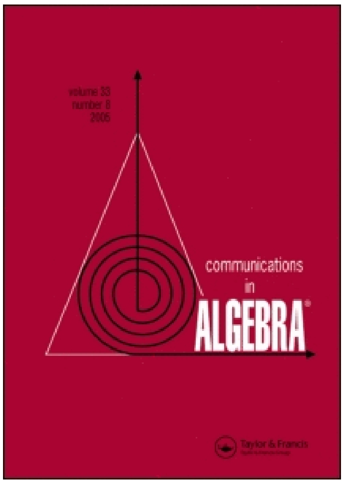

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Donu Arapura[a]; Su-Jeong Kang[b]
[a] Department of Mathematics, Purdue University, West Lafayette, Indiana, USA [b] Department of Mathematics and Computer Sciences, Providence College, Providence, Rhode Island, USA

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KÄHLER–DE RHAM COHOMOLOGY AND CHERN CLASSES

Donu Arapura[1] and Su-Jeong Kang[2]
[1]*Department of Mathematics, Purdue University, West Lafayette, Indiana, USA*
[2]*Department of Mathematics and Computer Sciences, Providence College, Providence, Rhode Island, USA*

Constraints on the Chern classes of a vector bundle on a possibly singular algebraic variety are found, which are stronger than the obvious Hodge theoretic constraints. This is done by showing that these lift to Chern classes in the hypercohomology of the complex of Kähler differentials.

Key Words: Chern classes; de Rham filtration; Hodge filtration; Kähler–de Rham cohomology.

2000 Mathematics Subject Classification: 14F43; 14C30.

Given a complex projective variety X, algebraic cycles give rise to homology classes. When X is nonsingular, Poincaré duality allows us to view these as cohomology classes in $H^*(X, \mathbb{Q})$. By standard arguments, the subspace spanned by algebraic cycles coincides with the subspace spanned by monomials in the Chern classes of algebraic vector bundles. (This can be deduced from the isomorphism $\mathrm{CH}^*(\mathrm{X}) \otimes \mathbb{Q} \cong \mathrm{K}^0(\mathrm{X}) \otimes \mathbb{Q}$ [9, 15.2.16].) If X is singular, then algebraic cycles and Chern classes go their separate ways. The Hodge theoretical issues for algebraic cycles are better understood. Jannsen [13] has formulated a natural Hodge conjecture characterizing algebraic cycles on singular varieties. By contrast, for Chern classes, it is not even clear (to us) what a reasonable conjecture would be. In general, it is not difficult to see that the pth Chern class of a vector bundle is again a Hodge cycle in the sense that it lies in the $F^pH^{2p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{Q})$, where F^p is the Hodge filtration of the canonical mixed Hodge structure (cf. [5, §9.1]). However, it was pointed out by Barbieri-Viale and Srinivas [4], that the first Chern classes satisfy a stronger condition. In this article, we are able to generalize this restriction to classes of higher degree. The key ingredient is Kähler–de Rham cohomology which is the hypercohomology $H^i_{\mathrm{KD}}(X, \mathbb{C}) = \mathbb{H}^i(X, \Omega^\bullet_X)$. The formal study of this cohomology is carried out here. Although generally $H^i_{\mathrm{KD}}(X, \mathbb{C}) \neq H^i(X, \mathbb{C})$, it does behave like ordinary cohomology in many respects. This cohomology carries a natural filtration $\mathcal{F}^pH^i_{\mathrm{KD}}(X, \mathbb{C})$. Work of Du Bois implies that there is a projection $H^i_{\mathrm{KD}}(X, \mathbb{C}) \to H^i(X, \mathbb{C})$ mapping $\mathcal{F}$ to a subfiltration of F that we call the de Rham filtration F_{DR}. Then we show that the classes $c_p(\mathcal{E})$ of a vector bundle $\mathcal{E}$ lie $F^p_{\mathrm{DR}}H^{2p}(X, \mathbb{C})$

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Address correspondence to Dr. Su-Jeong Kang, Department of Mathematics, Central Michigan University, Mt. Pleasant, MI 48859, USA; E-mail: skang2@providence.edu

because they lift to classes with values in $\mathcal{F}^p H^{2p}_{\mathrm{KD}}(X)$. We give examples to show that in general $F^q_{\mathrm{DR}} H^i(X) \subsetneq F^q H^i(X)$. So this is indeed a stronger condition.

All varieties will be defined over $\mathbb{C}$. Sheaves will be usually considered with respect to the classical topology.

1. KÄHLER–DE RHAM COHOMOLOGY

Let X be an algebraic variety. Let Ω^1_X be the sheaf of holomorphic Kähler differential 1-forms on X, and let $\Omega^p_X = \wedge^p \Omega^1_X$. These sheaves fit into a complex $(\Omega^\bullet_X, d)$ [12, §16.6]. When X is embeddable into a smooth variety M, $(\Omega^\bullet_X, d)$ is a quotient of the usual holomorphic de Rham complex $(\Omega^\bullet_M, d)$ by the differential ideal generated by the ideal sheaf $\mathcal{I}_X$ of X.

We call the hypercohomology of the complex

$$H^i_{\mathrm{KD}}(X, \mathbb{C}) = \mathbb{H}^i(X, \Omega^\bullet_X)$$

Kähler–de Rham cohomology rather than de Rham cohomology, which has other meanings in the singular case. Kähler–de Rham cohomology is a contravariant functor on the category of varieties.

There is a spectral sequence

$$E_1^{pq} = H^q(X, \Omega^p_X) \Rightarrow H^{p+q}_{\mathrm{KD}}(X, \mathbb{C}), \tag{1}$$

where $d_1 : H^q(X, \Omega^p_X) \to H^q(X, \Omega^{p+1}_X)$ is given by differentiation. This implies that Kähler–de Rham cohomology is finite dimensional when X is a complete variety. It also follows by GAGA, that the hypercohomology of the algebraic de Rham complex would be isomorphic to it. This cohomology carries a natural filtration

$$\mathcal{F}^p H^i_{\mathrm{KD}}(X, \mathbb{C}) = \mathrm{im}[\mathbb{H}^i(X, F^p\Omega^\bullet_X) \to \mathbb{H}^i(X, \Omega^\bullet_X)]$$

induced by the stupid filtration $F^\bullet$ on $\Omega^\bullet_X$:

$$F^p \Omega^n_X = \begin{cases} 0 & \text{for } n < p \\ \Omega^n_X & \text{for } n \geq p. \end{cases}$$

Let $\mathbb{C} = \mathbb{C}_X$ denote the constant sheaf on X. Then there is a natural map $\epsilon : \mathbb{C} \to \Omega^\bullet_X$ given by the inclusion $\mathbb{C} \hookrightarrow \mathcal{O}_X$. When X is smooth, $\mathbb{C} \to \Omega^\bullet_X$ is a resolution by the Poincaré lemma, but this need not hold in general. (See [3] for discussion of these matters). If X is both smooth and complete, then $(H^i_{\mathrm{KD}}(X, \mathbb{C}), \mathcal{F}^\bullet)$ coincides with $(H^i(X, \mathbb{C}), F^\bullet)$, where $F^\bullet$ is the Hodge filtration.

