SOME APPLICATIONS OF POSITIVE CHARACTERISTIC TECHNIQUES TO VANISHING THEOREMS

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To Joe Lipman

These are notes to my talk at Lipman's birthday conference. Some details have appeared in [A1, A2]. The rest hopefully will someday.

1. Differential Forms

Let X be a smooth complete variety over $\mathbb{C}$. Then as a consequence of Hodge theory + GAGA:

$$H^i(X^{an}, \mathbb{C}) \cong H^i(X, \Omega_X^\bullet) \cong \bigoplus_{a+b=i} H^b(X, \Omega_X^a)$$

If X is complete but *singular*, then both these isomorphisms will usually *fail* for the complex of Kähler differentials. However:

Theorem 1 (Du Bois)**.** *There exists well defined objects $\mathbf{\Omega}_X^p \in ObD(X) = ObD^bCoh(X)$ such that*

$$\mathbf{\Omega}_{X-Sing}^p \cong \Omega_{X-Sing}^p$$

and

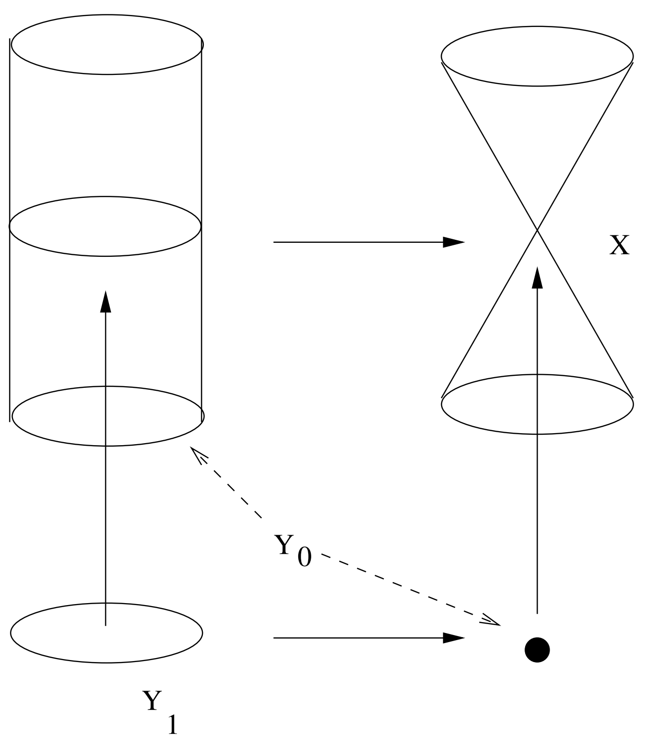
$$H^i(X^{an}, \mathbb{C}) \cong \bigoplus_{a+b=i} H^b(X, \mathbf{\Omega}_X^a)$$

(Furthermore, the dimensions of the summands on the right coincide with Deligne's mixed Hodge numbers.)

Rough idea of construction: Using resolution of singularities, we can construct a simplicial diagram of smooth projective varieties

$$\dots Y_1 \rightrightarrows Y_0 \xrightarrow{\pi_0} X$$

with the same cohomology as X i.e. the map $\pi_\bullet$ satisfies cohomological descent. Given a sheaf F, the direct images $\mathbb{R}\pi_{i*}\pi_i^*F$ fit into a "double complex", the descent condition is that F is quasi-isomorphic to the total complex $Tot(\mathbb{R}\pi_{i*}\pi_i^*F)$ for every F. Then $\boldsymbol{\Omega}_X^p$ is defined as $Tot(\mathbb{R}\pi_*\Omega^p_{Y_\bullet})$.



Note that mixed Hodge structures are constructed in a similar way. So compatibility is not surprising.

If $Z \subseteq X$ is closed, we get a restriction map $\mathbf{\Omega}^p_X \to \mathbf{\Omega}^p_Z$. We can complete this to a distinguished triangle

$$\mathbf{\Omega}^p_{X,Z} \to \mathbf{\Omega}^p_X \to \mathbf{\Omega}^p_Z \to \mathbf{\Omega}^p_{X,Z}[1]$$

2. Frobenius amplitude

Let X be a complete variety defined over a field k. Let $Z \subset X$ be closed, and $E = E^\bullet \in ObD(X)$. The *Frobenius or F-amplitude* $\phi(E)$ is an integer which measures *positivity* of the complex in an inverse sense (*smaller is better*).

Then ϕ has the following properties:

(1) If E is a coherent sheaf in degree 0,

$$0 \le \phi(E) \le \dim X$$

(2) If $\phi(E[i]) = \phi(E) - i$.

(3) If E fits into a distinguished triangle

$$E' \to E \to E'' \to E'[1]$$

then $\phi(E) \le max(\phi(E'), \phi(E''))$.

(4) If E' has a finite resolution by locally free sheaves, then $\phi(E' \otimes E'') \le \phi(E') + \phi(E'')$.

