

On Step 2:

Define a map $V^\alpha M \to V^1 M_{\alpha,p}$

$$m \mapsto \sum_{0 \leq k \leq p} [-(\partial_t t - \alpha)]^k m \otimes e_{\alpha,k}$$

To see why this is plausible, suppose that $(\partial_t t - \alpha)^2 \cdot m = 0$. Then, using the key formula,

$$(\partial_t t - 1)(m \otimes e_{\alpha,0} - (\partial_t t - \alpha)m \otimes e_{\alpha,1})$$

$$= (\partial_t t - \alpha)m \otimes e_{\alpha,0} - (\partial_t t - \alpha)^2 m \otimes e_{\alpha,1} - (\partial_t t - \alpha)m \otimes e_{\alpha,0} = 0$$

This map induces a map

$$\rho_p : \mathrm{gr}_V^\alpha M \to \mathrm{gr}_V^1 M_{\alpha,p}$$

On Step 2:

Lemma

For N admitting a V-filtration, there is an isomorphism

$$i^*N \xrightarrow{\sim} [0 \to \mathrm{gr}_V^1 N \xrightarrow{\partial_t} \mathrm{gr}_V^0 N \to 0]$$

where $\mathrm{gr}_V^0 N$ is in degree 0. □

Remark: Proving this lemma requires an understanding of how V and gr_V^α interact with duality. Modulo that, it is equivalent to the claim that

$$i^\dagger N \xrightarrow{\sim} [0 \to \mathrm{gr}_V^0 N \xrightarrow{t} \mathrm{gr}_V^1 N \to 0]$$

half of which was proved last time.

On Step 2:

In view of this lemma, we can regard $\rho_p : \mathrm{gr}_V^\alpha M \to \mathrm{gr}_V^1 M_{\alpha,p}$ as a morphism

$$\rho_p : \mathrm{gr}_V^\alpha M \to i^*(M_{\alpha,p})[-1]$$

Claim: For p sufficiently large, ρ_p is a quasi-isomorphism.

There are two parts to this claim:

1. $\mathrm{gr}_V^\alpha M \cong \mathcal{H}^0(i^*(M_{\alpha,p})[-1])$ $(p >> 0)$
2. $\mathcal{H}^1(i^*(M_{\alpha,p})[-1]) = 0$ $(p >> 0)$

We will prove the first part and omit proof of the second part.

We have that

$$\begin{aligned}\mathcal{H}^0(i^*(M_{\alpha,p})[-1]) &= \ker(\partial_t : \mathrm{gr}_V^1 M_{\alpha,p} \to \mathrm{gr}_V^0 M_{\alpha,p}) \\ &= \ker(t\partial_t : \mathrm{gr}_V^1 M_{\alpha,p} \to \mathrm{gr}_V^1 M_{\alpha,p})\end{aligned}$$

The key formula tells us that

$$t\partial_t(m \otimes e_{\alpha,k}) = (\partial_t t - \alpha)m \otimes e_{\alpha,k} + m \otimes e_{\alpha,k-1}$$

Therefore $\sum_{k=0}^{p} m_k \otimes e_{\alpha,k} \in \ker(t\partial_t)$ iff (for $0 \leq k \leq p-1$)

$$(t\partial_t - \alpha)m_k + m_{k+1} = 0 \text{ and } (t\partial_t - \alpha)m_p = 0$$

iff (for $0 \leq k \leq p$)

$$m_k = [-(t\partial_t - \alpha)]^k m_0 \text{ and } (\partial_t t - \alpha)^p m_0 = 0$$

So for p such that $(\partial_t t - \alpha)^p$ acts by zero on $\mathrm{gr}_V^\alpha M$, ρ_p induces the desired isomorphism. □

On Step 3:

Step 3 generalizes a result from SGA 7, discussed two weeks ago. We recall the setup:

$$\begin{array}{ccccc} X_0 & \xrightarrow{\bar{i}} & \overline{X} & \xleftarrow[\bar{j}]{} & \overline{X}^* \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X_0 & \xrightarrow{i} & X & \xleftarrow[j]{} & X^* \end{array}$$

For a coherent sheaf $\mathcal{F}$ on X, whose restriction $\mathcal{F}^*$ to X^* is locally free, let $\overline{\mathcal{F}}$ denote its restriction to $\overline{X}^*$. Define

$$\psi_\eta^{mqu}(\mathcal{F}) \hookrightarrow \bar{i}^{-1}\bar{j}_*\overline{\mathcal{F}}$$

to be the subsheaf generated by images of sections of $\overline{\mathcal{F}}$ of "moderate growth and quasi-unipotent finite determination".

On Step 3:

Rather than define this condition, we remark that any such section f of $\overline{\mathcal{F}}$ can be written as a (finite) sum

$$f = \sum_{\alpha,k} (p')^{-1}(f_{\alpha,k}) t^{\alpha} \log^{k}(t)$$

where $f_{\alpha,k}$ is a section of $\mathcal{F}^*$, $k \geq 0$, $\alpha \in \mathbb{Q}$, and $-1 \leq \alpha < 0$. In fact, this decomposition is unique, and we have an isomorphism

$$\psi_{\eta}^{mqu}(\mathcal{F}) \cong \bigoplus_{0<\alpha\leq 1} i^{-1}\left(j_* j^* \mathcal{F} \otimes_{t^{-1}(\mathcal{O}_{\mathbb{A}^1})} t^{-1}(N_{\alpha})\right)$$

On Step 3:

In particular:

$$\psi_\eta^{mqu}(\Omega_X^\bullet) \cong i^{-1} \bigoplus_{0<\alpha\leq 1} j_*\Omega_{X^*}^\bullet \otimes_{t^{-1}(\mathcal{O}_{\mathbb{A}^1})} t^{-1}(N_\alpha)$$

$$\cong i^{-1} \bigoplus_{0<\alpha\leq 1} (\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_X[t^{-1}]) \otimes_{t^{-1}(\mathcal{O}_{\mathbb{A}^1})} t^{-1}(N_\alpha)$$

$$\cong i^{-1} \bigoplus_{0<\alpha\leq 1} \Omega_X^\bullet \otimes_{\mathcal{O}_X} (\mathcal{O}_X[t^{-1}] \otimes_{t^{-1}(\mathcal{O}_{\mathbb{A}^1})} t^{-1}(N_\alpha))$$

$$\cong i^{-1} \bigoplus_{0<\alpha\leq 1} \mathrm{DR}((\mathcal{O}_X)_\alpha^{mod})$$

On Step 3:

Deligne's result from SGA 7, II, Exp XIV, section 4, gave an isomorphism

$$\psi_\eta^{mqu}(\Omega^\bullet_{X^*}) \xrightarrow{\sim} {}^p\psi(\mathrm{DR}(\mathcal{O}_X))$$

One shows that (for general M) there is a natural isomorphism

$$i^{-1}\mathrm{DR}(M^{mod}_{t,\alpha}) \xrightarrow{\sim} \mathrm{DR}(\psi^{\mathrm{mod}}_{t,e(\alpha)} M)$$

Combining this with the above remarks, we get isomorphisms

$$i^{-1}\mathrm{DR}((\mathcal{O}_X)^{mod}_\alpha) \xrightarrow{\sim} {}^p\psi_{e(\alpha)}(\mathrm{DR}(\mathcal{O}_X))$$

as claimed in step 3. Deligne actually handles the more general case of a vector bundle, and the general (regular holonomic) case can be reduced to this one by devissage.