Recall that the *Du Bois complex* [6, 16] is a filtered complex $(\widetilde{\Omega}^\bullet_X, \widetilde{F}^\bullet)$ (or more accurately an object in a filtered derived category) with the following properties:

(i) There is a canonical morphism of filtered complexes $\rho : (\Omega^\bullet_X, F^\bullet) \to (\widetilde{\Omega}^\bullet_X, \widetilde{F}^\bullet)$;
(ii) The composition $\mathbb{C} \to \Omega^\bullet_X \to \widetilde{\Omega}^\bullet_X$ gives a quasi-isomorphism $\mathbb{C} \cong \widetilde{\Omega}^\bullet_X$;
(iii) When X is complete, the filtration on $H^i(X, \mathbb{C})$ induced by the filtration $\widetilde{F}^\bullet$ coincides with the Hodge filtration. In this case, the spectral sequence on hypercohomology associated to the filtration $\widetilde{F}^\bullet$ degenerates at E_1.

Morphisms ϵ and ρ induce canonical morphisms on hypercohomologies

$$H^i(X, \mathbb{C}) \xrightarrow{\epsilon} H^i_{\mathrm{KD}}(X, \mathbb{C}) \xrightarrow{\rho} \mathbb{H}^i(X, \widetilde{\Omega}^\bullet_X) = H^i(X, \mathbb{C}) \tag{2}$$

such that the composition is the identity [5]. The *de Rham filtration* $F^\bullet_{\mathrm{DR}}$ on the singular cohomology $H^i(X, \mathbb{C})$ [1] is defined to be

$$F^p_{\mathrm{DR}} H^i(X, \mathbb{C}) = H^i(X, \mathbb{C}) \cap \rho\left(\mathscr{F}^p H^i_{\mathrm{KD}}(X, \mathbb{C})\right). \tag{3}$$

2. PROPERTIES OF KÄHLER–DE RHAM COHOMOLOGY

Lemma 2.1. *Let $p : X \times \mathbb{A}^1 \to X$ be the projection and $i : X \to X \times \mathbb{A}^1$ defined by $i(x) = (x, 0)$. Fix an affine open cover $\mathscr{U}$ of $X \times \mathbb{A}^1$. The induced map on the total complex of Čech complexes with respect to $\mathscr{U}$*

$$p^* \circ i^* : C^\bullet(\mathscr{U}, \Omega^\bullet_{X\times\mathbb{A}^1}) \to C^\bullet(\mathscr{U}, \Omega^\bullet_{X\times\mathbb{A}^1})$$

is homotopy equivalent to the identity.

Proof. Let $(sC^\bullet, D)$ be the total complex of the double complex $C^\bullet(\mathscr{U}, \Omega^\bullet_{X\times\mathbb{A}^1})$:

$$sC^n = \bigoplus_{p+q=n} C^p(\mathscr{U}, \Omega^q_{X\times\mathbb{A}^1}), \qquad D(\alpha_{pq}) = d(\alpha_{pq}) + (-1)^q \delta(\alpha_{pq}),$$

where $d : C^p(\mathscr{U}, \Omega^q_{X\times\mathbb{A}^1}) \to C^p(\mathscr{U}, \Omega^{q+1}_{X\times\mathbb{A}^1})$ and $\delta : C^p(\mathscr{U}, \Omega^q_{X\times\mathbb{A}^1}) \to C^{p+1}(\mathscr{U}, \Omega^q_{X\times\mathbb{A}^1})$. We claim that there is a chain homotopy $H : sC^n \to sC^{n-1}$ satisfying

$$DH + HD = \mathrm{id}^* - (p^* \circ i^*). \tag{4}$$

Let t be a coordinate of $\mathbb{A}^1$ and U be an affine open set in $X \times \mathbb{A}^1$. For any $\beta + (\alpha \wedge dt) \in \Omega^q_{U\times\mathbb{A}^1} = \Omega^q_U \oplus (\Omega^{q-1}_U \wedge dt)$, define

$$\begin{aligned} h_q : \Omega^q_{U\times\mathbb{A}^1} &\longrightarrow \Omega^{q-1}_{U\times\mathbb{A}^1} \\ \beta + (\alpha \wedge dt) &\mapsto (-1)^{q-1}\alpha t \end{aligned}$$

This induces a map $H : sC^n \to sC^{n-1}$ via $H(\alpha_{pq}) = h_q(\alpha_{pq})$. Direct calculation shows that (4) is satisfied. □

Corollary 2.2 ($\mathbb{A}^1$-Homotopy Invariance).

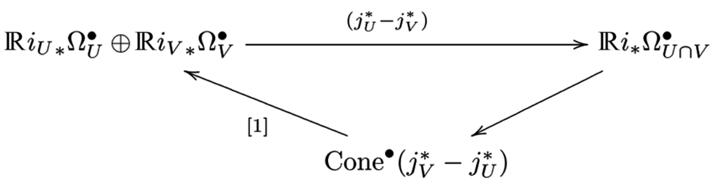
$$H^i_{\mathrm{KD}}(X \times \mathbb{A}^1, \mathbb{C}) = \mathbb{H}^i(X \times \mathbb{A}^1, \Omega^\bullet_{X\times\mathbb{A}^1}) \cong \mathbb{H}^i(X, \Omega^\bullet_X) = H^i_{\mathrm{KD}}(X, \mathbb{C}).$$

Proof. Since $p \circ i = \mathrm{id}$, we have $i^* \circ p^* = \mathrm{id}^*$. Lemma 2.1 implies that $p^* \circ i^* = \mathrm{id}^*$ on cohomology. □

Lemma 2.3. *Let $X = U \cup V$, where U, V are Zariski open subsets of X. Then there is a Mayer–Vietoris sequence*

$$\cdots \to H^{i-1}_{\mathrm{KD}}(U \cap V, \mathbb{C}) \to H^{i}_{\mathrm{KD}}(X, \mathbb{C}) \to H^{i}_{\mathrm{KD}}(U, \mathbb{C}) \oplus H^{i}_{\mathrm{KD}}(V, \mathbb{C}) \\ \to H^{i}_{\mathrm{KD}}(U \cap V, \mathbb{C}) \to \cdots .$$