(5) If $f : Y \to X$ is morphism, then

$$\phi(f^*E) \leq \phi(E) + max \dim f^{-1}(x)$$

(6) If $f : Y \to X$ is etale, then $\phi(f_*E) = \phi(E)$.

(7) If $char\, k = 0$ and E is an *ample vector bundle*,

$$\phi(E) \leq rank(E)$$

More generally, if E is d-ample in Sommese's sense, then $\phi(E) \leq d + rank(E)$.

(8) When $k = \mathbb{C}$ and E a vector bundle, $\phi(E)$ can be estimated in terms of *curvature or convexity*. Specifically, if E is $q+1$-convex, and in particular admits a Hermitean metric such that form $\Theta(\xi, \bar{\xi}, -, -)$, on T_x, has at most q nonpositive eigenvalues for all $\xi \neq 0$ and all $x \in X$, then

$$\phi(E) \leq rank(E) + q.$$

Here

$$\Theta \in C^\infty(E^* \otimes \overline{E}^* \otimes T_X^* \otimes \overline{T_X}^*)$$

is the curvature tensor.

(9) If $\{\mathcal{E}_t\}$ is a flat family of vector bundles, then $\phi(E_t)$ is upper semicontinuous.

Items (4), (7), (8) and (9) are really theorems. I consider these deeper than the main theorem given below. The remaining properties are elementary consequences of the definition (given later).

3. The vanishing theorem

MAIN THEOREM

Let X be a complete variety over a field of characteristic 0 with a closed subset Z. Then for any complex $E \in ObD(X)$,

$$H^a(X, \mathbf{\Omega}^b_{X,Z} \otimes E) = 0$$

for $a + b > \dim X + \phi(E)$.

When combined with the previous estimates, these yield a number of specific vanishing theorems, including previously known results:

(1) If E is an ample line bundle, $Z = \emptyset$ and X smooth, then this is the Akizuki-Kodaira-Nakano vanishing theorem.

(2) For E as above, $Z = \emptyset$ but X general, this due Navarro Aznar et. al.

(3) For an ample vector bundle E, $Z = \emptyset$ and X smooth this due to Le Potier.

(4) For a d-ample vector bundle E, $Z = \emptyset$ and X smooth this due to Sommese.

(5) For E an ample line bundle, $b = \dim X$, Z a divisor normal crossings, this is a special case of Kawamata-Viehweg. (The full version needs a modification of ϕ...)

4. Definition of F-amplitude

I will start with the definition in positive characteristic, where it is most natural.

Let k be a field of characteristic $p > 0$. If $X = Spec\, R$ is an affine scheme over k, the (absolute) Frobenius $F : X \to X$ is induced by the pth power map $r \mapsto r^p$ on R.

In general, given a scheme X over k, $F : X \to X$ is the identity on the underlying space, and the pth power map on O_X. If E is a locally free sheaf on X, let

$$E^{(p)} = F^* E.$$

If E given by a cocycle $g_{ij} \in GL_r(O_X(U_{ij}))$, $E^{(p)}$ is given by the cocycle g_{ij}^p. It follows that $E^{(p)} = E^{\otimes p}$ if E is line bundle. This operation extends the derived category in the obvious way:

$$E^{(p)} = \mathbb{L}F^* E$$

Let $E^{(p^N)}$ be the Nth iterate of this operation.

Then $\phi(E)$ is the smallest integer such that for every locally free sheaf $\mathcal{F}$,

$$H^i(X, E^{(p^N)} \otimes \mathcal{F}) = 0$$

for $i > \phi(E)$ and $N \gg 0$

Now suppose $char\, k = 0$. The definition is given by specialization. Given a variety X over k and a bounded complex $E^\bullet$ of coherent sheaves, we can find a finitely generated subalgebra $A \subset k$, and a scheme $\mathcal{X} \to Spec\, A$ with a complex $\mathcal{E}^\bullet$ such that

$$\begin{array}{ccc} \mathcal{X} & \leftarrow & X \\ \downarrow & & \downarrow \\ Spec\, A & \leftarrow & Spec\, k \end{array}$$

is cartesian, and $E^\bullet = \mathcal{E}^\bullet|_X$. The closed fibers of $\mathcal{X}$ are schemes over finite fields, thus $\mathcal{X}$ forms a bridge between the worlds of characteristic 0 and p. Then we define

$\phi(E)$ is the smallest integer such that $\phi(\mathcal{E}|_{X_q}) \leq \phi(E)$ for almost all closed points $q \in Spec\, A$.

This is independent of $(\mathcal{X}, \mathcal{E})$, so this notion is well defined.

5. Bootstrapping lemma

Theorem 2 (Deligne-Illusie). *Suppose that X is a smooth projective variety over a perfect field k of characteristic $p > 0$. If $\dim X < p$ and X lifts modulo p^2 (i.e to $W(k)/(p^2)$), then*

$$F_*\Omega^\bullet_X \cong \bigoplus \Omega^i_X[-i]$$

holds in $D(X)$.