Proof. Let $i_U : U \hookrightarrow X$, $i_V : V \hookrightarrow X$, $j_U : U \cap V \hookrightarrow U$, $j_V : U \cap V \hookrightarrow V$, and $i : U \cap V \hookrightarrow X$ be inclusions. We have a distinguished triangle



where $\mathrm{Cone}^\bullet(j_U^* - j_V^*) = \mathbb{R}i_{U*}\Omega_U^{\bullet+1} \oplus \mathbb{R}i_{V*}\Omega_V^{\bullet+1} \oplus \mathbb{R}i_*\Omega^\bullet_{U\cap V}$ with a differential $D(\alpha, \beta, \gamma) = (-d\alpha, -d\beta, d\gamma + \alpha|_{U\cap V} - \beta|_{U\cap V})$. By checking at the stalk level, it can be shown that we have a quasi-isomorphism $\iota^\bullet : \Omega_X^{\bullet+1} \to \mathrm{Cone}^\bullet(j_U^* - j_V^*)$ defined by $\iota^p = ((-1)^p i_U^*, (-1)^p i_V^*, 0)$. Now we get the desired long exact sequence. □

Remark 2.4. It is easy to see that the maps in the long exact sequence

$$\cdots \to H^{i-1}_{\mathrm{KD}}(U \cap V, \mathbb{C}) \to H^{i}_{\mathrm{KD}}(X, \mathbb{C}) \to H^{i}_{\mathrm{KD}}(U, \mathbb{C}) \oplus H^{i}_{\mathrm{KD}}(V, \mathbb{C}) \\ \to H^{i}_{\mathrm{KD}}(U \cap V, \mathbb{C}) \to \cdots$$

preserve the natural filtration $\mathcal{F}^\bullet$.

The product on the complex $\Omega^\bullet_X$ gives rise to a cup product on $H^*_{\mathrm{KD}}(X, \mathbb{C})$ via

$$\mathbb{H}^i(X, \Omega^\bullet_X) \times \mathbb{H}^j(X, \Omega^\bullet_X) \longrightarrow \mathbb{H}^{i+j}(X, \Omega^\bullet_X).$$

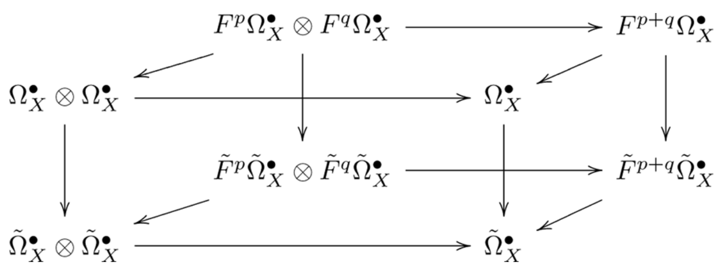
Lemma 2.5.

(i) $\mathcal{F}^p H^i_{\mathrm{KD}}(X, \mathbb{C}) \times \mathcal{F}^q H^j_{\mathrm{KD}}(X, \mathbb{C}) \subseteq \mathcal{F}^{p+q} H^{i+j}_{\mathrm{KD}}(X, \mathbb{C})$.
(ii) *This cup product is compatible with cup product on singular cohomology of X under ρ.*

Proof. By construction, the Du Bois complex $\widetilde{\Omega}^\bullet_X = \mathbb{R}\pi_*\Omega^\bullet_{X_\bullet}$, where $\pi : X_\bullet \to X$ is a simplicial resolution. Therefore, it carries product $\widetilde{\Omega}^\bullet_X \otimes \widetilde{\Omega}^\bullet_X \to \widetilde{\Omega}^\bullet_X$ given by the composition

$$\mathbb{R}\pi_*(\Omega^\bullet_{X_\bullet}) \otimes \mathbb{R}\pi_*(\Omega^\bullet_{X_\bullet}) \to \mathbb{R}\pi_*(\Omega^\bullet_{X_\bullet} \otimes^{\mathbb{L}}_{\mathbb{C}} \Omega^\bullet_{X_\bullet}) \\ \cong \mathbb{R}\pi_*(\Omega^\bullet_{X_\bullet} \otimes_{\mathbb{C}} \Omega^\bullet_{X_\bullet}) \to \mathbb{R}\pi_*\Omega^\bullet_{X_\bullet},$$

where the last map is the usual product on Kähler differentials. This fits into the following commutative diagram:



and the lemma follows immediately. □

We can now summarize all the basic properties of the de Rham filtration.

Proposition 2.6. *Let X be a projective algebraic variety over $\mathbb{C}$. Then*

(i) *If X is smooth projective, then $F^p_{\mathrm{DR}}H^i(X, \mathbb{C}) = F^pH^i(X, \mathbb{C})$;*
(ii) *The de Rham filtration is functorial with respect to pullbacks and products*;
(iii) $F^p_{\mathrm{DR}}H^i(X, \mathbb{C}) \subseteq F^pH^i(X, \mathbb{C})$ *for any i, p.*

Proof. (i) and (ii) are clear by definition and Lemma 2.5. (iii) is an immediate consequence of the splitting (2). □

3. THE CHERN–DE RHAM CLASS

For a complex projective variety X, there is a morphism

$$d\log : H^1(X, \mathcal{O}_X^*) \;\to\; \mathbb{H}^2(X, \Omega^\bullet_X) = H^2_{\mathrm{KD}}(X, \mathbb{C})$$

induced by a morphism of complexes $d\log : \mathcal{O}_X^*[-1] \to \Omega^\bullet_X$. For a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ on X, we define

$$\mathrm{c}_1^{\mathrm{dr}}(\mathcal{L}) = d\log(\ell) \in H^2_{\mathrm{KD}}(X, \mathbb{C}), \tag{5}$$

where $\ell \in H^1(X, \mathcal{O}_X^*)$ is the class corresponding to $\mathcal{L}$. Since the morphism $d\log : \mathcal{O}_X^*[-1] \to \Omega^\bullet_X$ factors through $F^1\Omega^\bullet_X$, this implies that

$$\mathrm{c}_1^{\mathrm{dr}}(\mathcal{L}) \in \mathrm{im}[\mathbb{H}^2(X, F^1\Omega^\bullet_X) \to H^2_{\mathrm{KD}}(X, \mathbb{C})] = \mathcal{F}^1 H^2_{\mathrm{KD}}(X, \mathbb{C}). \tag{6}$$

There is an alternate way to define this invariant. Consider the exponential exact sequence $0 \to \mathbb{Z}(1) \to \mathcal{O}_X \stackrel{\exp}{\to} \mathcal{O}_X^* \to 0$, and let $\partial : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}(1))$ be the boundary map. Via the following compositions

$$H^1(X, \mathcal{O}_X^*) \stackrel{\partial}{\to} H^2(X, \mathbb{Z}(1)) \stackrel{\alpha}{\to} H^2(X, \mathbb{C}) \stackrel{\epsilon}{\to} H^2_{\mathrm{KD}}(X, \mathbb{C}),$$

where $\alpha : \mathbb{Z}(1) \hookrightarrow \mathbb{C}$ is an inclusion, we can define

$$(\mathrm{c}_1^{\mathrm{dr}})'(\mathscr{L}) = (\epsilon \circ \alpha \circ \partial)(\ell). \tag{7}$$

The relation between (5) and (7) can be seen via the following diagram:

$$\begin{array}{ccccc} H^2(X,\mathbb{C}) & \xrightarrow{\ \epsilon\ } & H^2_{\mathrm{KD}}(X,\mathbb{C}) & \longleftarrow & \mathbb{H}^2(X,F^1\Omega^\bullet_X) \\ \Vert & & & & \Vert \\ H^2(X,\mathbb{C}) & \longleftarrow & \mathbb{H}^2(X,[\mathbb{C}\to\mathscr{O}_X]) & \xrightarrow{\ d\ } & \mathbb{H}^2(X,F^1\Omega^\bullet_X) \\ \uparrow{\scriptstyle \alpha} & & \uparrow & & \uparrow{\scriptstyle d\log} \\ H^2(X,\mathbb{Z}(1)) & \longleftarrow & \mathbb{H}^2(X,[\mathbb{Z}(1)\to\mathscr{O}_X]) & \xrightarrow[\cong]{\ \exp\ } & H^1(X,\mathscr{O}^*_X) \end{array} \tag{8}$$

i.e.,

$$\mathrm{c}_1^{\mathrm{dr}}(\mathscr{L}) = d\log(\ell) = -\epsilon \circ \alpha \circ \partial(\ell) = -(\mathrm{c}_1^{\mathrm{dr}})'(\mathscr{L}). \tag{9}$$

Lemma 3.1. *Let $\mathscr{L}$ be a line bundle on X and $c_1(\mathscr{L}) \in H^2(X, \mathbb{Z})$ be the classical 1st Chern class. Then,*

$$\mathrm{c}_1^{\mathrm{dr}}(\mathscr{L}) = \epsilon(c_1(\mathscr{L})), \qquad \rho(\mathrm{c}_1^{\mathrm{dr}}(\mathscr{L})) = c_1(\mathscr{L}),$$

where ϵ and ρ are maps in (2).

Proof. Note that there is an embedding $\iota : X \hookrightarrow M$ into a smooth variety and a line bundle $\mathscr{F}$ on M such that $\mathscr{L} = \iota^*(\mathscr{F})$ [8, Lemma, p. 159]. By considering the diagram for M similar to (8), we get $c_1(\mathscr{F}) = -(\alpha \circ \partial)(f)$, where $f \in H^1(M, \mathscr{O}_M^*)$ the class of $\mathscr{F}$. Since all maps in the diagram behave well under ι^*,

$$(\mathrm{c}_1^{\mathrm{dr}})'(\mathscr{L}) = (\mathrm{c}_1^{\mathrm{dr}})'(\iota^*\mathscr{F}) = (\epsilon \circ \alpha \circ \partial)(\iota^*(f)) = -\epsilon(\iota^*(c_1(\mathscr{F}))) = -\epsilon(c_1(\mathscr{L})).$$

Now (9) implies the first identity. The second identity follows immediately from the first one and the splitting (2). $\square$

Now let $\mathscr{E}$ be a vector bundle of rank $r \geq 1$ on X, and $\mathbb{P}(\mathscr{E})$ be the associated projective bundle with the projection $\pi : \mathbb{P}(\mathscr{E}) \to X$. Let $\xi = \mathrm{c}_1^{\mathrm{dr}}(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)) \in H^2_{\mathrm{KD}}(\mathbb{P}(\mathscr{E}), \mathbb{C})$.

Lemma 3.2. *The pull back map $\pi^* : H^*_{\mathrm{KD}}(X, \mathbb{C}) \to H^*_{\mathrm{KD}}(\mathbb{P}(\mathscr{E}), \mathbb{C})$ gives $H^*_{\mathrm{KD}}(\mathbb{P}(\mathscr{E}), \mathbb{C})$ a free $H^*_{\mathrm{KD}}(X, \mathbb{C})$-module structure generated by* $1, \xi, \xi^2, \ldots, \xi^{r-1}$.

Proof. First assume $\mathscr{E}$ is a trivial bundle of rank $r \geq 1$, then $\mathbb{P}(\mathscr{E}) \cong X \times \mathbb{P}^{r-1}$. Since $\mathbb{P}^{r-1}$ is smooth, we can see by a local calculation that

$$\Omega^\bullet_{X\times\mathbb{P}^{r-1}} = \bigoplus_{\ell=0}^{r-1} \left(\pi^*\Omega_X^{\bullet-\ell} \wedge q^*\Omega^\ell_{\mathbb{P}^{r-1}}\right),$$

where $\pi : X \times \mathbb{P}^{r-1} \to X$ and $q : X \times \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ are projections. Hence

$$H^i_{\mathrm{KD}}(X \times \mathbb{P}^{r-1}, \mathbb{C}) \cong \bigoplus_{\ell=0}^{r-1} \mathbb{H}^i\left(X \times \mathbb{P}^{r-1},\ \pi^*\Omega_X^{\bullet-\ell} \wedge q^*\Omega^{\ell}_{\mathbb{P}^{r-1}}\right). \tag{10}$$

For a fixed ℓ, we have a spectral sequence

$$\begin{aligned} {}_{\ell}E_1^{pq} &= H^q(X \times \mathbb{P}^{r-1}, \pi^*\Omega_X^{p-\ell} \wedge q^*\Omega^{\ell}_{\mathbb{P}^{r-1}}) \\ &\Rightarrow \mathbb{H}^{p+q}\left(X \times \mathbb{P}^{r-1},\ \pi^*\Omega_X^{\bullet-\ell} \wedge q^*\Omega^{\ell}_{\mathbb{P}^{r-1}}\right). \end{aligned}$$

By the Künneth formula,

$$\begin{aligned} {}_{\ell}E_1^{pq} &= H^q(X \times \mathbb{P}^{r-1}, \pi^*\Omega_X^{p-\ell} \wedge q^*\Omega^{\ell}_{\mathbb{P}^{r-1}}) \\ &= \bigoplus_{t+s=q} \pi^*H^t(X, \Omega_X^{p-\ell}) \cup q^*H^s(\mathbb{P}^{r-1}, \Omega^{\ell}_{\mathbb{P}^{r-1}}) \\ &= \pi^*H^{q-\ell}(X, \Omega_X^{p-\ell}) \cup q^*H^{\ell}(\mathbb{P}^{r-1}, \Omega^{\ell}_{\mathbb{P}^{r-1}}) \\ &= \pi^*H^{q-\ell}(X, \Omega_X^{p-\ell}) \cup \xi^{\ell}. \end{aligned}$$