As a corollary, we obtain the following bootstrapping lemma:

Lemma 1.

$$\sum_{a+b=n} h^a(X, \Omega^b_X \otimes E) \le \sum_{a+b=n} h^a(X, \Omega^b_X \otimes E^{(p)})$$

As usual, we write $h^i = \dim H^i$.

Proof. The expression

$$\sum_{a+b=n} h^a(X, \Omega^b_X \otimes E^{(p)})$$

is the dimension of

$$\bigoplus H^{n-i}(\Omega^i_X \otimes E)$$

which is isomorphic

$$H^n(F_*\Omega^\bullet_X \otimes E).$$

This last group is the abutment of a spectral sequence with

$$E_1^{ab} = H^b(F_*\Omega^a_X \otimes E) \cong H^b(\Omega^a_X \otimes E^{(p)}),$$

and this yields the above inequality. □

With a little bit of work, the bootstrapping lemma can be extended to the singular case:

Lemma 2. *If X is a complete variety with a finite (strict) simplicial resolution $Y_\bullet \to X$ such that the whole diagram lifts mod p^2 and $\dim Y_\bullet < p$, then*

$$\sum_{a+b=n} h^a(X, \mathbf{\Omega}_X^b \otimes E) \leq \sum_{a+b=n} h^a(X, \mathbf{\Omega}_X^b \otimes E^{(p)})$$

The vanishing theorem can easily be deduced from this and specialization.

6. Estimates on F-amplitude

I want to say some words about proofs of the inequalities listed earlier in section 2.

Let V be a finite vector dimensional vector defined over a field k. When $char\, k = 0$, the *Schur* functors $S^\lambda(V)$, as λ varies over all partitions of $r = \dim V$, give the positive irreducible representations of $GL(V)$. The Schur functors can be defined (with care) even when $char\, k = p > 0$, but they need no longer be irreducible. In particular, the symmetric power $S^p(V) = S^{(p,0,\ldots)}(V)$ contains a submodule $V^{(p)}$ consisting of pth powers of elements of V. This inclusion can be extended to a $GL(V)$-equivariant resolution

$$0 \to V^{(p)} \to S^p(V) \to S^{(p-1,1)}(V) \to \ldots S^{(p-r,1,\ldots 1)}(V) \to 0$$

by work of Carter-Lusztig. This extends automatically to locally free sheaves. Thus vanishing statements for Schur powers lead to vanishing statements for Frobenius powers, and this is one of the key ingredient in the proof of inequalities (7) and (8).

There is another ingredient needed above, and also for (9). The fact that F-amplitude involves an infinite number of conditions creates technical problems for specialization arguments. We define the *level* $\lambda = \lambda(E)$ of the E on $\mathbb{P}^N$ to be the smallest integer such that

$$H^{\lambda+1}(E(-1)) = H^{\lambda+2}(E(-2)) = \ldots = 0$$

λ is upper semicontinuous in flat families. Notice that $\lambda = 0$ iff the Castelnuovo-Mumford regularity is 0. We clearly have

$$\lambda(E^{(p^n)}) \leq \phi(E)$$

for all $n >> 0$. In the opposite direction, we have

Theorem 3. *If $X \subseteq \mathbb{P}^N$ is embedded by a sufficiently high (in a precise sense) power of a very ample line bundle then*

$$\phi(E) \leq \lambda(E(\dim X))$$

(9) follows easily from these inequalities.

7. Notes

These were added in response to questions during and after the talk.

(1) In which derived category does E live? I'm still trying to sort this out. The safest answer for now is to take E in the bounded derived category of vector bundles (which is an exact category, so this makes sense [BBD]. These are perfect complexes in a strong sense.)

(2) How do Du Bois' [D] and Hartshorne's [H2] procedures compare? Du Bois gives a bit more than I stated. In fact, there is a filtered complex which computes the cohomology $H^i(X^{an}, \mathbb{C})$ which is unique in some filtered derived category. I merely described the associated graded. Hartshorne's procedure of embedding $X \subset Y$ in something smooth and completing along X also yields $H^i(\widehat{\Omega}^\bullet_Y) \cong H^i(X^{an}, \mathbb{C})$, but one can't get the Hodge filtration this way. To put it another way, the complexes of Du Bois and Hartshorne are not filtered quasi-isomorphic.

(3) How does this relate to p-ampleness? For a vector bundle in char p, $\phi(E) = 0$ implies p-ampleness [H1], but not conversely. Some people (perhaps starting with Hartshorne though he doesn't remember) call the stronger condition "cohomological p-ampleness".

(4) Is $\phi(\mathcal{E}|_{X_q})$ essentially constant in the specialization? This would great, but I have no idea how $\phi(\mathcal{E}|_{X_q})$ behaves as q varies.

(5) Is there a good bound for $\phi(S^\lambda(E))$? I hope so. This one of the things I've been thinking about.

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