Since it converges to $\pi^*\mathbb{H}^{p+q-\ell}(X, \Omega_X^{\bullet-\ell}) \cup \xi^{\ell} = \pi^*H^{p+q-2\ell}_{\mathrm{KD}}(X, \mathbb{C}) \cup \xi^{\ell}$, we have

$$\mathbb{H}^i\left(X \times \mathbb{P}^{r-1},\ \pi^*\Omega^{\bullet-\ell} \wedge q^*\Omega^{\ell}_{\mathbb{P}^{r-1}}\right) \cong \pi^*H^{i-2\ell}_{\mathrm{KD}}(X, \mathbb{C}) \cup \xi^{\ell}.$$

By combining this with (10), we have the lemma for the trivial bundle

$$H^i_{\mathrm{KD}}(X \times \mathbb{P}^{r-1}, \mathbb{C}) \cong \bigoplus_{\ell=0}^{r-1} \pi^*H^{i-2\ell}_{\mathrm{KD}}(X, \mathbb{C}) \cup \xi^{\ell}. \tag{11}$$

For a general vector bundle $\mathscr{E}$ on X, choose an open cover $\{U_i\}_{i\in I}$ trivializing the bundle $\mathscr{E}$. We show that this decomposition is compatible with glueing along this open cover. Let U_1, U_2 be any two open sets in this cover. Note $\mathscr{E}\big|_{U_i} \cong U_i \times \mathbb{A}^r$ and $\mathbb{P}(\mathscr{E}\big|_{U_i}) = \mathbb{P}(\mathscr{E})\big|_{U_i} \cong U_i \times \mathbb{P}^{r-1}$ for $i = 1, 2$. The Mayer–Vietoris exact sequence (Lemma 2.3) induces a long exact sequence

$$\begin{aligned} \cdots &\to \bigoplus_{\ell} \pi^*H^{i-2\ell}_{\mathrm{KD}}(U_1 \cup U_2) \cup \xi^{\ell} \\ &\to \left(\bigoplus_{\ell} \pi^*H^{i-2\ell}_{\mathrm{KD}}(U_1) \cup \xi^{\ell}\right) \oplus \left(\bigoplus_{\ell} \pi^*H^{i-2\ell}_{\mathrm{KD}}(U_2) \cup \xi^{\ell}\right) \\ &\to \bigoplus_{\ell} \pi^*H^{i-2\ell}_{\mathrm{KD}}(U_1 \cap U_2) \cup \xi^{\ell} \\ &\to \bigoplus_{\ell} \pi^*H^{i-2\ell+1}_{\mathrm{KD}}(U_1 \cup U_2) \cup \xi^{\ell} \to \cdots, \end{aligned}$$

where we abuse the notation by writing ξ for $\xi|_*$ where $* = U_1, U_2, U_1 \cap U_2, U_1 \cup U_2$ (restriction of ξ to a relevant space). Combining this with the Mayer–Vietoris exact sequence on the Kähler–de Rham cohomology of $\mathbb{P}(\mathscr{E})$ and the decomposition (10) over open sets U_1, U_2 and $U_1 \cap U_2$, we get a commutative diagram

$$\begin{array}{ccccc}
H^i_{\mathrm{KD}}(\mathbb{P}(\mathscr{E})|_{U_1\cup U_2}) & \longrightarrow & H^i_{\mathrm{KD}}(\mathbb{P}(\mathscr{E})|_{U_1}) \oplus H^i_{\mathrm{KD}}(\mathbb{P}(\mathscr{E})|_{U_2}) & \longrightarrow & H^i_{\mathrm{KD}}(\mathbb{P}(\mathscr{E})|_{U_1\cap U_2}) \\
\uparrow \phi & & \uparrow \cong & & \uparrow \cong \\
\bigoplus_\ell \pi^* H^{i-2\ell}_{\mathrm{KD}}(U_1 \cup U_2) \cup \xi^\ell & \longrightarrow & A_1 \oplus A_2 & \longrightarrow & \bigoplus_\ell \pi^* H^{i-2\ell}_{\mathrm{KD}}(U_1 \cap U_2) \cup \xi^\ell,
\end{array}$$

where $A_i = \bigoplus_\ell \pi^* H^{i-2\ell}_{\mathrm{KD}}(U_i) \cup \xi^\ell$. Since there are two more isomorphisms on the left side of ϕ, 5-lemma implies that ϕ is an isomorphism. Thus, the decomposition (10) is compatible with glueing, and this proves the lemma for general case. □

Corollary 3.3. *Given a sequence of projective space bundles $X_n \to X_{n-1} \to \cdots \to X_1$, the map $H^*_{\mathrm{KD}}(X_1, \mathbb{C}) \to H^*_{\mathrm{KD}}(X_n, \mathbb{C})$ is injective.*

Remark 3.4. This corollary applies when X_n is given by the splitting construction [9, p. 51].

We can now adapt Grothendieck's definition [11] to this setting.

Definition 3.5. Let $\mathscr{E}$ be a vector bundle of rank r on X. For each $i = 0, 1, \ldots, r$, we define the *ith Chern–de Rham-class* $\mathrm{c}_i^{\mathrm{dr}}(\mathscr{E}) \in H^{2i}_{\mathrm{KD}}(X, \mathbb{C})$ as the unique sequence of classes such that:

CD1. $\mathrm{c}_0^{\mathrm{dr}}(\mathscr{E}) = 1$;
CD2. $\sum_{i=0}^r (-1)^i \pi^* \mathrm{c}_i^{\mathrm{dr}}(\mathscr{E}) \xi^{r-i} = 0$ in $H^{2r}_{\mathrm{KD}}(\mathbb{P}(\mathscr{E}), \mathbb{C})$.

Since in $H^{2r}_{\mathrm{KD}}(\mathbb{P}(\mathscr{E}), \mathbb{C})$, ξ^r has a unique linear combination of $1, \xi, \xi^2, \ldots, \xi^{r-1}$ with the coefficients in $\pi^* H^*_{\mathrm{KD}}(X, \mathbb{C})$, this definition makes sense. Furthermore, by Lemma 3.1 and the usual theory of Chern classes, we have the following lemma.

Lemma 3.6. *For a vector bundle $\mathscr{E}$,*

$$\mathrm{c}_i^{\mathrm{dr}}(\mathscr{E}) = \epsilon \circ c_i(\mathscr{E}) \in \epsilon(H^{2i}(X, \mathbb{Q}))$$

$$\mathrm{c}_i^{\mathrm{dr}}(\mathscr{E}) \in H^{2i}_{\mathrm{KD}}(X, \mathbb{C}) \cap \epsilon(H^{2i}(X, \mathbb{Q})) \cong H^{2i}_{\mathrm{KD}}(X, \mathbb{C}) \cap H^{2i}(X, \mathbb{Q}).$$

Definition 3.7. For a vector bundle $\mathscr{E}$ of rank r on X, we define *the Chern–de Rham-polynomial* $\mathrm{c}_t^{\mathrm{dr}}$ to be

$$\mathrm{c}_t^{\mathrm{dr}}(\mathscr{E}) = \mathrm{c}_0^{\mathrm{dr}}(\mathscr{E}) + \mathrm{c}_1^{\mathrm{dr}}(\mathscr{E})t + \cdots + \mathrm{c}_r^{\mathrm{dr}}(\mathscr{E})t^r.$$

Lemma 3.8. *The Chern–de Rham-class satisfies the following axioms*:

C1. *If $f : X' \to X$ is a morphism and $\mathscr{E}$ a vector bundle on X, then for each i*

$$\mathrm{c}_i^{\mathrm{dr}}(f^*(\mathscr{E})) = f^*\mathrm{c}_i^{\mathrm{dr}}(\mathscr{E});$$

C2. *If $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ is an exact sequence of vector bundles on X, then*

$$\mathrm{c}_t^{\mathrm{dr}}(\mathscr{E}) = \mathrm{c}_t^{\mathrm{dr}}(\mathscr{E}')\mathrm{c}_t^{\mathrm{dr}}(\mathscr{E}'');$$

C3. *If $\mathscr{E}$ has a filtration with line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_r$ as quotients, then*

$$\mathrm{c}_t^{\mathrm{dr}}(\mathscr{E}) = \prod_{i=1}^{r} \mathrm{c}_t^{\mathrm{dr}}(\mathscr{L}_i).$$

Proof. This follows from the usual properties of Chern classes [11] and the previous lemma. □

Corollary 3.9. *The pth Chern–de Rham-class determines a homomorphism*

$$\mathrm{c}_p^{\mathrm{dr}} : K^0(X) \longrightarrow H^{2p}_{\mathrm{KD}}(X, \mathbb{C})$$

on the Grothendieck group of vector bundles on X such that $\mathrm{c}_p^{\mathrm{dr}}([\mathscr{E}]) = \mathrm{c}_p^{\mathrm{dr}}(\mathscr{E})$.

Corollary 3.10. *For a vector bundle $\mathscr{E}$ of rank r on X, $\mathrm{c}_p^{\mathrm{dr}}(\mathscr{E}) \in \mathscr{F}^p H^{2p}_{\mathrm{KD}}(X, \mathbb{C})$.*

Proof. By the splitting principle [9, 3.2.3], there is a variety P and a morphism $\pi : P \to X$ such that $\pi^* : H^*_{\mathrm{KD}}(X, \mathbb{C}) \to H^*_{\mathrm{KD}}(P, \mathbb{C})$ is injective (Remark 3.4) and $\pi^*(\mathscr{E})$ has a filtration by sub-bundles

$$\pi^*(\mathscr{E}) = \mathscr{E}_0 \supset \mathscr{E}_1 \supset \cdots \supset \mathscr{E}_r \supset 0$$

with line bundle quotients $\mathscr{E}_i/\mathscr{E}_{i+1} = \mathscr{L}_i$ for each i. Then by Lemma 3.8, we have

$$\mathrm{c}_t^{\mathrm{dr}}(\mathscr{E}) \cong \pi^*\mathrm{c}_t^{\mathrm{dr}}(\mathscr{E}) = \mathrm{c}_t^{\mathrm{dr}}(\pi^*\mathscr{E}) = \prod_{i=0}^{r-1}(1 + \mathrm{c}_1^{\mathrm{dr}}(\mathscr{L}_i)t)$$

and hence by (6) and Lemma 2.5,

$$\mathrm{c}_p^{\mathrm{dr}}(\mathscr{E}) = \sum_{i_1<\cdots<i_p} \left(\prod_{j=1}^{p} \mathrm{c}_1^{\mathrm{dr}}(\mathscr{L}_{i_p}) \right) \in \mathscr{F}^p H^{2p}_{\mathrm{KD}}(X, \mathbb{C}).$$ □

This immediately yields the refinement of [4] mentioned in the introduction.

Proposition 3.11. *For a vector bundle $\mathscr{E}$ on X, $c_p(\mathscr{E}) \in F^p_{\mathrm{DR}} H^{2p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{Q})$.*

First Proof. This is an immediate consequence of Corollary 3.10 and Lemma 3.6. □

We can give alternate proof (in fact our original proof) using only the results of the first section.

Second Proof. Note that for any vector bundle $\mathcal{E}$ on X, there is a smooth variety M, embedding $i : X \to M$ and a vector bundle $\mathcal{F}$ on M, such that $i^*\mathcal{F} \cong \mathcal{E}$ [8, Lemma, p. 159]. Then,

$$c_p(\mathcal{E}) = i^* c_p(\mathcal{F}) \in i^*(F^p H^{2p}(M, \mathbb{C})) = i^*(F^p_{\mathrm{DR}} H^{2p}(M, \mathbb{C})) \subset F^p_{\mathrm{DR}} H^{2p}(X, \mathbb{C}). \qquad \square$$

Let $\mathrm{ch}_p(\mathcal{E})$ denoted the pth component of the Chern character homomorphism [9, 3.2.3]

$$K^0(X) \otimes \mathbb{Q} \xrightarrow{\mathrm{ch}} H^{\mathrm{even}}(X, \mathbb{Q}) \to H^{2p}(X, \mathbb{Q}).$$

Lemma 3.12. *The image of* ch_p *also lies in* $F^p_{\mathrm{DR}} H^{2p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{Q})$.

Proof. By the previous proposition the image of the pth Chern class map lies in $F^p_{\mathrm{DR}} H^{2p}(X, \mathbb{C})$. Since $\mathrm{ch}_p(\mathcal{E})$ can be written as polynomial in the Chern classes, the lemma follows from Lemma 2.5. $\square$

Remark 3.13. When X is nonsingular, the Hodge conjecture for X is equivalent to the surjectivity of

$$\mathrm{ch}_p : K^0(X) \otimes \mathbb{Q} \to \bigoplus_p F^p_{\mathrm{DR}} H^{2p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{Q}).$$

It is not clear if the corresponding statement is reasonable in general, so we leave it as a question: Is there a counterexample to the surjectivity when X is singular (or not!)?

Following [7, 14], we can define the naive Deligne cohomology of X by $H^i_{\mathcal{D}}(X, \mathbb{Z}(p)) = \mathbb{H}^i(X, \mathbb{Z}_{\mathcal{D}}(p))$, where $\mathbb{Z}_{\mathcal{D}}(p) : \mathbb{Z}(p) \to \mathcal{O}_X \to \cdots \to \Omega^{p-1}_X$.

Lemma 3.14.

(i) *The natural map* $H^i_{\mathcal{D}}(X, \mathbb{Z}(p)) \to F^p_{\mathrm{DR}} H^i(X, \mathbb{C}) \cap H^i(X, \mathbb{Z}(p))$ *is surjective.*
(ii) $\mathrm{im}(c_1) = F^1_{\mathrm{DR}} H^2(X, \mathbb{C}) \cap H^2(X, \mathbb{Z}(1))$.

Proof. For (i), consider the exact sequence

$$\cdots \to H^i_{\mathcal{D}}(X, \mathbb{Z}(p)) \xrightarrow{q} H^i(X, \mathbb{Z}(p)) \to \mathbb{H}^i(X, \Omega^\bullet_X / F^p \Omega^\bullet_X) \to \cdots$$

induced by an exact sequence $0 \to (\Omega^\bullet_X / F^p \Omega^\bullet_X)[-1] \to \mathbb{Z}_{\mathcal{D}}(p) \to \mathbb{Z}(p) \to 0$. Since $H^i(X, \mathbb{Z}(p)) \hookrightarrow H^i(X, \mathbb{C}) \xrightarrow{\epsilon} H^i_{\mathrm{KD}}(X, \mathbb{C})$, we have

$$\begin{aligned} \mathrm{im}(q) &= \ker[H^i(X, \mathbb{Z}(p)) \to \mathbb{H}^i(X, \Omega^\bullet_X / F^p \Omega^\bullet_X)] \\ &= H^i(X, \mathbb{Z}(p)) \cap F^p_{\mathrm{DR}} H^i(X, \mathbb{C}) \end{aligned}$$

(ii) follows from the surjectivity of $H^1(X, \mathscr{O}_X^*) = H^2_{\mathscr{D}}(X, \mathbb{Z}(1)) \to F^1_{\mathrm{DR}} H^2(X, \mathbb{C}) \cap H^2(X, \mathbb{Z}(1))$ in (i), since $\mathbb{Z}_{\mathscr{D}}(1)$ is quasi-isomorphic to $\mathscr{O}_X^*[-1]$. □

4. EXAMPLES

By the splitting (2), we see that $H^i(X, \mathbb{C})$ injects into $H^i_{\mathrm{KD}}(X, \mathbb{C})$. We present a couple of one dimensional varieties with an isolated singularity such that $H^i_{\mathrm{KD}}(X, \mathbb{C})$ is strictly larger than $H^i(X, \mathbb{C})$. We check this by comparing dimensions. First, we recall that there are two basic invariants of a complex analytic isolated hypersurface singularity (X, p): the Milnor number μ and the Tjurina number τ. If f is the local equation, the Milnor number μ is the nth Betti number (where $n = \dim X$) of the Milnor fiber $f^{-1}(t) \cap \{z \mid ||z|| < \epsilon\}$. The Tjurina number $\tau = \dim Ext^1(\Omega^1_{X,p}, \mathscr{O}_{X,p})$ is the dimension of the space of first order deformations of (X, p). We have $\tau \leq \mu$.

Throughout this section, we will use the following notation:

$$H^i(X, \Omega^j_X)_{\text{d-closed}} = \ker[H^i(X, \Omega^j_X) \xrightarrow{d} H^i(X, \Omega^{j+1}_X)].$$

We first prove the following lemma which we will need in Proposition 4.2.

Lemma 4.1. *Let X be an irreducible affine curve with an isolated hypersurface singularity at p. Then*

$$\dim\left(\frac{H^0(X, \Omega^1_X)}{H^0(X, \Omega^1_X)_{\text{d-closed}}}\right) \leq \tau.$$

Proof. The sheaf Ω^2_X is a skyscraper sheaf supported at p. If $f(x, y)$ is a local equation of X, then

$$\Omega^2_{X,p} = \mathscr{O}_p dx \wedge dy/(df \wedge dx, df \wedge dy) \cong \mathscr{O}_p/(f_x, f_y).$$

The dimension of the right equals τ by the formula [10, p. 159]. Therefore, it follows that

$$\dim\left(\frac{H^0(X, \Omega^1_X)}{H^0(X, \Omega^1_X)_{\text{d-closed}}}\right) \leq \dim H^0(X, \Omega^2_X) = \tau.$$ □

Proposition 4.2. *Let X be an irreducible affine curve with an isolated hypersurface singularity at p. Then,*

$$\dim H^1_{\mathrm{KD}}(X, \mathbb{C}) \geq \dim H^1(X, \mathbb{C}) + \mu - \tau.$$

Proof. The spectral sequence (1) gives rise to an exact sequence

$$0 \to E^{1,0}_\infty \to H^1_{\mathrm{KD}}(X, \mathbb{C}) \to E^{0,1}_\infty \to 0, \tag{12}$$

where

$$E^{0,1}_\infty = E^{0,1}_3 \subset E^{0,1}_2 = H^1(X, \mathscr{O}_X)_{\text{d-closed}} = 0,$$

$$E_{\infty}^{1,0} = E_2^{1,0} = \frac{H^0(X, \Omega_X^1)_{\text{d-closed}}}{dH^0(X, \mathcal{O}_X)}.$$

The first group vanishes since X is affine and therefore Stein. Lemma 4.1, (12), and the exact sequence

$$0 \to \frac{H^0(X, \Omega_X^1)_{\text{d-closed}}}{dH^0(X, \mathcal{O}_X)} \to \frac{H^0(X, \Omega_X^1)}{dH^0(X, \mathcal{O}_X)} \to \frac{H^0(X, \Omega_X^1)}{H^0(X, \Omega_X^1)_{\text{d-closed}}} \to 0$$

imply that

$$\dim \left(\frac{H^0(X, \Omega_X^1)}{dH^0(X, \mathcal{O}_X)} \right) \leq \dim H^1_{\text{KD}}(X, \mathbb{C}) + \tau.$$

Now by [2, Theorem 1.2], we have

$$\mu + \dim H_1(X, \mathbb{C}) = \dim \left(\frac{H^0(X, \Omega_X)}{dH^0(X, \mathcal{O}_X)} \right) \leq \dim H^1_{\text{KD}}(X, \mathbb{C}) + \tau. \tag{13}$$

Since $\dim H_1(X, \mathbb{C}) = \dim H^1(X, \mathbb{C})$ by the universal coefficient theorem, the proposition is proved. □

Corollary 4.3. *Let X be an irreducible projective curve with an isolated hypersurface singularity p with invariants μ and τ. Then*

$$\dim H^1_{\text{KD}}(X, \mathbb{C}) \geq \dim H^1(X, \mathbb{C}) + \mu - \tau.$$

Proof. Let $\{U, V\}$ be an affine covering of X with $p \in U$ and $p \notin V$. Then from the usual Mayer–Vietoris sequence, the fact that second Betti number of an affine curve is 0 and the connectedness of these sets, we see that

$$\begin{aligned} \dim H^1(X, \mathbb{C}) &= \dim H^1(U, \mathbb{C}) + \dim H^1(V, \mathbb{C}) \\ &\quad - \dim H^1(U \cap V, \mathbb{C}) + \dim H^2(X, \mathbb{C}). \end{aligned}$$

We can see directly that $H^0_{\text{KD}}(U, \mathbb{C}) = \mathbb{C}$. Since affine varieties are Stein, the spectral sequence (1) implies that $H^2_{\text{KD}}(U, \mathbb{C}) = 0$. Thus Lemma 2.3 yields a very similar formula

$$\begin{aligned} \dim H^1_{\text{KD}}(X, \mathbb{C}) &= \dim H^1_{\text{KD}}(U, \mathbb{C}) + \dim H^1(V, \mathbb{C}) \\ &\quad - \dim H^1(U \cap V, \mathbb{C}) + \dim H^2_{\text{KD}}(X, \mathbb{C}). \end{aligned}$$

The corollary now follows from these formulas and the Lemma 4.2. □

By using the Maple subroutines of Rossi and Terracini [17] we can compute these invariants on a machine to obtain the following example.

Example 4.4. Let X be a curve defined by an affine equation $F(x, y) = x^5 + y^5 + x^2y^2$. It has a unique isolated singularity at the origin, $\mu = 11$ and $\tau = 10$. Hence by Proposition 4.2, $H^1_{\mathrm{KD}}(X, \mathbb{C}) \supsetneq H^1(X, \mathbb{C})$.

It is still conceivable that the Hodge and de Rham filtrations could coincide in this example. We give a different example, where the filtrations are different.

Example 4.5. Let C be a projective plane cuspidal curve with a normalization $f : \widetilde{C} \to C$ with $g(\widetilde{C}) > 0$. Then, $F^1_{\mathrm{DR}}H^1(C, \mathbb{C})$ is strictly contained in $F^1H^1(C, \mathbb{C})$.

Proof. We have an exact sequence

$$0 \to \mathscr{K} \to f^*\Omega^1_C \to \Omega^1_{\widetilde{C}} \to \Omega^1_{\widetilde{C}/C} \to 0, \tag{14}$$

where $\mathscr{K}$ is a kernel of $f^*\Omega^1_C \to \Omega^1_{\widetilde{C}}$. Let p be the cusp on C. Let $x^3 - y^2$ be a local analytic equation of C in a neighborhood of $p = (0, 0)$. Then the normalization is given by setting $x = t^2$ and $y = t^3$, i.e.,

$$\mathbb{C}\{x, y\}/(x^3 - y^2) \longrightarrow \mathbb{C}\{t\}; \qquad (x, y) \mapsto \frac{y}{x} = \frac{t^3}{t^2} = t.$$

By direct calculation, we can see that $\Omega^1_{\widetilde{C}/C}$ and $\mathscr{K}$ are skyscraper sheaves supported at $q = f^{-1}(p)$ with one dimensional stalk. In fact, $\mathscr{K}$ is generated by $\alpha = 3ydx - 2xdy$. Observe that $d\alpha \neq 0$, so that the space of closed forms on C can be identified with the image $\mathrm{im}[H^0(C, \Omega^1_C) \to H^0(\widetilde{C}, \Omega^1_{\widetilde{C}})]$. Now the exact sequence (14) can be broken into

$$0 \to \mathscr{K} \to f^*\Omega^1_C \to \mathscr{T} \to 0 \quad \text{and} \quad 0 \to \mathscr{T} \to \Omega^1_{\widetilde{C}} \to \mathbb{C}_q \to 0,$$

where $\mathscr{T} = f^*\Omega^1_C/\mathscr{K}$. From the second exact sequence, we obtain $\mathscr{T} \cong \Omega^1_{\widetilde{C}}(-q)$. Then

$$\begin{aligned} F^1_{\mathrm{DR}}H^1(C, \mathbb{C}) &= \mathrm{im}[\mathbb{H}^1(C, \Omega^{\geq 1}_C) \to H^1(C, \mathbb{C})] \\ &\subset \mathrm{im}[H^0(C, \Omega^1_C)_{\text{d-closed}} \to H^1(C, \mathbb{C})] \\ &= \mathrm{im}[H^0(C, \Omega^1_C) \to H^0(\widetilde{C}, \Omega^1_{\widetilde{C}})] \\ &\overset{(*)}{\cong} H^0(\widetilde{C}, \mathscr{T}) = H^0(\widetilde{C}, \Omega^1_{\widetilde{C}}(-q)). \end{aligned}$$

The isomorphism marked $(*)$ follows from the diagram

$$\begin{array}{ccccccccccc} & & H^0(C, \Omega^1_C) & & & & & & & & \\ & & \downarrow & \searrow & & & & & & & \\ 0 & \to & H^0(\tilde{C}, \Omega^1_{\tilde{C}}(-q)) & \to & H^0(\tilde{C}, \Omega^1_{\tilde{C}}) & \to & H^0(q, \mathbb{C}) & \to & H^1(\tilde{C}, \Omega^1_{\tilde{C}}(-q)) & \to & H^1(\tilde{C}, \Omega^1_{\tilde{C}}) \to 0. \end{array}$$

From the above long exact sequence, we get

$$\dim H^0(\widetilde{C}, \Omega^1_{\widetilde{C}}(-q)) - g(\widetilde{C}) + 1 - \dim H^1(\widetilde{C}, \Omega^1_{\widetilde{C}}(-q)) + 1 = 0. \tag{15}$$

Since $\dim H^1(\widetilde{C}, \Omega^1_{\widetilde{C}}(-q)) = \dim H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(q)) = 1$ by Serre duality, (15) implies

$$\dim F^1_{\mathrm{DR}} H^1(C, \mathbb{C}) \leq \dim H^0(\widetilde{C}, \Omega^1_{\widetilde{C}}(-q)) = g(\widetilde{C}) - 1 = \dim F^1 H^1(C, \mathbb{C}) - 1.$$

This shows that $F^1_{\mathrm{DR}} H^1(C, \mathbb{C})$ is strictly smaller than $F^1 H^1(C, \mathbb{C})$. □

We can get examples where $H^i_{\mathrm{KD}}(X, \mathbb{C}) \neq H^i(X, \mathbb{C})$ and $F^p_{\mathrm{DR}} H^i(X, \mathbb{C}) \neq F^p H^i(X, \mathbb{C})$ for arbitrary i and p by taking products of the above examples. Barbieri-Viale and Srinivas [4, pp. 39–40] have given an example of a normal surface where $F^1_{\mathrm{DR}} H^2(X, \mathbb{C}) \neq F^1 H^2(X, \mathbb{C})$. Note that the codimensions of the singular loci of all of the examples discussed are at most 2. When $i > \dim X + \dim X_{sing} + 1$, $H^i_{\mathrm{KD}}(X, \mathbb{C}) \cong H^i(X, \mathbb{C})$ as filtered groups [1].

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